Advances in Pure Mathematics, 2011, 1, 170-183
doi:10.4236/apm.2011.14031 Published Online July 2011 (http://www.SciRP.org/journal/apm)

Non-integrability of Painlevé V Equations in the Liouville Sense and Stokes Phenomenon

Tsvetana Stoyanova
Tsvetana Stoyanova, *Department of Mathematics and Informatics*, *Sofia University*, *Sofia*, *Bulgaria*
E-mail: *cveti@fmi.uni-sofia.bg*
Received March 15, 2011; *revised May* 2, 2011; *accepted May* 15, 2011

Abstract

In this paper we are concerned with the integrability of the fifth Painlevé equation (P_V) from the point of view of the Hamiltonian dynamics. We prove that the Painlevé V equation (2) with parameters $\kappa_\infty = 0, \kappa_0 = -\theta$ for arbitrary complex θ (and more generally with parameters related by Bäclund transformations) is non integrable by means of meromorphic first integrals. We explicitly compute formal and analytic invariants of the second variational equations which generate topologically the differential Galois group. In this way our calculations and Ziglin-Ramis-Morales-Ruiz-Simó method yield to the non-integrable results.

Keywords: Differential Galois theory, Painlevé V equation, Hamiltonian Systems, Stokes Phenome-nonAsymptotic Theory

1. Introduction

The six Painlevé equations ($P_I - P_{VI}$) were introduced and first studied by Paul Painlevé [1] and his student B. Gambier [2] who classified all the rational differential equations of the second order

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = R\left(t, y, \frac{\mathrm{d}y}{\mathrm{d}t}\right)$$

free of movable critical points. The solutions of these equations define some new functions, the so-called Painlevé transcendents or Painlevé functions.

Although the Painlevé equations were discovered from strictly mathematical considera-tions they have recently appeared in several physical applications. Among field-theoretical problems which can be solved in terms of the considered below Painlevé V transcendent we mention the two-point correlation functions at zero temperature for the one-dimensional impenetrable Bose gas (Jimbo, Miwa, Mori, Sato [3]).

In the present article we deal with the fifth Painlevé equation, P_V,

$$\ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\dot{y}^2 - \frac{1}{t}\dot{y} + \frac{(y-1)^2}{2t^2}\left(\kappa_\infty^2 y - \frac{\kappa_0^2}{y}\right) - (\theta+1)\frac{y}{t} + \frac{\delta y(y+1)}{(y-1)} \tag{1}$$

where $t \in \mathbb{C}$ and $\kappa_\infty, \kappa_0, \theta, \delta$ are arbitrary complex parameters. It is well known that when $\delta = 0, \theta = -1$ equation (1) can be solved by quadratures, [4]. If $\theta \neq -1$ and $\delta = 0$ the fifth Painlevé equation (1) is equivalent to the third Painlevé equation P_{III}. In this paper we investigate the generic case of P_V when $\delta \neq 0$. Hence, by replacing t by $\sqrt{-2\delta t}$ one can normalize the constant as $\delta = -1/2$ [5], and so we consider

$$\ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\dot{y}^2 - \frac{1}{t}\dot{y} + \frac{(y-1)^2}{2t^2}\left(\kappa_\infty^2 y - \frac{\kappa_0^2}{y}\right) - (\theta+1)\frac{y}{t} - \frac{y(y+1)}{2(y-1)} \tag{2}$$

As the Painlevé equations can be written as time-dependent Hamiltonian systems of $1+1/2$ degrees of freedom (see Malnquist [6] and Okamoto [5]) their integrability should be considered in the context of Hamiltonian systems. We recall that by this we mean the existence of enough meromorphic first integrals (in our case-two). In [7] Morales-Ruiz raises the question about the integrability of the Painlevé transcendents as Hamil-tonian systems. Later Morales-Ruiz in [8], and Stoya-nova and Christov in [9] obtain a non-integrable result for Painlevé II family. Non-integrability of the Painlevé

VI equation for some particular values of the parameters is proved by Horozov and Stoyanova in [10] and by Stoyanova in [11]. In the present note we continue the study of Painlevé transcendents with the fifth Painlevé equation and obtain an analogous result for one family of the parameters. Our method uses the differential Galois approach to non-integrability of Hamiltonian systems [12] which is an extension of the Ziglin theory [13, 14]. In particular, studying the differential Galois group of the first and second variational equations along a particular rational non-equilibrium solution we can find non-integrable results. It appears that the corresponding variational equations have an irregular singularity and new difficulty have to be overcome.

Our main result is the following theorem:

main Assume that $\kappa_\infty = 0, \kappa_0 = -\theta$ where θ is an arbitrary complex parameter. Then the fifth Painlevé equation (2) is not integrable. We chose to investigate P_V (2) for these values of the parameters because then the Hamiltonian system (11) possesses a simple rational solution. The point is our method requires one single-valued solution.

By Bäcklund transformations of P_V we can extend the result of main for an infinite subfamily of P_V (2): genm Assume that $\kappa_\infty + \kappa_0 = \pm\theta + m$ where m is even and at least one κ_∞, κ_0 is integer. Then the fifth Painlevé equation (2) is not integrable.

The paper is organized as follows. In section 2 we recall the main results of the Ziglin-Ramis-Morales-Ruiz -Simó theory of non-integrability of the Hamiltonian systems, of the differential Galois theory and asymtotic theory for ordinary differential equations needed in the proofs. In section 3 we prove the non-integrability of the fifth Painlevé equation (2) for $\kappa_\infty = 0, \kappa_0 = -\theta$ and $\theta \notin -\mathbb{N}^*$ (witt). In n1 we prove the non-integrability of P_V (2) for $\kappa_\infty = 0, \kappa_0 = -\theta = 1$. In section 4 using Bäcklund transformations of Painlevé V equation we extend the results of section 3 to the entire orbit of the parameters and establish the main results of this article.

2. Preliminaries

2.1. Non-Integrability and Differential Galois Theory

In this section we briefly recall Ziglin-Ramis-Morales-Ruiz-Simó theory of non-integrability of Hamiltonian systems following [12] and [15].

Consider a Hamiltonian system

$$\dot{x} = X_H(x) \tag{3}$$

with a Hamiltonian H on a complex analytic $2n$ -dimensional manifold M. Let $x = x(t)$ be a particular solution of (3), which is not an equilibrium point of the vector field X_H. Denote by Γ the phase curve corresponding to $x = x(t)$. We can write the variational equations VE along Γ

$$\dot{\xi} = \frac{\partial X_H}{\partial x}(x(t))\xi$$

We can always reduce the order of this system by one restricting VE to the normal bundle of Γ on the level variety $M_h = \{x \mid H(x) = h\}$. The new-obtained system is called *normal variational equation* (NVE). In his papers [13], [14], Ziglin showed that if system (3) has a meromorphic first integral, then the monodromy group $\mathcal{M}$ of the normal variational equations has a rational integral.

Morales-Ruiz and Ramis generalized the Ziglin approach replacing the monodromy group $\mathcal{M}$ by the differential Galois group of NVE, [12]. Solutions of NVE define the an extension $K \subset L_1$ of the coefficient field K of NVE. The group of all the differential automorphisms of L_1 which leave fixed the elements of K defines *the differential Galois group* $Gal(L_1/K)$ of L_1 over K (or of equations NVE). Then the main result of Ziglin-Morales-Ramis theory is 0.1 (Morales-Ramis) Suppose that the Hamiltonian system (3) possesses n independent first integrals in involution. Then the connected component G^0 of the unit element of the Galois group $Gal(L_1/K)$ is Abelian.

The opposite is not true in general, that is, if the connected component G^0 of the unit element of the Galois group is Abelian it is not sure that the Hamiltonian system is integrable. This means that we need other obstruction to integrability. Already in [12] Morales-Ruiz suggested the quite natural conjecture that the *higher Galois groups* are also responsible for non-integrability. In [7] he announced this result and recently in a joint paper of Morales, Ramis and Simó [15] it was proved. Let us recall the corresponding notions and results.

Again we take a solution $x(t)$ of the Hamiltonian system (3). We write the general solution as $x(t,z)$, where z parametrizes the solutions near $x(t)$, with z_0 corresponding to it. Then we write (3) as

$$\dot{x}(t,z) = X_H(x(t,z))$$

Denote the derivatives of $x(t,z)$ with respect to z by $x^{(1)}(t,z), x^{(2)}(t,z)\cdots$, etc. Let us differentiate the last equation with respect to z and evaluate it at z_0. The functions $x^{(k)}(t,z)$ satisfy equations of the type

$$\begin{aligned}\dot{x}^{(k)}(t,z) = {} & X_H^{(1)}(x(t,z))x^{(k)}(t,z) \\ & + P\left(x^{(1)}(t,z),\cdots,x^{(k-1)}(t,z)\right)\end{aligned} \tag{4}$$

where P denotes polynomial term in the components of its arguments. The coefficients depend on t through $X_H^{(j)}\left(x(t,z)\right), j < k$. One can show that the system of non-homogene-ous equations for $x^{(k)}(t,z)$, (4), can be arranged to a homogeneous linear system of higher dimension. These recently built systems define succe-ssive extensions of the main field K of the coefficients the NVE, *i.e.* we have $K \subset L_1 \subset L_2 \cdots \subset L_k$ where L_1 is as above, L_2 is the extension obtained by adjoining the solutions of (4) for $k = 2$, etc. We can define the Galois groups $Gal(L_1/K), Gal(L_2/K), \cdots$. Then the theorem from [15] asserts that 0.2 (Morales-Ramis-Simó) If the Hamiltonian system (3) is integrable then for each $m \in \mathbb{N}^*$ the connected component $(G_m)^0$ of the unit element of the Galois group $Gal(L_m/K)$ is commu-tative.

