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Real Hypersurfaces in Complex Two-Plane Grassmannians whose Jacobi Operators Corresponding to $D^{\perp}$-Directions are of Codazzi Type

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Abstract

We prove the non-existence of Hopf real hypersurfaces in complex two-plane Grassmannians whose Jacobi operators corresponding to the directions in the distribution $D^{\perp}$ are of Codazzi type if they satisfy a further condition. We obtain that that they must be either of type (*A*) or of type (*B*) (see [1]), but no one of these satisfies our condition. As a consequence, we obtain the non-existence of Hopf real hypersurfaces in such ambient spaces whose Jacobi operators corresponding to $D^{\perp}$-directions are parallel with the same further condition.

Keywords: Real Hypersurfaces, Complex Two-Plane Grassmannians, Jacobi Operators, Codazzi Type

1. Introduction

The geometry of real hypersurfaces in complex space forms or in quaternionic space forms is one of interesting parts in the field of differential geometry. Now let us consider consider real hypersurfaces in complex two-plane Grassmannian $G_2\left(\mathbb{C}^{m+2}\right)$, which consists of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. It is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathbb{J}$ (see Berndt and Suh [2]). Let M be a real hypersurface in $G_2\left(\mathbb{C}^{m+2}\right)$ and N a local normal unit vector field. We can define the structure vector field of M by $\xi = -JN$. Moreover, if $\{J_1, J_2, J_3\}$ is a local basis of $\mathbb{J}$, we define $\xi_i = -J_i N$, $i = 1,2,3$. Thus we can consider two natural geometric conditions: that both $[\xi] = Span\{\xi\}$ and $D^{\perp} = Span\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A corresponding to N. Berndt and Suh, [1] proved the following:

Theorem A *Let M be a connected real hypersurface in $G_2\left(\mathbb{C}^{m+2}\right)$, $m \geq 3$. Then both $[\xi]$ and $D^{\perp}$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2\left(\mathbb{C}^{m+1}\right)$ in $G_2\left(\mathbb{C}^{m+2}\right)$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2\left(\mathbb{C}^{m+2}\right)$.*

The structure vector field ξ of a real hypersurface M in $G_2\left(\mathbb{C}^{m+2}\right)$ is said to be a *Reeb* vector field. If the *Reeb* vector field ξ of a real hypersurface M in $G_2\left(\mathbb{C}^{m+2}\right)$ is invariant by the shape operator, M is said to be a *Hopf hypersurface*. In such a case the integral curves of the *Reeb* vector field ξ are geodesics (see Berndt and Suh [2]). Moreover, the flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2\left(\mathbb{C}^{m+2}\right)$ is said to be *geodesic Reeb flow*. Moreover, if the corresponding principal curvature α corresponding to ξ is non-vanishing we say M is with non-vanishing *geodesic Reeb flow*.

Jacobi fields along geodesics of a given Riemannian manifold $\left(\tilde{M}, \tilde{g}\right)$ satisfy a very well-known differential equation. This classical differential equation naturally inspires the so-called Jacobi operators. That is, if $\tilde{R}$ is the curvature operator of $\tilde{M}$, the Jacobi operator (with respect to X) at $p \in \tilde{M}$, $\tilde{R}_X \in End\left(T_p\tilde{M}\right)$, is defined as $\left(\tilde{R}_X(Y)\right)(p) = \left(\tilde{R}(Y, X)X\right)(p)$, for all $Y \in T_p\tilde{M}$, being a self-adjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$. Clearly each tangent vector field X to $\tilde{M}$ provides a Jacobi operator with respect to X.

Let $\bar{R}$ denote the Riemannian curvature tensor of the complex two-plane Grassmannian $G_2\left(\mathbb{C}^{m+2}\right)$. Now if M is a real hypersurface in $G_2\left(\mathbb{C}^{m+2}\right)$ with normal vector field N we can consider the normal Jacobi operator $\bar{R}_N$ on $G_2\left(\mathbb{C}^{m+2}\right)$. Moreover, it is clear that $\bar{R}_N(N)=0$, so we can consider $\bar{R}_N$ as a self adjoint endomorphism of the tangent bundle TM of M. We will call it the *normal Jacobi operator* on M. The Jacobi operator associated to the Reeb vector field R_ξ is called the *structure Jacobi operator* on M, where R denotes the curvature tensor of M.

Recently, Jeong, Pérez and Suh, (see [3]) have proved the non-existence of real hypersurfaces M in $G_2\left(\mathbb{C}^{m+2}\right)$ with parallel structure Jacobi operator when a further condition is satisfied. Also, Jeong, Kim and Suh, (see [3]) have proved the non-existence of real hypersurfaces M in $G_2\left(\mathbb{C}^{m+2}\right)$ with parallel normal Jacobi operator. Further results can also be seen in [4].

In this paper we will consider the Jacobi operators associated to a basis of the distribution $D^\perp$, R_{ξ_i}, $i=1,2,3$. A type $(1,1)$ tensor T on M is called of Codazzi type if $(\nabla_X T)Y=(\nabla_Y T)X$ for any X,Y tangent to M, where ∇ denotes the covariant derivative on M. In this paper we will study real hypersurfaces M in $G_2\left(\mathbb{C}^{m+2}\right)$ whose Jacobi operators R_{ξ_i}, $i=1,2,3$ are of Codazzi type. Namely, we will prove the following.

