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Extremum Principle for Very Weak Solutions of *A*-Harmonic Equation with Weight*

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Abstract

Extremum principle for very weak solutions of *A*-harmonic equation $div\, A(x,\nabla u)=0$ is obtained, where the operator $A:\Omega\times R^n\to R^n$ satisfies some coercivity and controllable growth conditions with Muckenhoupt weight.

Keywords: *A*-Harmonic Equation, Muckenhoupt Weight, Extremum Principle, Hodge Decomposition

1. Introduction

Throughout this paper Ω will stands for a bounded regular domain in R^n, $n\geq 2$. By a regular domain we understand any domain of finite measure for which the estimates (1.6) and (1.7) for the Hodge decomposition are justified, see [1]. A Lipschitz domain, for example, is regular.

Given a nonnegative locally integrable function w, we say that w belongs to the A_p class of Muckenhoupt, $1<p<\infty$, if

$$\sup_Q\left(\frac{1}{|Q|}\int_Q w\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w^{1/(1-p)}\mathrm{d}x\right)^{p-1}=A_p(w)<\infty \quad (1)$$

where the supremum is taken over all cubes Q of R^n. When $p=1$, replace the inequality (1.1) with

$$Mw(x)\leq cw(x)$$

for some fixed constant c and a.e. $x\in R^n$, where M is the Hardy-Littlewood maximal operator.

It is well-known that $A_1\subset A_p$ whenever $p>1$, see [2]. We will denote by $L^p(\Omega,w)$, $1<p<\infty$, the Banach space of all measurable functions f defined on Ω for which

$$\|f\|_{L^p(\Omega,w)}=\left(\int_\Omega |f(x)|^p w(x)\mathrm{d}x\right)^{1/p}<\infty$$

The weighted Sobolev class $W^{1,p}(\Omega,w)$ consists of all functions f for which f and its first generalized derivatives belong to $L^p(\Omega,w)$.

We will need the following definition. Given $u,v\in W^{1,r}(\Omega)$, $1\leq r<\infty$, the function

$$\eta=\frac{1}{2}(u-v-|u-v|)=\min\{0,u-v\}$$

also belong to $W^{1,r}(\Omega)$. The chain rule gives $\nabla\eta=\nabla u-\nabla v$ if $u(x)<v(x)$ and $\nabla\eta=0$ if $u(x)>v(x)$. We say $u(x)\geq v(x)$ on $\partial\Omega$ in sobolev sense, or symbolically, $u|_{\partial\Omega}\geq v|_{\partial\Omega}$ if the function η defined above lies in $W_0^{1,r}(\Omega)$.

Consider the following second order divergence type elliptic equation (also called *A*-harmonic equation or Leray-Lions equation)

$$\mathrm{div}\, A(x,\nabla u)=0 \quad (2)$$

where $A:\Omega\times R^n\to R^n$ is a Carathéodory function and satisfies

1) $\langle A(x,\xi),\xi\rangle\geq\alpha w(x)|\xi|^p$,

2) $|A(x,\xi)|\leq\beta w(x)|\xi|^{p-1}$,

where $1<p<\infty$, $0<\alpha\leq\beta<\infty$ are fixed constants, and $w(x)\in A_1$ be a Muckenhoupt weight. The prototype of Equation (2) is the p-harmonic equation with weight

$$\mathrm{div}\left(w(x)|\nabla u|^{p-2}\nabla u\right)=0$$

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Definition: *A function* $u \in W^{1,r}(\Omega, w)$ *with* $\max\{1, p-1\} \le r < p$ *is called a very weak solution of* (2) *if*

$$\int_\Omega \langle A(x, \nabla u), \nabla \phi \rangle \mathrm{d}x = 0 \quad (3)$$

for all $\varphi \in W^{1,r/(r-p+1)}(\Omega, w)$ with compact support.

Recall that $u \in W^{1,p}(\Omega, w)$ is a weak solution of (2) if (3) holds for all $\varphi \in W^{1,p}(\Omega, w)$ with compact support. The word *very weak* in the above definition means that the Sobolev integrable exponent r of u is smaller than the *natural* exponent p.

Extremum principle for weak and very weak solutions of elliptic equations is an important and basic property. It is closely related to the uniqueness results for some boundary value problems of elliptic PDEs, see [4]. Motivated by this property, Gao, Li and Deng showed in [3] the extremum principle for very weak solutions of (2) with the weight $w(x) \equiv 1$. In the present paper, we generalize the result obtained in [3] to weighted case, and prove the extremum principle for very weak solutions of (2). The main result of this paper is the following theorem.

Theorem 1: *Suppose that* $w \in A_1$ *be a Muckenhoupt weight. There exists* $r_1 = r_1(\alpha, \beta, n, p, w) < p < r_2 = r_2(\alpha, \beta, n, p, w)$, *such that if* $u \in W^{1,r}(\Omega, w)$ *is a very weak solution of the* A*-harmonic Equation* (1), *and* $m \le u(x) \le M$ *on* $\partial\Omega$ *in the sobolev sense, then* $m \le u(x) \le M$ *almost everywhere in* Ω, *provided that* $r_1 < r < r_2$.