2.2. Galois Group and Irregular Singularities

In this section we review some definiti-ons, facts and notations from the theory of the differential equations with an irregular point, as well as, from the differential Galois theory of such equations which is required to prove our main theorem. For the basic facts on differential Galois group at the irregular points we refer to Martinet and Ramis [16,17], van der Put and Singer [18], Morales-Ruiz [12], Mitschi [19], Singer [20]. For the basic facts on the analytic theory (formal solutions, formal power series, asymptotic and summability) we refer to Ramis [21, 22], Balser [23], Wasov [24].

We consider a linear homogeneous differential equa-tion

$$y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_0(x)\, y = 0 \qquad (5)$$

with coefficients $a_j(x)$ in $\mathbb{C}(x)$. From now on we shall assume that equation (5) admits over $\mathbb{CP}^1$ one irregular point of rank one at zero and one or more regular points. That is enough for our purpose. Classical theory says, [24], that in this case equation (5) has a formal fundamental matrix at 0

$$\boldsymbol{Y}(x) = \boldsymbol{H}(x)\boldsymbol{T}x^{L}\boldsymbol{T}^{-1}\mathrm{e}^{Q(1/x)} \qquad (6)$$

where $\boldsymbol{H} \in \mathrm{GL}_n\left(\mathbb{C}((x))\right)$, $L \in \mathrm{gl}_n(\mathbb{C})$ is in Jordan form, $\boldsymbol{T}$ is a non-singular constant matrix and $\boldsymbol{Q} = \mathrm{diag}\left(q_1(1/x), \cdots, q_n(1/x)\right)$ with $q_j(1/x)$-polynomials. In general case of rank one at zero the polynomials $q_j(1/x)$ are of maximal degree 1 with respect to $1/x$ but they could be polynomials in a fractional power of $1/x$, [24]. Here we assume that the polynomials $q_j(1/x)$ be monomials of degree 1 in $1/x$ some of them being possibly zero. Then the matrix $\boldsymbol{Q}(1/x)$ in (6) is $\boldsymbol{Q} = \mathrm{diag}\left(\frac{q_1}{x}, \cdots, \frac{q_n}{x}\right)$ with $q_j \in \mathbb{C}$, not necessary distinct.

We now turn to the Galois group of equation (5) over the field of formal Laurent series $\mathbb{C}((x))$. FGG ([18, 20]) The formal differential Galois group of equation (5) over $\mathbb{C}((x))$ is the Zariski closure of the group generated by the formal monodromy and the exponential torus. FM ([17]) The formal monodromy matrix $\boldsymbol{\gamma} \in \mathrm{GL}_n(\mathbb{C})$ relative to $\hat{Y}(x)$ in (6) is defined by

$$\boldsymbol{Y}\left(x\mathrm{e}^{2\pi i}\right)(x) \cdot \boldsymbol{\gamma}$$

and the formal monodromy group is the closed subgroup of the corresponding Galois group topologically gene-rated by $\boldsymbol{\gamma}$. The formal monodromy group is inde-pendent on the choice of the fundamental solution $\boldsymbol{Y}(x)$ and it is a formal invariant of the differential equation (5). ET ([20]) The exponential torus $\mathcal{T}$ relative to the solution $\boldsymbol{Y}(x)$ (6) is the group of differential $\mathbb{C}((x))$-automorphism $\mathcal{T} = Gal\left(E/\mathbb{C}((x))\right)$ where $E = \mathbb{C}((x))\left(\mathrm{e}^{Q}\right) = \mathbb{C}((x))\left(\mathrm{e}^{q_1/x}, \cdots, \mathrm{e}^{q_n/x}\right)$. We may iden-tify $\mathcal{T}$ with the subgroup of $\left(\mathbb{C}^*\right)^n$. Next lemma gives the relationship between the formal monodromy $\boldsymbol{\gamma}$ and the exponential torus $\mathcal{T}$. m-t ([19]) The formal mono-dromy $\boldsymbol{\gamma}$ acts by conjugation on the exponential torus $\mathcal{T}$. Hence $\mathcal{T}$ is a normal subgroup of the differential Galois group of equation (5) over $\mathbb{C}((x))$.

Now we turn to the Galois group over the field of convergent Laurent series $\mathbb{C}(\{x\})$. The general theory of summability ensures that the matrix $\boldsymbol{H}(x)$ in (6) is multisummable along any non-singular ray d. In the case when all non-zero polynomials $q_i(1/x) - q_j(1/x)$ j have the same degree 1 this means that $\boldsymbol{H}(x)$ is either convergent or 1-summable.

We need to recall some definitions and theoretical results. All angular directions and sectors are to be con-sidered on the Riemann surface of the (natural) loga-rithm.

Section ([23]) 1. A sector is defined to be a set of the form

$$S = S(d,\alpha,\rho) = \left\{ x = r\mathrm{e}^{i\varphi} \,\middle|\, 0 < r < \rho, d - \frac{\alpha}{2} < \varphi < d + \frac{\alpha}{2} \right\}$$

where d is an arbitrary real number (bisecting direc-tion of S), α is a positive real (the opening of S), and ρ is either a positive real number or $+\infty$ (the radius of S).

2. A closed sector is a set of the form

$$\overline{S} = \overline{S}(d,\alpha,\rho) = \left\{ x = r\mathrm{e}^{i\varphi} \,\middle|\, 0 < r < \rho, d - \frac{\alpha}{2} < \varphi < d + \frac{\alpha}{2} \right\}$$

with d and α as before, but ρ is a positive real number (*i.e.* never equal to $+\infty$). sd ([19]) A singular direction for equation (5) relative to $\boldsymbol{Y}(x)$ in (6) is a bisecting ray of any maximal angular sector where $Re\left(\frac{q_i - q_j}{x}\right) < 0$ for some $i, j = 1, \cdots, n$.

Following Balser [23] we define a Gevrey function and a Gevrey series. ga Let $f = f(x)$ be analytic in some sector $S(d, \alpha, \rho)$ at $x = 0$. We say that f is asymptotic to $f(x) = \sum f_n x^n \in \mathbb{C}[[x]]$ in Gevrey order1 sense, if for every closed subsector $\overline{S}_1$ of S there exist positive constants $C_{S_1}, A_{S_1} > 0$ such that for every non-negative integer N and every $x \in \overline{S}_1$ one has

$$\left| f(x) - \sum_{n=0}^{N-1} f_n x^n \right| \le C_{S_1} A_{S_1}^N N! |x|^N$$

We, according to standard, denote by $\mathbb{A}_1(S)$ the ring of all Gevrey functions of order 1 in S.

Corresponding to the notion of a Gevrey function is the notion of a Gevrey series. ga1 A formal power series $f \sum_{n \ge 0} f_n x^n$ is said to be of Gevrey order 1 if there exist two positive constants $C, A > 0$ such that

$$|f_n| \le CA^n n! \quad \text{for every } n \in \mathbb{N}$$

We denote by $\mathbb{C}((x))_1$ the set of all power series of Gevrey order 1.

R(Ramis) ([22]) 1. Let $f \in \mathbb{C}\{x\}_1$ such that there exists an open sector V whose opening $> \pi$ and a holomorphic function $f \in \mathbb{A}_1(V)$ such that f is asymptotic to f on V in Gevrey 1 sense. We will say that f is 1-summable in the direction d (d being the bisecting line of V) and f is the 1-sum of f in the direction d. 2. If $f \in \mathbb{C}\{x\}_1$ is 1-summable in all but a finite number of directions, we will say that it is 1-summable. We will denote $f \in \mathbb{C}\{x\}_1$. This summability definition is very useful but it does not say how to compute the sum. Another definition of 1-summability is gives in terms of Borel and Laplace transforms. In the next two definitions we follow Balser [23], van der Put [18] and Singer [20].

Borel The formal Borel transform $\mathcal{B}$ of order 1 to a formal power series $f(t) = \sum_{n=0}^{\infty} f_n x^n$ is called the formal series

$$\mathcal{B}\left(\sum f_n x^n\right) = \sum_{n=0}^{\infty} \frac{f_n}{n!} \zeta^n \tag{7}$$

Then $f \in \mathbb{C}((x))_1 \in \mathbb{C}((x))_1 \Leftrightarrow \mathcal{B}(f) \in \mathbb{C}\{\xi\}$ (*i.e.* convergent) [20].

The inverse of the Borel transform is the Laplace transform: laplac Let f be analytic and of exponential size 1, *i.e.* $|f(\zeta)| \le A \exp(B|\zeta|)$ along the ray r_d from 0 to 0 in direction d. Then the integral

$$\mathcal{L}_d f(x) = \int_{r_d} f(\zeta) \exp\left(-\frac{\zeta}{x}\right) \mathrm{d}\left(\frac{\zeta}{x}\right) \tag{8}$$

is said to be the Laplace complex transform $\mathcal{L}_d$ of order 1 in the direction d of f. The following preposition gives useful criteria for a Gevrey series of order 1 to be 1-summable cri([18]) Let $f \in \mathbb{C}((x))_1$ and let d be a direction. Then the following are equivalent:

1. f is 1-summable in the direction d.
2. The convergent power series $\mathcal{B}(f)$ has an analytic continuation h in a full sector $\{\zeta \in \mathbb{C} \mid 0 < |\zeta| < \infty, |\arg(\zeta) - d| < \varepsilon\}$. In addition, this analytic continuation has exponential growth of order ≤ 1 at ∞ on this sector, *i.e.* $|h(\zeta)| \le A \exp(B|\zeta|)$. In this case, $f = \mathcal{L}_d(h)$ is its 1-sum. The good is that the two definitions R and cri are in fact equivalent [22].