Theorem 1.1 *There do not exist any connected Hopf real hypersurfaces* M *in* $G_2\left(\mathbb{C}^{m+2}\right)$, $m\geq 3$, *such that* $\left(\nabla_X R_{\xi_i}\right)Y=\left(\nabla_Y R_{\xi_i}\right)X$, $i=1,2,3$, *for any* $X,Y\in TM$ *if the distribution* D *or the* $D^\perp$*-component of the Reeb vector field is invariant by the shape operator.*

As a consequence of Theorem 1.1, we immediately obtain the following.

Theorem 1.2 *There do not exist any connected Hopf real hypersurfaces* M *in* $G_2\left(\mathbb{C}^{m+2}\right)$, $m\geq 3$, *whose Jacobi operators* R_{ξ_i}, $i=1,2,3$, *are parallel if the distribution* D *or the* $D^\perp$*-component of the Reeb vector field is invariant by the shape operator.*

2. Preliminaries

For the study of Riemannian geometry of $G_2\left(C^{m+2}\right)$ see [1]. All the notations we will use since now are the ones in [1] and [2]. We will suppose that the metric g of $G_2\left(C^{m+2}\right)$ is normalized for the maximal sectional curvature of the manifold to be eight. Then the Riemannian curvature tensor $\bar{R}$ of $G_2\left(C^{m+2}\right)$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &\quad - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y \\ &\quad - 2g(J_\nu X,Y)J_\nu Z\} \\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{2.1}$$

where J_1,J_2,J_3 is any canonical local basis of $\mathbb{J}$.

Let M be a real hypersurface of $G_2\left(C^{m+2}\right)$, that is, a submanifold of $G_2\left(C^{m+2}\right)$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M,g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2\left(C^{m+2}\right)$ induces on M an almost contact metric structure (ϕ,ξ,η,g). Furthermore, let J_1,J_2,J_3 be a canonical local basis of J. Then each J_ν induces an almost contact metric structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$ on M. Using the above expression for the curvature tensor $\bar{R}$, the Gauss and Codazzi equations are respectively given by

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y \\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ &+ \sum_{\nu=1}^{3}\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\} \\ &+ \sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\} \\ &- \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\} \\ &- \sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y,Z) - \eta(Y)g(\phi_\nu\phi X,Z)\}\xi_\nu \\ &+ g(AY,Z)AX - g(AX,Z)AY\end{aligned}$$

and

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu\} \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\ &+ \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu,\end{aligned}$$

where R denotes the curvature tensor of M in $G_2\left(C^{m+2}\right)$.

In [2] the following Proposition is obtained.

Proposition 2.1 *If* M *is a connected orientable real hypersurface in* $G_2\left(\mathbb{C}^{m+2}\right)$ *with geodesic Reeb flow,* *then*

$$\alpha g\left(\left(A\phi+\phi A\right)X,Y\right)-2g\left(A\phi X,Y\right)+2g\left(\phi X,Y\right)$$
$$=2\sum_{\nu=1}^{3}\left(\eta_\nu\left(X\right)\eta_\nu\left(\phi Y\right)-\eta_\nu\left(Y\right)\eta_\nu\left(\phi X\right)-g\left(\phi_\nu X,Y\right)\eta_\nu\left(\xi\right)\right.$$
$$\left.-2\eta\left(X\right)\eta_\nu\left(\phi Y\right)\eta_\nu\xi+2\eta\left(Y\right)\eta_\nu\left(\phi X\right)\eta_\nu\left(\xi\right)\right)$$

for any $X,Y\in TM$ where $\alpha=g\left(A\xi,\xi\right)$.

Recently Lee and Suh (see [5]) have proved the following.

Proposition 2.2 *Let* M *be a connected orientable Hopf real hypersurface in* $G_2\left(\mathbb{C}^{m+2}\right)$, $m\geq 3$. *Then the Reeb vector* ξ *belongs to the distribution* D *if and only if* M *is locally congruent to an open part of a tube around a totally geodesic* $\mathbb{H}P^n$ *in* $G_2\left(\mathbb{C}^{m+2}\right)$, *where* $m=2n$.