With the extremum principle at hand, we can consider the 0-Dirichlet problem

$$\begin{cases} div A(x, \nabla u) = 0 \\ u \in W_0^{1,r}(\Omega) \end{cases} \quad (4)$$

Theorem 2: *Let* r_1 *and* r_2 *be the exponents in Theorem* 1 *and* $r_1 < r < r_2$. *Then the* 0*-Dirichlet boundary value problem* (4) *has only zero solution.*

We will need the following lemma in the proof of the main theorem, which is a Hodge decomposition in weighted spaces.

Lemma: [5] *Let* Ω *be a regular domain and* $w(x)$ *be an* A_1 *weight. If* $u \in W_0^{1,p-\varepsilon}(\Omega, w)$, $1 < p < \infty$, $-1 < \varepsilon < p-1$, *then there exist* $\varphi \in W_0^{1,(p-\varepsilon)/(1-\varepsilon)}(\Omega, w)$ *and a divergence-free vector field* $H \in L^{(p-\varepsilon)/(1-\varepsilon)}(\Omega, w)$ *such that*

$$|\nabla u|^{-\varepsilon} \nabla u = \nabla \phi + h \quad (5)$$

and

$$\|\nabla \varphi\|_{L^{(p-\varepsilon)/(1-\varepsilon)}(\Omega,w)} \le C A_p(w)^\gamma \|\nabla u\|_{L^{p-\varepsilon}(\Omega,w)}^{1-\varepsilon} \quad (6)$$

$$\|h\|_{L^{(p-\varepsilon)/(1-\varepsilon)}(\Omega,w)} \le C A_p(w)^\gamma |\varepsilon| \|\nabla u\|_{L^{p-\varepsilon}(\Omega,w)}^{1-\varepsilon} \quad (7)$$

where $\gamma = \gamma(p)$ and $C = C(n, p, w)$ depending only on p and n, p, w, respectively.

2. Proof of Theorem 1 and Theorem 2

Proof of Throrem 1. If $u(x)$ is a very weak solution of the Equation (2), then also is $v(x) = u(x) - m \in W^{1,r}(\Omega, w)$. For every test function $\psi \in W_0^{1,r/L^{(p-\varepsilon)/(1-\varepsilon)}}(\Omega, w)$, we have

$$\int_\Omega \langle A(x, \nabla u), \nabla \psi \rangle \mathrm{d}x = 0 \quad (8)$$

Now let $\phi = \min\{0, v\}$. It is easy to see that $\phi \in W_0^{1,r}(\Omega, w)$. Consider the Hodge decomposition of $|\nabla \varphi|^{r-p} \nabla \varphi$,

$$|\nabla \varphi|^{r-p} \nabla \varphi = \nabla \psi + h$$

By Lemma, we have the following estimate

$$\|h\|_{L^{r/(r-p+1)}(\Omega,w)} \le C A_p(w)^\gamma |r-p| \|\nabla \varphi\|_{L^r(\Omega,w)}^{r-p+1} \quad (9)$$

The integral identity (8) with ψ as a test function of class $W_0^{1,r/(r-p+1)}(\Omega, w)$ takes the form

$$\int_\Omega \left\langle A(x, \nabla u), |\nabla \phi|^{r-p} \nabla \phi \right\rangle \mathrm{d}x = \int_\Omega \langle A(x, \nabla u), h \rangle \mathrm{d}x \quad (10)$$

Let us put

$$X = \{x \in \Omega : v(x) < 0\}$$

Since the gradient of ϕ is equal to ∇v on X, while it vanishes on Ω / X, then for this choice of ϕ the integral in (10) reduces to

$$\int_X \left\langle A(x, \nabla v), |\nabla v|^{r-p} \nabla v \right\rangle \mathrm{d}x = \int_X \langle A(x, \nabla v), h \rangle \mathrm{d}x$$

By the conditions 1) and 2), the above equality yields

$$\alpha \int_X |\nabla v|^r w \mathrm{d}x \le \beta \int_X |\nabla v|^{r-p} |h| w \mathrm{d}x$$

Using Hölder's inequality and (9) we obtain

$$\begin{aligned} \alpha \int_X |\nabla v|^r w \mathrm{d}x &\le \beta \|\nabla \varphi\|_{L^r(\Omega,w)}^{p-1} \|h\|_{L^{r/(r-p+1)}(\Omega,w)} \\ &\le C\beta A_p(w)^\gamma |p-r| \int_X |\nabla v|^r w \mathrm{d}x \end{aligned} \quad (11)$$

Taking $r_1 < p < r_2$ sufficiently close to p to satisfy $\alpha = C\beta A_p(w)^\gamma (p - r_1)$ and $\alpha = C\beta A_p(w)^\gamma (r_2 - p)$.

Thus, if $r_1 < r < r_2$, then $\theta = \dfrac{C\beta A_p(w)^\gamma |p-r|}{\alpha} < 1$ (11)

yields $\|\nabla v\|_{L^r(\Omega,w)} = 0$, from which we deduce $\phi(x) = 0$ almost everywhere in Ω, and this simply means that $m \le u(x)$ almost everywhere in Ω.

Similarly, by the same method we used above, we can also derive $u(x) \le M$ almost everywhere in Ω. This completes the proof of Theorem 1.

Proof of Throrem 2. By Theorem 1, we know that $u(x) \ge 0$ and $u(x) \le 0$ almost everywhere in Ω. This simply means that $u(x) = 0$ almost everywhere in Ω. This completes the proof of Theorem 2.

3. References

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