To define the Stokes multipliers relative to the solution (6) we need the following fundamental result of Ramis sum(Ramis, [22]) Let $Ly = g$ be a linear non homogeneous ordinary differenti-al equation of order n with polynomial coefficients and $g(x) \in \mathbb{C}\{x\}$. We suppose the Newton polygon at the origin admits only one strictly positive slope 1. If the series $f \in \mathbb{C}((x))$ is a formal power series solution of $Ly = g$ then f is 1-summable or convergent. In this paper we will not apply the theory of the Newton polygon. We note that when $q_j(1/x) = q_j/x, j = 1, \cdots, n$ with $q_j \in \mathbb{C}$ the Newton polygon of equation (5) admits only one strictly positive slope 1. sum says that there exists a unique holomorphic function $f(x)$ in all but a finite number of directions, solution of the differential equation $Ly = g$ such that asymptotic to $f(x)$ in Gevrey-1 sense. Moreover $f(x)$ can be obtained from $f(x)$ by a Borel-Laplace transform.

Let us assume that the matrix $\boldsymbol{L}$ in (6) is in a diagonal form, *i.e.* there are no logarithms in the solutions of equation (5). Due to our assumption the fundamental set of solutions of equation (5) is spanned by the functions

$$y_j(x) = x^{l_j} \mathrm{e}^{\frac{q_j}{x}} \hat{y}_j(x), \quad j = 1, \cdots, n \tag{9}$$

where $l_j, q_j \in \mathbb{C}$ and $\hat{y}_j(x) \in x\mathbb{C}[[x]]$. The formal series $\hat{y}_j(x)$ are "in general" divergent. It is well known fact that each of these divergent series $\hat{y}_j(x)$ is a solution of a linear homogeneous differential equation with polynomial coefficients

$$b_n(x) y^{(n)} + b_{n-1}(x) y^{(n-1)} + \cdots + b_0(x) y = 0 \tag{10}$$

whose other solutions are $x^{l_i - l_j} \mathrm{e}^{\frac{q_i - q_j}{x}} \hat{y}_j(x)$. Next, the series $\hat{y}_j(x)$, $j = 1 \div n$ lie at the first row of the matrix $\boldsymbol{H}(x)$. We remark that the behaviour of the elements of the first row of the matrix $\boldsymbol{H}(x)$ (and from here of their derivations) is enough for our purpose. This is based on the important result that the set $\mathbb{C}\{x\}_{1,d}$ of all $f(x)$ such that are 1-summable in a direction d is a differential algebra over $\mathbb{C}$ (see Balser [23], Chapter 3.3, Theorem 2). We we will consider the equations (10) (not equation (5)) and we will apply sum to equation (10). In this way for any open sector V with vertex 0, with opening $> \pi$ and bisected by non-singular direction d to any series $\hat{y}_j(x)$ we associate a unique holomorphic function $f_j(x)$ solutions of the equation (10) and 1-sum of the corresponding series $\hat{y}_j(x)$ along d such that $f_j(x)$ is asymptotic to $\hat{y}_j(x)$ in Gevrey-1 sense (R). Replacing the series $\hat{y}_j(x)$ (and their derivations) in matrix $\boldsymbol{H}(x)$ by their 1-sums we obtain a holomorphic matrix $\boldsymbol{H}(x)$ on a sector with opening $> \pi$, bisected by a non-singular direction d, asymptotic to $\boldsymbol{H}(x)$ in Gevrey-1 sense on this sector and denotes the 1-sum of $\boldsymbol{H}(x)$. The new-obtained matrix $\boldsymbol{Y}(x) = \boldsymbol{H}(x)x^L e^{Q(1/x)}$ is an actual fundamental matrix of equation (5) and denotes the 1-sum of $\boldsymbol{Y}(x)$ along any nonsingular ray d.

Further, following [18] and [20], relative to equations (5), (10) and to the solution (6) (the matrix $\boldsymbol{Q}(1/x)$) we define:

1. Eigenvalues $q_{ij} = \dfrac{q_i - q_j}{x}$ of equation (10), where q_i/x, q_j/x are the eigenvalues of equation (5);
2. A Stokes direction d for q_{ij} is a direction such that $Re(q_{ij}) = 0$;
3. Let d_1, d_2 be consecutive Stokes directions. We say that the pair (d_1, d_2) is a negative Stokes pair if $Re(q_{ij}) < 0$ for $\arg(x) \in (d_1, d_2)$;
4. A singular direction is the bisector of a negative Stokes pair.

sd1 Equations (10) have at least one zero eigenvalue, this corresponding to the series $\hat{y}_j(x)$. In this way the singular directions corresponding to the series $\hat{y}_j(x)$, from here to equations (5) and (10) are exact these defined by (4) (this defined more exactly the expression "for some $i, j = 1, \cdots, n$" in sd).

Let d be a singular direction for equations (10) at $x = 0$ and let $d^+ = d + \varepsilon$ and $d^- = d - \varepsilon$ where $\varepsilon > 0$ is small be two non-singular neighboring directions of d. Let Y_{d^+} and Y_{d^-} denote the 1-sums of Y along d^+ and d^- respectively. st Wit0h respect to a given formal fundamental solution Y as in (6) the Stokes matrix (or multiplier) $\mathrm{St}_d \in \mathrm{GL}_n(\mathbb{C})$ corresponding to the singular line d at $x = 0$ is defined by $Y_{d^-} = Y_{d^+} \cdot \mathrm{St}_d$. R (Ramis) ([16]) With respect to a formal solution Y given as in (6) the analytic differential Galois group of equation (5) at 0 is the Zariski closure in $\mathrm{GL}_n(\mathbb{C})$ of the subgroup generated by the formal monodromy γ, the exponential torus $\mathcal{T}$ and the Stokes matrices St_d for all singular rays d. It is possible to generalize the above theorem to a global linear differential equation: Mi ([19]) The global Galois group G of equation (5) is topologically genera-ted in $\mathrm{GL}_n(\mathbb{C})$ by the local Galois subgroup G_a where a runs over the set of singular points of (5).

2.3. P_V as a Hamiltonian System

The fifth Painlevé equation, P_V, is equivalent to the Hamiltonian system, [25],

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{11}$$

with the Hamiltonian H,

$$H = \frac{1}{t}\left[p(p+t)q(q-1) + \alpha_2 qt - \alpha_3 pq - \alpha_1 p(q-1)\right] \tag{12}$$

In fact, putting

$$\kappa_\infty = \alpha_1, \kappa_0 = \alpha_3, \theta = \alpha_2 - \alpha_0 - 1 \tag{13}$$

with

$$\alpha_0 = 1 - \alpha_1 - \alpha_2 - \alpha_3 \tag{14}$$

we see that equation for $y = 1 - 1/q$ is nothing but P_V (2). The non-autonomous system (11) can be turned into autonomous one with two degrees of freedom by introducing new dynamic variables— t and the conjugate to it— F. The new Hamiltonian becomes

$$H = H + F$$

The symplectic structure ω is canonical in the variables (q, p, t, F), *i.e.* $\omega = dp \wedge dq + dF \wedge dt$. In what follows we denote by s the time variable.

3. Non-Integrability of P_V for $\kappa_\infty = 0, \kappa_0 = -\theta$

In this section we begin studying P_V with the following values of the constants: $\kappa_\infty = 0, \kappa_0 = -\theta$ where θ is an arbitrary parameter. The results of this part are the base of the main theorems.

For these values of the constants the corresponding autonomous Hamiltonian system becomes

$$\dot{q} = \frac{1}{t}\left[(2p+t)q(q-1) + \theta q\right]$$

$$\dot{p} = -\frac{1}{t}\left[(2q-1)p(p+t) + (\theta+1)t + \theta p\right]$$

$$\dot{t} = 1$$

$$\dot{F} = -\frac{1}{t}\left[pq(q-1)+(\theta+1)q\right] + \frac{1}{t^2}\left[p(p+t)q(q-1)+(\theta+1)qt+\theta pq\right]$$

We choose as our non-equilibrium particular solutions — $q = 0, p = -s, t = s, F = 0$.

As we plan to compute the second variational equations it would be convenient to put

$$q = 0 + \varepsilon q_1 + \varepsilon^2 q_2 + \cdots$$

$$p = -s + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots$$

$$t = s + \varepsilon t_1 + \varepsilon^2 t_2 + \cdots$$

$$F = 0 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots$$

It is easy to see that as a normal to the phase curves of these solutions we can pick the $(q, p)-$plane in the hypersurface $\boldsymbol{H} = \text{Const}$, *i.e.* the equations in normal variations are just the equations for q, p-variables. Because of the equation $\dot{t} = 1$ we can use t as an independent variable instead of s.

For the first variational equations we obtain the system

$$\dot{q}_1 = \left[1+\frac{\theta}{t}\right]q_1, \quad \dot{p}_1 = -\left[1+\frac{\theta}{t}\right]p_1$$

This system can be solved by quadratures and

$$\Phi(t) = \begin{bmatrix} t^{\theta}\mathrm{e}^{t} & 0 \\ 0 & t^{-\theta}\mathrm{e}^{-t} \end{bmatrix}$$

is a fundamental matrix solution. Thus the differential Galois group G of the first variatio-nal equation is Abelian, it is conjugated to the diagonal matrices

$$G = \left\{\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, c \in \mathbb{C}^*\right\}$$

and there is no obstruction to integrability. Next, for the second variational equations we obtain

$$\dot{q} = \left[1+\frac{\theta}{t}\right]q - q_1^2 - \frac{2}{t}p_1q_1$$

$$\dot{p} = -\left[1+\frac{\theta}{t}\right]p + \frac{1}{t}p_1^2 + 2p_1q_1$$

where we have replaced $q(t) := q_2(t), p(t) := p_2(t)$.

why As our approach uses the Stokes multipliers we find that it is more suitable to study the second variational equations as a scalar equation than a system. Further, as the equation for $q(t)$ does not depend on $p(t)$ it is enough for our purpose to consider only the scalar homogeneous equation corresponding to this equation. Furthermore, we find our problem will get more simple if we write the equation for $q(t)$ as a fourth order linear homogeneous equation, not as a third order (the natural method). Then the equations of $q(t)$ will have a Galois group contained in $\mathrm{GL}_4(\mathbb{C})$ (not in $\mathrm{GL}_3(\mathbb{C})$). This apparent complication preserves the non-commutativity of the unit element of the corresponding differential Galois group. The equation of $q(t)$ can be written as a fourth order linear homogeneous equation

$$\begin{aligned} L(q) = {} & \frac{\mathrm{d}^4 q}{\mathrm{d}t^4} - \left[5+\frac{5\theta-3}{t}\right]\frac{\mathrm{d}^3 q}{\mathrm{d}t^3} \\ & + \left[8+\frac{16\theta-11}{t}+\frac{2\theta(4\theta-3)}{t^2}\right]\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} \\ & - 2\left[2+\frac{6(\theta-1)}{t}+\frac{\theta(6\theta-7)}{t^2}+\frac{\theta(\theta-1)(2\theta+1)}{t^3}\right]\frac{\mathrm{d}q}{\mathrm{d}t} \\ & - 2\left[\frac{2}{t}+\frac{4\theta}{t^2}+\frac{\theta(2\theta+1)}{t^3}\right]q = 0 \end{aligned} \tag{15}$$

As it said from here on we will study the differential Galois group only of Equation (15).

Equation (15) has over $\mathbb{CP}^1$ two singular points - the point $t = 0$ is a regular singularity and the point $t = \infty$ is an irregular singularity of rank one. Furthermore, this equation is reducible $L = L_1L_2$ and it is solvable in terms of second order linear differential equation

$$L_2(q) = f(t), \quad L_2 = \partial^2 - \left[1+\frac{\theta-1}{t}\right]\partial^1 - \frac{1}{t} \tag{16}$$

The second equation $L_1(q) = 0$ is

$$\begin{aligned} L_1(q) = {} & \ddot{q} - \left[\frac{2(2\theta-1)}{t}+4\right]\dot{q} \\ & + \left[4+\frac{4(2\theta-1)}{t}+\frac{2\theta(2\theta-1)}{t^2}\right]q = 0 \end{aligned} \tag{17}$$

and the system $\left\{t^{2\theta}\mathrm{e}^{2t}, t^{2\theta-1}\mathrm{e}^{2t}\right\}$ is a fundamental system of solution of (17).

Let us consider the homogeneous equation $L_2(q) = 0$. Changing the dependent variable

$$q = y\exp\left[\frac{1}{2}\int\left(1+\frac{\theta-1}{t}\right)\mathrm{d}t\right]$$

we transform equation $L_2(q) = 0$ to the reduced form

$$\ddot{y} - \left[\frac{1}{4}+\frac{\theta+1}{2t}+\frac{\theta^2-1}{4t^2}\right]y = 0 \tag{18}$$

Equation (18) is known as the Whittaker equation [26]

$$\ddot{y}-\left(\frac{1}{4}-\frac{\kappa}{t}+\frac{4\mu^2-1}{4t^2}\right)y=0$$

with parameters $\kappa=-\frac{\theta+1}{2}$ and $\mu=\frac{\theta}{2}$. Then we have

witt The identity component of the differential Galois group of equation (18) is Abelian if and only if $\theta\in-\mathbb{N}^*$.

It is well known from the work of Martinet and Ramis [16] (see [12] too) that the identity component G^0 of the differential Galois group of the Whittaker equation is Abelian if, and only if, $\left(\kappa+\mu-\frac{1}{2},\kappa-\mu-\frac{1}{2}\right)$ belong to $\left(\mathbb{N}\times\left(-\mathbb{N}^*\right)\right)\cup\left(\left(-\mathbb{N}^*\right)\times\mathbb{N}\right)$ (i.e. $\kappa+\mu-\frac{1}{2},\kappa-\mu-\frac{1}{2}$ are integers, one of them being positive and the other negative). In our case $\kappa+\mu-\frac{1}{2}=-1,\kappa-\mu-\frac{1}{2}=-1-\theta.$ Hence the identity component G^0 of the differential Galois group of equation (18) is Abelian if, and only if, $-1-\theta\in\mathbb{N}$. This proves the lemma.

We finish with same notes. The fundamental set of solutions of equation (18) is spanned by the set $\left\{y_1=t^{\frac{\theta+1}{2}}e^{\frac{t}{2}},y_2=t^{\frac{\theta+1}{2}}e^{-\frac{t}{2}}\int_0^t x^{-\theta-1}e^{-x+t}dx\right\}$, *i.e.* only one of the Stokes multipliers of the Whittaker equation (18) is different from zero when $\theta\notin-\mathbb{N}^*$.

Observe that the identity component G^0, as well the Galois group G of equation (18) is a subgroup of $SL_2(\mathbb{C})$. In particular, when $\theta\notin-\mathbb{N}^*$

$$G=G^0=\left\{\begin{pmatrix}\lambda & \mu\\ 0 & \lambda^{-1}\end{pmatrix},\lambda\in\mathbb{C}^*,\mu\in\mathbb{C}\right\}$$

and when $\theta\in-\mathbb{N}^*$

$$G=G^0=\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix},\lambda\in\mathbb{C}^*\right\}$$

re Equation $L_2(q)=0$ has a solution space that is spanned by functions $q_1=t^{\theta}e^{t},q_2=t^{\theta}\int_0^t x^{-\theta-1}e^{-x+t}dx$. Therefore the identity component of its Galois group is not a subgroup of $SL_2(\mathbb{C})$, *i.e.* equations $L_2(q)=0$ and (18) have different identity components of the corresponding Galois groups. As in equation (18) only one of the Stokes multipliers of equation $L_2(q)=0$ is different of zero when $\theta\notin-\mathbb{N}^*$. Furthermore, the identity components of the Galois group of both of the equations $L_2(q)=0$ and (18) are Abelian (resp. not Abelian) in the same way, as both of them depend on the same integral $\int_0^t x^{-\theta-1}e^{-x+t}dx$. In particular, when $\theta\in-\mathbb{N}^*$ for the Galois group of equation $L_2(q)=0$ we have

$$G=G^0=\left\{\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix},\lambda\in\mathbb{C}^*\right\}$$

When $\theta\notin-\mathbb{N}^*$ to define the identity component G^0 of the Galois group is a complicated task. But even in the worst case when

$$G^0=\left\{\begin{pmatrix}\lambda & \mu\\ 0 & 1\end{pmatrix},\lambda\in\mathbb{C}^*,\mu\in\mathbb{C}\right\}$$

G^0 is not Abelian group. However, as witt gives us necessary and sufficient condition for Abelian differential Galois group of equation (18) (and from above remark of equation $L_2(q)=0$), thus as a corollary we have the following theorem. witt Assume that $\kappa_\infty=0,\kappa_0=-\theta\notin\mathbb{N}^*$. Then the Painlevé V equation (2) is not integrable in the Liouville sense.

Due to the reducibility of $L,L=L_1L_2$, with equation (15) one can associate a matrix equation

$$\dot{\boldsymbol{Q}}+\begin{pmatrix}\boldsymbol{A}_2 & \boldsymbol{C}\\ 0 & \boldsymbol{A}_1\end{pmatrix}\boldsymbol{Q}=0 \tag{19}$$

where $\boldsymbol{Q}=(\boldsymbol{Q}_2,\boldsymbol{Q}_1)$, the matrix equation $\dot{\boldsymbol{Q}}_1+A_1\boldsymbol{Q}_1=0$ is completely reducible and $\boldsymbol{C}$ may be taken to be the matrix $\boldsymbol{C}=\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix}$, [27]. In this way the matrix

equation $\dot{\boldsymbol{Q}}_1+A_1\boldsymbol{Q}_1=0$ and the matrix $\boldsymbol{A}_1$ may be taken to be the corresponding matrix equation and the corresponding matrix to the equation (17) in the standard sense. Namely, if we put $q^1=q,q^2=\dot{q}$ then a function q is a solution of the scalar equation (17) if and only if the column vector $\boldsymbol{Q}_1=\left(q^1,q^2\right)^{\tau}$ is a solution of the following matrix equation

$$\boldsymbol{Q}_1'+\boldsymbol{A}_1\boldsymbol{Q}_1=0$$

$$\boldsymbol{A}_1=\begin{pmatrix}0 & -1\\ 4+\frac{4(2\theta-1)}{t}+\frac{2\theta(2\theta-1)}{t^2} & -\frac{2(2\theta-1)}{t}-4\end{pmatrix}$$

We will not fix on the equation $\dot{\boldsymbol{Q}}_1+A_1\boldsymbol{Q}_1=0$ but will note that its differential Galois group is Abelian. Hence, there is no obstruction to integrability. Note that the equation $\dot{\boldsymbol{Q}}_2+\boldsymbol{A}_2\boldsymbol{Q}_2=0$ and the matrix $\boldsymbol{A}_2$ may be taken to be the corresponding equation and the corresponding matrix to equation $L_2(q)=0$ in the standard sense, *i.e.*

$$\boldsymbol{A}_2=\begin{pmatrix}0 & -1\\ -\frac{1}{t} & -1-\frac{\theta-1}{t}\end{pmatrix}$$

Next, the matrix

$$Q = \begin{pmatrix} Q_2 & UQ_1 \\ 0 & Q_1 \end{pmatrix} \quad (20)$$

is a fundamental matrix solution of equation (19) if and only if U satisfies $\dot{U} + (A_2U - UA_1) = -C$, [27]. Furthermore, the matrices Q_2 and Q_1 in (20) are the fundamental matrix solutions of equations $\dot{Q}_1 + A_1Q_1 = 0$ and $\dot{Q}_2 + A_2Q_2 = 0$ respectively.

Now, it is easy to see that the identity component of the differential Galois group of the equation (15) is not Abelian for $-\theta \notin \mathbb{N}^*$. Indeed, let us denote G to be the differential Galois group of equation (19). Then we have that $\sigma(Q), \sigma \in G$ is again a fundamental matrix solution of equation (19), and a calculation shows that $\left(Q^{-1}\sigma(Q)\right)' = 0$. Therefore $\sigma(Q) = QR(\sigma)$ for some $R(\sigma) \in \mathrm{GL}_4(\mathbb{C})$. Expressing any such $R(\sigma)$ in block notation,

$$R(\sigma) = \begin{pmatrix} G_2 & G_3 \\ G_4 & G_1 \end{pmatrix}$$

let us write the equation $\sigma(Q) = QR(\sigma)$ explicitly

$$\begin{aligned} \sigma(Q) &= \begin{pmatrix} Q_2 & UQ_1 \\ 0 & Q_1 \end{pmatrix}\begin{pmatrix} G_2 & G_3 \\ G_4 & G_1 \end{pmatrix} \\ &= \begin{pmatrix} Q_2G_2 + UQ_1G_4 & Q_2G_3 + UQ_1G_1 \\ Q_1G_4 & Q_1G_1 \end{pmatrix} \\ &= \begin{pmatrix} \sigma(Q_2) & \sigma(UQ_1) \\ \sigma(0) & \sigma(Q_1) \end{pmatrix} \end{aligned}$$

The last two equations imply $G_4 = 0$ and $\sigma(Q_1) = Q_1G_1$. Hence one can identify the matrix G_1 with the representation of the differential Galois group of equation (17) in $\mathrm{GL}_2(\mathbb{C})$. Next, from $G_4 = 0$ follows that $\sigma(Q_2) = Q_2G_2$ and one can identify the matrix $G_2 \in \mathrm{GL}_2(\mathbb{C})$ with the differential Galois group of equation $L_2(q) = 0$. Now from witt and re it results that for $\theta \notin -\mathbb{N}^*$ the identity component G_2^0 of the Galois group G_2 is not commutative and as a corollary the identity component of the differential Galois group of equation (21) is not Abelian. Thus from the Morales - Ramis-Simó theorem [15] and why the corresponding Hamiltonian system is not integrable. This proves witt.

3.1. Non-Integrability for $\theta = -1$

To prove non-integrability for $\theta \notin -\mathbb{N}^*$ we will study the matrix UQ_1 in the matrix solution (20). Let for simplicity $\theta = -1$. In the last section from this particular case, $\theta = -1$, we will extend the present results over $\theta \notin -\mathbb{N}^*$ by Bäcklund transformations.

As we are going to apply the theory of irregular points at $t = 0$ we change the dependent variable $t = -1/z$. Then for $\theta = -1$ equations (15), $L_2(q) = 0$ and (17) become

$$\begin{aligned} L(q) &= \frac{\mathrm{d}^4q}{\mathrm{d}z^4} + \left[\frac{4}{z} - \frac{5}{z^2}\right]\frac{\mathrm{d}^3q}{\mathrm{d}z^3} + \left[\frac{2}{z^2} - \frac{3}{z^3} + \frac{8}{z^4}\right]\frac{\mathrm{d}^2q}{\mathrm{d}z^2} \\ &\quad - \left[\frac{2}{z^4} + \frac{8}{z^5} + \frac{4}{z^6}\right]\frac{\mathrm{d}q}{\mathrm{d}z} + \left[\frac{2}{z^5} + \frac{8}{z^6} + \frac{4}{z^7}\right]q = 0 \end{aligned} \quad (21)$$

$$L_2(q) = q'' - \frac{1}{z^2}q' + \frac{1}{z^3}q = 0, \qquad ' = \frac{\mathrm{d}}{\mathrm{d}z} \quad (22)$$

$$L_1(q) = q'' + \left[\frac{4}{z} - \frac{4}{z^2}\right]q' + \left[\frac{2}{z^2} - \frac{4}{z^3} + \frac{4}{z^4}\right]q = 0 \quad (23)$$

Equations (22) and (23) have solution spaces spanned by the sets $\{z, z\mathrm{e}^{-1/z}\}$ and $\{(2z^{-2} + z^{-1})\mathrm{e}^{-2/z}, (z^{-2} + z^{-1})\mathrm{e}^{-2/z}\}$ respectively. As a formal fundamental set of solutions around the irregular singularity at 0 of the equation (21) we can take the fundamental set of solutions $\{q_1 = z\mathrm{e}^{-1/z}, q_2 = z\}$ of equation $L_2(q) = 0$ and

$$q_3 = z\mathrm{e}^{-\frac{2}{z}}\varphi(z), \quad q_4 = z\mathrm{e}^{-\frac{2}{z}}\psi(z)$$

where $\varphi(z)$ and $\psi(z)$ are two formal series

$$\varphi(z) = z - z^2 + 2!z^3 - 3!z^4 + \cdots = \sum_{n=0}^{\infty}(-1)^n n! z^{n+1}$$

$$\begin{aligned} \psi(z) &= \frac{1}{2}z - \frac{1}{2^2}z^2 + 2!\frac{1}{2^3}z^3 - 3!\frac{1}{2^4}z^4 + \cdots \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n n!}{2^{n+1}}z^{n+1} \end{aligned}$$

Then a formal fundamental matrix of equation (21) is

$$Q = H(z)z^J\mathrm{e}^{R(1/z)} \quad (24)$$

where

$$R = \mathrm{diag}\left(-\frac{1}{z}, 0, -\frac{2}{z}, -\frac{2}{z}\right), \quad J = \mathrm{diag}(1, 1, -2, -2) \quad (25)$$

and

$$H(z) = \begin{pmatrix} 1 & 1 & z^3\varphi(z) & z^3\psi(z) \\ \frac{1}{z} + \frac{1}{z^2} & \frac{1}{z} & h_{23} & h_{24} \\ 0 & 0 & 2 + z & 1 + z \\ 0 & 0 & -1 - \frac{2}{z} + \frac{4}{z^2} & -1 + \frac{2}{z^2} \end{pmatrix} \quad (26)$$

The precise form of h_{23} and h_{24} are not important for our purpose but we do note that they are elements of

$\mathbb{C}((z))$. We remark that the formal series $\varphi(z)$ and $\psi(z)$ are the result of formal operations that have nothing to do with sectors.

We now turn to the Galois group of equation (5) over $\mathbb{C}((z))$. The formal Galois group over $\mathbb{C}((z))$ is the Zariski closure of the group generated by the formal monodromy γ and the exponential torus $\mathcal{T}$. The last ones on the other hand represent differential automorphisms of the extension F of the field $\mathbb{C}((z))$ by the entries of the matrix $\mathbb{C}((z))z^J e^{R(1/z)}$ over $\mathbb{C}((z))$; $F=\mathbb{C}((z))(e^{-1/z}, e^{-2/z})$ in our case. Furthermore, the formal monodromy γ is trivial. Therefore $Gal(F/\mathbb{C}((z)))$ is equal to the exponential torus $\mathcal{T}$

$$Gal(F/\mathbb{C}((z)))=\mathcal{T}=\left\{\begin{pmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & c^2 \end{pmatrix} \middle| c\in\mathbb{C}^*\right\} \quad (27)$$

Let now F be the Picard-Vessiot extension of $\mathbb{C}(\{z\})$ for the equation (21). To determine the convergent Galois group at $z=0$ over $\mathbb{C}(\{z\})$ we will need to compute the Stokes matrices at $z=0$ relative to the solution $Q(z)$, (24).

Let us focus now on $\varphi(z)=\sum_{n\ge 0} a_n z^{n+1}$ and $\psi(z)=\sum_{n\ge 0} b_n z^{n+1}$. The first series $\varphi(z)=\sum_{n=0}^{\infty}(-1)^n n! z^{n+1}$ is the so called Euler series, the second is a modified Euler series. The properties of the Euler series and the corresponding Stokes phenomenon are well studied, for example in a paper of Ramis [22] and in a paper of Singer [20]. In the next following these two papers we will compute the Stokes matrix relative to the series $\varphi(z)$ and $\psi(z)$.

These series are divergent and they obviously satisfy a Gevrey-1 growth condition. Then we can consider the formal Borel transforms of $\varphi(z)$ and $\psi(z)$

$$\varphi(\zeta)=\mathcal{B}_1\varphi(z)=\sum_0^{\infty}\frac{(-1)^n\zeta^{n+1}}{n+1}=\log(1+\zeta)$$

$$\psi(\zeta)=\mathcal{B}_1\psi(z)=\sum_0^{\infty}\frac{(-1)^n\zeta^{n+1}}{2^{n+1}(n+1)}=\log(2+\zeta)-\ln 2$$

The Gevrey-1 growth condition ensures that $\varphi(\zeta)$ and $\psi(\zeta)$ are analytic in the neighborhood of the origin of the ζ plane. For any ray $d\neq\mathbb{R}^-$ the functions $\log(1+\zeta)$ and $\log(2+\zeta)$ have analytic continuations along d. For such a ray their Laplace transforms (see Example 1.4.22 in [20])

$$\varphi_d(z)=\mathcal{L}_{1,d}(\log(1+\zeta))(z)$$
$$=\int_d \log(1+\zeta)e^{-\frac{\zeta}{z}}d\left(\frac{\zeta}{z}\right)=\int_d\frac{1}{1+\zeta}e^{-\frac{\zeta}{z}}d\zeta$$

$$\psi_d(z)=\mathcal{L}_{1,d}(\log(2+\zeta)-\ln 2)(z)$$
$$=\int_d\log(2+\zeta)e^{-\frac{\zeta}{z}}d\left(\frac{\zeta}{z}\right)$$
$$=-\ln 2\int_d e^{-\frac{\zeta}{z}}d\left(\frac{\zeta}{z}\right)$$
$$=\int_d\frac{1}{2+\zeta}e^{-\frac{\zeta}{z}}d\zeta$$

define the corresponding 1-sum of the series $\varphi(z)$ and $\psi(z)$ in the direction d. We note that functions $q_3(z)=ze^{-\frac{2}{z}}\varphi_d(z)$ and $q_4(z)=ze^{-\frac{2}{z}}\psi_d(z)$ again satisfy equation (21). Indeed, one can complete the set $\{q_1=ze^{-1/z}, q_2=z\}$ to the fundamental set of solutions of equation (21) by the particular solutions of equations

$$q''-\frac{1}{z^2}q'+\frac{1}{z^3}q=\left(\frac{2}{z^2}+\frac{1}{z}\right)e^{-2/z} \quad (28)$$

$$q''-\frac{1}{z^2}q'+\frac{1}{z^3}q=\left(\frac{1}{z^2}+\frac{1}{z}\right)e^{-2/z}$$

Looking for such solutions of the form $q(z)=ze^{-2/z}\varphi(z)$ and $q(z)=ze^{-2/z}\psi(z)$ respectively, we obtain the following non homogeneous differential equations with polynomial coefficients

$$z^4\varphi''+(2z^3+3z^2)\varphi'+2\varphi=2z+z^2 \quad (29)$$

$$z^4\psi''+(2z^3+3z^2)\psi'+2\psi=z+z^2$$

respectively with unique formal solutions

$$\varphi(z)=\sum(-1)^n n! z^{n+1} \quad\text{and}\quad \psi(z)==\sum\frac{(-1)^n n!}{2^{n+1}}z^{n+1}$$

respectively. Further, solving equations (29) by the variation of constants we get a particular solution

$$\varphi(z)=e^{-1/z}\int_0^z\frac{e^{-1/x}}{x}dx$$

of the first equation and a particular solution

$$\psi(z)=e^{2/z}\int_0^z\frac{e^{-2/u}}{u}du$$

of the second equation, where for convenience the integrals are taken in the direction $\mathbb{R}^+$. Next, let $\varphi(z)=\int_0^z\frac{e^{-1/x+1/z}}{x}dx$ and define a new variable ζ by $-\zeta/z=-1/x+1/z$ gives $\varphi(z)=\int_0^{+\infty}\frac{e^{-\zeta/z}}{1+\zeta}d\zeta$. In the same manner let $\psi(z)=\int_0^z\frac{e^{-2/u+2/z}}{u}du$ and setting

$-\zeta/z = -2/u + 2/z$ gives $\psi(z) = \int_0^{+\infty} \frac{e^{-\zeta/z}}{2+\xi} d\xi$. More general, these integrals exist if $d \neq \mathbb{R}^-$. Therefore, applying to equations (29) Ramis' sum, for any ray d except the negative real axis, we are able to canonically associate unique functions $\varphi_d(z)$ and $\psi_d(z)$, analytic in a large sector around this ray, with the divergent series $\sum(-1)^n n! z^{n+1}$ and $\sum \frac{(-1)^n n!}{2^{n+1}} z^{n+1}$ respectively and so these series are 1-summable and $\varphi_d(z)$ and $\psi_d(z)$ are their 1-sum. Furthermore the functions $q_3(z) = z e^{-\frac{2}{z}} \varphi_d(z)$ and $q_4(z) = z e^{-\frac{2}{z}} \psi_d(z)$ will satisfy equations (28) respectively and from here the equation (21). In this way the matrix =

$$\boldsymbol{Q}(z) = \boldsymbol{H}(z) z^J e^{R(-1/z)}$$

with z^J and $e^{R(1/z)}$ as before but

$$\boldsymbol{H}(z) = \begin{pmatrix} 1 & 1 & z^3\varphi_d(z) & z^3\psi_d(z) \\ \frac{1}{z}+\frac{1}{z^2} & \frac{1}{z} & h_{23} & h_{24} \\ 0 & 0 & 2+z & 1+z \\ 0 & 0 & -1-\frac{2}{z}+\frac{4}{z^2} & -1+\frac{2}{z^2} \end{pmatrix}$$

where $h_{23} = z^3\left[\left(\frac{1}{z}+\frac{2}{z^2}\right)\varphi_d(z) + \frac{d\varphi_d(z)}{dz}\right]$ and $h_{24} = z^3\left[\left(\frac{1}{z}+\frac{2}{z^2}\right)\psi_d(z) + \frac{d\psi_d(z)}{dz}\right]$ is an actual fundantal matrix of equation (21). The matrix $\boldsymbol{H}(z)$ is holomorphic in an open angular sector π at $z=0$ of opening angle 2π ($-\pi < \arg z < \pi$) and $\boldsymbol{H}(z)$ is asymptotic to $\boldsymbol{H}(z)$ (26) on this sector in Gevrey-1 sense.

We are now in a position to describe the analytic elements which, together with the formal Galois group (27) determine the analytic Galois group of equation (21). As the Stokes multipliers depend on the functions $\varphi(z)$ and $\psi(z)$ (resp. on $\varphi_d(z)$ and $\psi_d(z)$) we consider the following homogeneous ODE equation with polymial coefficients obtained from equation (21) by changing $q(z)$ to $\varphi(z) = z^{-1}e^{2/z}q(z)$

$$z^6 q^{(4)}(z) + \left(8z^5+3z^4\right)q'''(z) + \left(14z^4+6z^3+2z^2\right)q''(z) + (4z^3-4z)q'(z) + 4q(z) = 0 \tag{*}$$

The functions $\varphi(z)$ and $\psi(z)$ are its formal solutions. The other entries of the fundamental set of solutions of equation (*) are $e^{1/z}$ and $e^{2/z}$. So the eigenvalues of this equation are $\left\{0, 0, \frac{1}{z}, \frac{2}{z}\right\}$.

Now we are almost ready to compute the Stokes constants corresponding to the singular direction $\mathbb{R}^-$ ($d = \pi$). Relative to equation (*) we define:

- eigenvalues of equation (*): $\left\{0, 0, \frac{1}{z}, \frac{2}{z}\right\}$;
- the Stokes direction $d = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ such that

$$Re\left(\frac{2}{z}\right) = 2Re\left(\frac{1}{z}\right) = 0;$$

- the negative Stokes pair $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ such that $Re\left(\frac{2}{z}\right) < 0$;
- the singular direction $d = \pi$ as the bisector of the negative Stokes pair $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

We select two directions d_+ and d_- which are respectively slightly to the left and to the right of the critical direction $d = \pi = \mathbb{R}^-$. Let

$$\varphi_d^+ = \int_{d_+} \frac{1}{1+\zeta} e^{-\frac{\zeta}{z}} d\zeta \quad and \quad \varphi_d^- = \int_{d_-} \frac{1}{1+\zeta} e^{-\frac{\zeta}{z}} d\zeta$$

$$\psi_d^+ = \int_{d_+} \frac{1}{2+\xi} e^{-\frac{\zeta}{z}} d\zeta \quad and \quad \psi_d^- = \int_{d_-} \frac{1}{2+\xi} e^{-\frac{\zeta}{z}} d\zeta$$

be the associated Laplace transforms of φ and ψ of the directions d_+ and d_-. The difference between them amounts to integrate on a path coming form infinity along the critical line on the right till the origin and then doing to infinity by following the critical line on the left. As there is no singularity between 0 and -1, and no other between -1 and ∞, Cauchy's formula immediately implies that the difference $\varphi_d^- - \varphi_d^+$ is given by $2\pi i \mathrm{Res}_{\zeta=-1}\left(e^{-\zeta/z}/(1+\zeta)\right) = 2\pi i e^{1/z}$. In the same manner, the difference between ψ_d^- and ψ_d^+ is given by $2\pi i \mathrm{Res}_{\zeta=-1}\left(e^{-\zeta/z}/(2+\zeta)\right) = 2\pi i e^{2/z}$.

Next, we must have $H_{d=\pi}^- z^J e^{R(1/z)} = H_{d=\pi}^+ z^J e^{R(1/z)} \cdot \mathrm{St}_\pi$ where $\mathrm{St}_\pi \in \mathrm{GL}_4(\mathbb{C})$ is the Stokes matrix in the direction $d = \pi$. Furthermore, $q_3^- - q_3^+ = \left(\varphi_d^- - \varphi_d^+\right) z e^{-2/z} = 2\pi i z e^{-1/z} = 2\pi i q_1(z)$. Then for the Stokes matrix St_π we have $\left(st_\pi\right)_{1,3} = 2\pi i$. In the same manner,

$q_4^- - q_4^+ = \left(\psi_d^- - \psi_d^+\right) z\mathrm{e}^{-2/z} = 2\pi iz = 2\pi i q_2\left(z\right)$. Then we have that $\left(st_\pi\right)_{2,4} = 2\pi i$.

From the above reasoning for the Stokes matrix St_π we obtain

$$\mathrm{St}_\pi = \begin{pmatrix} 1 & 0 & 2\pi i & 0 \\ 0 & 1 & 0 & 2\pi i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (30)$$

Thus we have n1 Assume that $\kappa_\infty = 0, \kappa_0 = 1, \theta = -1$. Then the Painlevé V equation (2) is not integrable in the Liouville sense.

By a theorem of Schlesinger [28] the local differential Galois group of equation (21) at infinity is generated topologically by the monodromy group at ∞. We note that the actual monodromy around ∞ and around 0 of equation (21) are the same. Therefore the local differential Galois group of equation (21) at infinity can be interpreted as a subgroup of the local differential Galois group at the origin. Next, observe that the differential Galois group of equation (21) is a connected group. As the formal monodromy γ is trivial then by the Ramis' theorem the Galois group is topologically generated by the exponential torus $\mathcal{T}$ and the Stokes matrix St_π. The Zariski closure of the subgroup $\mathcal{T}^n, n \in \mathbb{Z}$ is the same $\mathcal{T}$ and the elements f_c of $\mathcal{T}$ is

$$f_c = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & c^2 \end{pmatrix} \quad \textit{where} \quad c \in \mathbb{C}^*$$

The matrix $\left(\mathrm{St}_\pi\right)^n, n \in \mathbb{Z}$ is of the kind $I + nX$ where $I = Id$ and X is a unipotent matrix. Denote by S_π the Zariski closure of the subgroup $\left(\mathrm{St}_\pi\right)^n, n \in \mathbb{Z}$ then the elements $s_{\pi,\lambda}$ of S_π are

$$s_{\pi,\lambda} = \begin{pmatrix} 1 & 0 & 2\pi i\lambda & 0 \\ 0 & 1 & 0 & 2\pi i\lambda \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \textit{where} \quad \lambda \in \mathbb{C}$$

When λ is different from zero and $c \neq \pm 1$ the commutator of f_c and $s_{\pi,\lambda}$ is

$$f_c s_{\pi,\lambda} f_c^{-1} s_{\pi,\lambda}^{-1} = \begin{pmatrix} 1 & 0 & 2\pi i\lambda\left(c^{-1} - 1\right) & 0 \\ 0 & 1 & 0 & 2\pi i\lambda\left(c^{-2} - 1\right) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (31)$$

which is not identically equal to Id.

The case $c = 1$ implies that $\sigma\left(\mathrm{e}^{-1/z}\right) = \mathrm{e}^{-1/z}$ for any $\sigma \in Gal\left(L\left(q\right) = 0\right)$, *i.e.* $\mathrm{e}^{-1/z} \in \mathbb{C}(z)$, which is an obviously contradiction. In the same manner $c = -1$ implies that $\mathrm{e}^{-2/z}$ is invariant under any morphism $\sigma \in Gal\left(L\left(q\right) = 0\right)$, *i.e.* $\mathrm{e}^{-2/z} \in \mathbb{C}\left(z\right)$. From these remarks, formula (31) and the fact that $Gal\left(L\left(q\right) = 0\right)$ is a connected group follow that the identity component of the differential Galois group of equation (21) is not Abelian. Thus from why and 0.2 we have that for $\kappa_\infty = 0, \kappa_0 = 1, \theta = -1$ the corresponding Hamiltonian system does not possess two meromorphic first integrals. This proves n1. galois We will not determine precisely the differential Galois group of equation (21). But we can say that this group is not finite and it coincides with its connected component G^0 of the unit element. Furthermore the differential Galois group of equation (21) is conjugate to the following matrices

$$Gal\left(L\left(q\right) = 0\right) = \left\{ \begin{pmatrix} c & 0 & d & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & c^2 \end{pmatrix} \middle| c \neq 0, a, d \in \mathbb{C} \right\} \quad (32)$$

4. Generalization

In this paragraph we will extend the results of the previous section to the entire orbits of the parameters using the Bäcklund transformations of the Painlevé fifth equation, given by the following list of restriction of this group on the parameter space, [29]

$$\begin{aligned} &s_i\left(\alpha_i\right) = -\alpha_i, \quad s_i\left(\alpha_j\right) = \alpha_j + \alpha_i, (j = i \pm 1), \\ &s_i\left(\alpha_j\right) = \alpha_j, (j \neq i, i \pm 1) \end{aligned} \quad (33)$$

$$\begin{aligned} &\pi\left(\alpha_j\right) = \alpha_{j+1}, \ \sigma\left(\alpha_0\right) = \alpha_0, \ \sigma\left(\alpha_1\right) = \alpha_3, \\ &\sigma\left(\alpha_2\right) = \alpha_2, \ \sigma\left(\alpha_3\right) = \alpha_1 \end{aligned}$$

We note that the Bäcklund transformations group of the fifth Painlevé equation is isomor-phic to the extended affine Weyl group of $A_3^{(1)}$ type, [5]. It is well known (see Okamoto [5]) that the group of Bäcklund transformations of the P_V equation is represented as the group of birational canonical transformations of the Painlevé system (that is the corresponding non-autonomous Hamiltonian system) associated with the fifth Painlevé equation. In particular the Bäcklund transformations remain the property non-integrability.

We define (following Masuda *et al.* [29]) the translation operators T_i ($i = 0,1,2,3$) by

$$T_1 = \pi s_3 s_2 s_1, T_2 = s_1 \pi s_3 s_2, T_3 = s_2 s_1 \pi s_3, T_0 = s_3 s_2 s_1 \pi \quad (34)$$

These operators acts on parameters α_i as

$$T_i(\alpha_{i-1}) = \alpha_{i-1}+1, T_i(\alpha_i) = \alpha_i - 1, T_i(\alpha_j) = \alpha_j (j \neq i-1, i) \quad (35)$$

4.1. Generalization of the Results of the Paragraph 3

In n1 we have proved that for $\kappa_\infty = 0, \kappa_0 = -\theta = 1$ the Painlevé V equation (2) is not integrable. Let us recall that these values of the parameters are particular case of the family $\kappa_\infty = 0, \kappa_0 = -\theta$ taken at $\theta = -1$. In the following proposition using the operators T_j, (34), (35), we will show that the result of n1 can be extended for $\theta \in -\mathbb{N}^*$. p1 Assume that $\kappa_\infty = 0, \kappa_0 = -\theta = m$ where $m \in \mathbb{N}^*$. Then the Painlevé V equation (2) is not integrable.

We will prove the statement by building the appropriate transformations which extend the initial parameter family $\kappa_\infty = 0, \kappa_0 = -\theta = 1$ or $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0,0,0,1)$ to this of the proposition.

The Bäcklund transformation $T_2T_1T_0$ (starting from T_0) maps the parameter family $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ to $(\alpha_0, \alpha_1, \alpha_2 - 1, \alpha_3 + 1)$. Now, applying $m-1$ times $T_2T_1T_0$ to the initial parameter family we obtain $(\alpha_0, \alpha_1, \alpha_2 - m + 1, \alpha_3 + m - 1) = (0,0,1-m,m)$. If we recall that $\theta = \alpha_2 - \alpha_0 - 1$ then we obtain that $\theta = -m$. The proof follows from the fact that the Bäcklund transformations are birational canonical transformations [5] and n1.

As a corollary from witt and p1 we have the following generic non-integrable result: g1 Assume that $\kappa_\infty = 0, \kappa_0 = -\theta$ where θ is an arbitrary complex parameter. Then the Painlevé V equation (2) is not integrable.

The next lemma describes the orbit of the vector $\alpha^0 = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0,0,1+\theta,-\theta)$ under the Bäclund transformation group of the fifth Painlevé equation. orbit Let $(q,p) = (0,-t)$ be a rational solution of the Hamiltonian system (11), (12) with parameters $\alpha^0 = (0,0,1+\theta,-\theta)$. Then beginning with α^0 by the Bäcklund transformations (33) we obtain a rational solution of (11), (12) with new $\alpha_j, j = 0,1,2,3$ as at least two of them are integer and at least one of these integer is α_1 or α_3. Furthermore the parameters satisfy either $\alpha_1 + \alpha_2 + \alpha_3 - \alpha_2 - 1 \in 2\mathbb{Z}$ or $\alpha_0 + \alpha_1 + \alpha_3 - \alpha_2 + 1 \in 2\mathbb{Z}$ relations.

Let $\alpha^i = (\alpha_0^i, \alpha_1^i, \alpha_2^i, \alpha_3^i)$ be the vector of parameters obtained by i successive transformations s_i, π, σ from α^0. We will prove the statement inductively. At first for α^0 the statement is true.

Let $i = 1$, *i.e.* we have applied on α^0 some of the transformations (33). Under s_0 and s_1 the vector α^0 does not change, *i.e.* $\alpha^1 = \alpha^0$ and the statement is true. Let us denote $S_1^i = \alpha_1^i + \alpha_2^i + \alpha_3^i - \alpha_0^i - 1$ and $S_2^i = \alpha_0^i + \alpha_1^i + \alpha_3^i - \alpha_2^i + 1$. Under s_2, α^0 becomes $\alpha^1 = (0, 1+\theta, -1-\theta, 1)$ and $S_1^1 = 0 \in 2\mathbb{Z}$. Under s_3, α^0 becomes $\alpha^1 = (-\theta, 0, 1, \theta)$ and $S_2^1 = 0 \in 2\mathbb{Z}$. Under π, α^0 becomes $\alpha^1 = (0, 1+\theta, -\theta, 0)$ and $S_1^1 = 0 \in 2\mathbb{Z}$. Under σ, α^0 becomes $\alpha^1 = (0, -\theta, 1+\theta, 0)$ and $S_1^1 = 0 \in 2\mathbb{Z}$. Hence for $i = 1$ the statement is true.

Suppose that the statement is true for i. We will prove that it is true for $i+1$. Let us recall (14), that is

$$(A) \qquad \alpha_0^i + \alpha_1^i + \alpha_2^i + \alpha_3^i = 1 \ \textit{for every } i$$

Observe that the conditions $\{S_1^i \in 2\mathbb{Z}, A\}$ imply that $\alpha_0^i \in \mathbb{Z}$. In the same manner $\{S_2^i \in 2\mathbb{Z}, A\}$ imply that $\alpha_2^i \in \mathbb{Z}$. Under s_0 the vector α^i becomes $\alpha^{i+1} = (-\alpha_0^i, \alpha_1^i + \alpha_0^i, \alpha_2^i, \alpha_3^i + \alpha_0^i)$ and for the sums S_1^{i+1} and S_2^{i+1} we obtain: $S_1^{i+1} = 2\alpha_0^i, S_2^{i+1} = 2 - 2\alpha_2^i$. So if $S_1^i \in 2\mathbb{Z}$ then $S_1^{i+1} \in 2\mathbb{Z}$ and if $S_2^i \in 2\mathbb{Z}$ then $S_2^{i+1} \in 2\mathbb{Z}$. Next, if $\{\alpha_1^i \in \mathbb{Z}, S_1^i \in 2\mathbb{Z}\}\left(\{\alpha_3^i \in \mathbb{Z}, S_1^i \in 2\mathbb{Z}\}\right)$ then $\alpha_1^{i+1} \in \mathbb{Z}\left(\alpha_3^{i+1} \in \mathbb{Z}\right)$. Similarly, if $\{\alpha_1^i \in \mathbb{Z}, S_2^i \in 2\mathbb{Z}\}\left(\{\alpha_3^i \in \mathbb{Z}, S_2^i \in 2\mathbb{Z}\}\right)$ then $\alpha_3^{i+1} \in \mathbb{Z}$ from A ($\alpha_1^{i+1} \in \mathbb{Z}$ from A). Hence the statement is true.

We leave the proof of the statement for $i+1$ applying on α^i the rest of the transformations of (33) as an easy exercise similar to the case of s_0.

The following corollary, by virtue of orbit turns out to be a natural generalization of g1. g2 For values of the parameters satisfying orbit the Painlevé V equation (2) is not integrable.

The proof follows from orbit, g1 and the well-known fact that the Bäcklund transformations are birational canonical transformations on q, p and t [5].

If we recall that $\theta = \alpha_2 - \alpha_0 - 1, \kappa_\infty = \alpha_1, \kappa_0 = \alpha_3$, (13), then $\alpha_2^i - \alpha_0^i - 1, \alpha_1^i, \alpha_3^i$ show how the initial $\theta, \kappa_\infty, \kappa_0$ change under the Bäcklund transformation group (33). Hence as a corollary we obtain: M Assume that $\kappa_\infty + \kappa_0 = \pm\theta + m$ where m is even and at least one $\kappa_j \in \mathbb{Z}, j = \infty, 0$. Then the fifth Painlevé equation (2) is not integrable.

5. Concluding Remarks

We prove non-integrability of one parameters' family of the fifth Painlevé equation as a Hamiltonian system. The main tool to identify obstruction to complete integrability of this Hamiltonian system is Ziglin-Morales-Ramis-Simó theory reducing the question to differential Galois theory. We explicitly compute formal and analytic invariants of the second variational equation (in fact of the part of it) by a method based on the asymptotic analysis of its irregular singularity at zero. From these results we compute the Galois group of our differential equation.

We consider here only the case $\kappa_\infty = 0, \kappa_0 = -\theta$ with θ an arbitrary complex parameter. It is tempting to use the methods of Galois theory for P_V with other values of the parameters, as well as, for other Painlevé transcedents P_{III} and P_{IV}, where the variational equations along each particular solution will have an irregular singularity at zero. We can hope that in the case of Abelian differential Galois group of the first variational equation and one irregular singularity at 0, the reducibility of the second variational equation, considered as a linear homogeneous scalar differential equation, could be an efficient tool to write down the corresponding solution space expressly and therefore to compute formal monodromies, exponential tori and Stokes multipliers.

6. Acknowledgements

The results of this paper has been reported by the author to the 7th DEDS Conference 2010 at University of South Florida. This work, as well as the visit of Ts. Stoyanova to Tampa are partially supported by Grant 225/2010 of Sofia University.

I thank the referees for the careful reading of the manuscript and helpful remarks.

7. References

[1] P. Painlevé, "Sur les Équations Différentielles du Second Ordre et d'Ordre Supérieur dont l'Intégrale Générale est Uniforme," *Acta Mathematica*, Vol. 25, No. 1, December 1902, pp. 1-86.

[2] B. Gambier, "Sur les Équations Différentielles du Second Ordre et du Premier Degré dont l'Intégrale Générale est a Points Critiques Fixes," *Acta Mathematica*, Vol. 33, No. 1, December 1910, pp. 1-55. doi:10.1007/BF02393211

[3] M. Jimbo, T. Miwa, Y. Mori and M. Sato, "Density Matrix of an Impenetrable Bose Gas and the Fifth Painlevé Transcendent," *Physical D*, Vol. 1, No. 1, April 1980, pp. 80-158. doi:10.1016/0167-2789(80)90006-8

[4] N. Lukashevich, "Solutions of the Fifth Equation of Painlevé," *Differential Equations*, Vol. 4, 1968, pp. 732-735.

[5] K. Okamoto, "Studies on the Painlevé Equation II. Fifth Painlevé Equatin P_V," *Japanese Journal of Mathematics*, Vol. 13, No. 1, 1987, pp. 47-76.

[6] J. Malmquist, "Sur Les Équattions Différetielles du Second ordre dont l'Intégrale Générale a ses Points Critiques Fixes," *Astronomy Och Fysik*, Vol. 18, No. 8, 1922, pp. 1-89.

[7] J. Morales-Ruiz, "Kovalevskaya, Liapounov, Painlevé, Ziglin and Differential Galois Theory," *Regular and Chaotic Dynamics*, Vol. 5, No. 3, 2000, pp. 251-272. doi:10.1070/rd2000v005n03ABEH000148

[8] J. Morales-Ruiz, "A Remark about the Painlevé Transcendents," Théories Asymptotiques et Équations de Painlevé, Sémin. Congr Angiers, Society Mathematical, Paris, Vol. 14, 2006, pp. 229-235.

[9] Ts. Stoyanova and O. Christov, "Non-integrability of the second Painlevé Equation as a Hamiltonian System," *Compte Rendu de l'academie Bulgare des Sciences*, Vol. 60, No. 1, January 2007, pp. 13-18.

[10] E. Horozov and Ts. Stoyanova, "Non-integrability of some Painlevé VI-equations and Dilogarithms," *Regular and Chaotic Dynamics*, Vol. 12, No. 6, Desember 2007, pp. 622-629.

[11] T. Stoyanova, "Non-integrability of Painlevé *VI* in the Liouville Sense," *Nonlinearity*, Vol. 22, No. 9, September 2009, pp. 2201-2230. doi:10.1088/0951-7715/22/9/008

[12] J. Morales-Ruiz, "Differential Galois Theory and Non-integrability of Hamiltonian Systems," Progress in Mathematics 179, Birkhäuser, Basel, 1999.

[13] S. Ziglin, "Branching of Solutions and Non-Existence of First Integrals in Hamiltonian Mechanics I," *Functional Analysis and Its Applications*, Vol. 16, No. 3, July 1982, pp. 181-189. doi:10.1007/BF01081586

[14] S. Ziglin, "Branching of Solutions and Non-existence of first Integrals in Hamiltonian Mechanics II," *Functional Analysis and Its Applications*, Vol. 17, No. 1, January 1983, pp. 6-17. doi:10.1007/BF01083174

[15] J. Morales-Ruiz, J-P. Ramis and C. Simó, "Integrability of Hamiltonian Systems and Differential Galois Groups of Higher Variational Equations," *Annales Scientifiques de l'École Normale Sup'erieure*, Vol. 40, No. 6, November 2007, pp. 845-884.

[16] J. Martinet and J-P. Ramis, "Théorie de Galois différentielle et resommation," Computer Algebra and Diffrential Equations, Academic Press, London, 1989, pp. 117-214.

[17] J. Martinet and J-P. Ramis, "Elementary Acceleration and Multisummability," *Physique Théorique*, Vol. 54, No. 4, 1991, pp. 331-401.

[18] M. van der Put and M. Singer, "Galois Theory of Linear Differential Equations," Grundlehren der Mathematischen Wissenschaflen, Springer, 2003.

[19] C. Mitschi, "Differential Galois Groups of Confluent Generalized Hypergeometric Equations: An Approach Using Stokes Multipliers," *Pasific Journal of Mathematics*, Vol. 176, No. 2, 1996, pp. 365-405.

[20] M. Singer, "Introduction to the Galois Theory of Linear Differential Equations," Algebraic Theory of Differential Equations, London Mathematical Society Lecture Note Series, Cambridge University Press, Vol. 2, No. 357, 2009, pp. 1-82.

[21] J. -P. Ramis, "Séries Divergentes et Théories Asymptotiques," Bulletin Sociéte Mathématique de France, Panoramas et Synthéses, Vol. 121, 1993, p. 74.

[22] J.-P. Ramis, “Gevrey Asymptotics and Applications to Holomorphic Ordinary Differential Equations,” Differential Equations and Asymptotic Theory, Series in Analysis, World Scientific Singapore, Vol. 2, 2004, pp. 44-99.

[23] W. Balser, “From Divergent Power Series to Analytic Functions,” Lecture Notes in Mathematics, Springer-Verlag, Berlin Heidelberg, Vol. 1582, 1994.

[24] W. Wasov, “Asymtotic Expansions for Ordinary Differential Equations,” Dover, New York, 1965.

[25] H. Watanabe, “Solutions of the Fifth Painlev’e Equatins I,” *Hokkaido Mathematical Journal*, Vol. 24, No. 2, 1995, pp. 231-268.

[26] E. Whittaker and E. Watson, “A Course of Modern Analysis,” Cambridge University Press, Cambridge, 1989.

[27] P. Berman and M. Singer, “Calculating the Galois group of $L_1(L_2(y))=0, L_1, L_2$ Completely Reducible Operators,” *Journal of Pure and Applied Algebra*, Vol. 139, No. 1-3, June 1999, pp. 3-23. doi:10.1016/S0022-4049(99)00003-1

[28] L. Schlesinger, “Handbuch der Theorie der Linearen Differentialgleichungen,” Teubner, Leipzig, 1897.

[29] T. Masuda, Y. Ohta and K. Kajiwara, “A Determinant Formula for a Class of Rational Solutions of Painlevé V Equation,” *Nagoya Mathematical Journal*, Vol. 168, 2002, pp. 1-25.