3. Proof of Theorem 1.1

From the expression of the curvature tensor of $G_2\left(\mathbb{C}^{m+2}\right)$ we get

$$R_{\xi_i}\left(X\right)=X-\eta_i\left(X\right)\xi_i-3g\left(\phi X,\xi_i\right)\phi\xi_i$$
$$-3\sum_{\nu=1}^{3}g\left(\phi_\nu X,\xi_i\right)\phi_\nu\xi_i+\sum_{\nu=1}^{3}g\left(\phi_\nu\phi\xi_i,\xi_i\right)\left(\phi_\nu\phi X-\eta\left(X\right)\xi_\nu\right)$$
$$+\sum_{\nu=1}^{3}g\left(\phi_\nu\phi X,\xi_i\right)\left(\eta\left(\xi_i\right)\xi_\nu-\phi_\nu\phi\xi_i\right)$$
$$-\eta\left(\xi_i\right)\phi_i\phi X+\eta\left(X\right)\phi_i\phi\xi_i+g\left(A\xi_i,\xi_i\right)AX-g\left(AX,\xi_i\right)A\xi_i \tag{3.1}$$

for any tangent vector field X. From (3.1) we have

$$\left(\nabla_X R_{\xi_i}\right)Y=-g\left(Y,\nabla_X\xi_i\right)\xi_i-\eta_i\left(Y\right)\nabla_X\xi_i$$
$$-3\left\{\eta\left(Y\right)\eta_i\left(AX\right)-\eta\left(\xi_i\right)g\left(AX,Y\right)+g\left(\phi Y,\nabla_X\xi_i\right)\phi\xi_i+\eta_i\left(\phi Y\right)\left(\eta\xi_i AX-\eta_i AX\xi+\phi\nabla_X\xi_i\right)\right\}$$
$$-3\sum_{\nu=1}^{3}\left\{\left(-q_{\nu+1}\left(X\right)\eta_i\left(\phi_{\nu+2}Y\right)+q_{\nu+2}\left(X\right)\eta_i\left(\phi_{\nu+1}Y\right)+\eta_\nu\left(Y\right)\eta_i\left(AX\right)-\eta_\nu\left(\xi_i\right)g\left(AX,Y\right)+g\left(\phi_\nu Y,\nabla_X\xi_i\right)\right)\phi_\nu\xi_i\right.$$
$$\left.+\eta_i\left(\phi_\nu Y\right)\left(-q_{\nu+1}\left(X\right)\phi_{\nu+2}\xi_i+q_{\nu+2}\left(X\right)\phi_{\nu+1}\xi_i+\eta_\nu\left(\xi_i\right)AX-\eta_i\left(AX\right)\xi_\nu+\phi_\nu\nabla_X\xi_i\right)\right\}$$
$$+\sum_{\nu=1}^{3}\left\{\left(-g\left(AX,\phi_\nu\xi_i\right)\eta\left(\xi_i\right)+\eta_i\left(AX\right)\eta\left(\phi_\nu\xi_i\right)+g\left(\nabla_X\xi_i,\phi\phi_\nu\xi_i\right)-q_{\nu+1}\left(X\right)\eta_i\left(\phi\phi_{\nu+2}\xi_i\right)\right.\right.$$
$$\left.+q_{\nu+2}\left(X\right)\eta_i\left(\phi\phi_{\nu+1}\xi_i\right)-\eta_\nu\left(\xi_i\right)g\left(AX,\phi\xi_i\right)+\eta_i\left(AX\right)\eta_\nu\left(\phi\xi_i\right)+g\left(\phi_\nu\phi\xi_i,\nabla_X\xi_i\right)\right)\phi_\nu\phi Y$$
$$+\eta_i\left(\phi_\nu\phi\xi_i\right)\left(-q_{\nu+1}\left(X\right)\phi_{\nu+2}\phi Y+q_{\nu+2}\left(X\right)\phi_{\nu+1}\phi Y+\eta_\nu\left(\phi Y\right)AX\right.$$
$$\left.\left.-g\left(AX,\phi Y\right)\xi_\nu+\eta\left(Y\right)\phi_\nu AX-g\left(AX,Y\right)\phi_\nu\xi\right)\right\}$$
$$-\sum_{\nu=1}^{3}\left\{\left(-\eta\left(\xi_i\right)g\left(AX,\phi_\nu\xi_i\right)+\eta_i\left(AX\right)\eta_\nu\left(\phi\xi_i\right)+g\left(\nabla_X\xi_i,\phi\phi_\nu\xi_i\right)-q_{\nu+1}\left(X\right)\eta_i\left(\phi\phi_{\nu+2}\xi_i\right)\right.\right.$$
$$\left.+q_{\nu+2}\left(X\right)\eta_i\left(\phi\phi_{\nu+1}\xi_i\right)-\eta_\nu\left(\xi_i\right)g\left(AX,\phi\xi_i\right)+\eta_i\left(AX\right)\eta_\nu\left(\phi\xi_i\right)+g\left(\nabla_X\xi_i,\phi_\nu\phi\xi_i\right)\right)\eta\left(Y\right)\xi_\nu$$
$$\left.+\eta_i\left(\phi\phi_\nu\xi_i\right)\left(g\left(Y,\phi AX\right)\xi_\nu+\eta\left(Y\right)\nabla_X\xi_\nu\right)\right\}$$
$$+\sum_{\nu=1}^{3}\left\{\left(-q_{\nu+1}\left(X\right)g\left(Y,\phi\phi_{\nu+2}\xi_i\right)+q_{\nu+2}\left(X\right)g\left(Y,\phi\phi_{\nu+1}\xi_i\right)+\eta_\nu(\phi Y)\eta_i\left(AX\right)\right.\right.$$
$$\left.-g\left(AX,\phi Y\right)\eta_\nu\left(\xi_i\right)+\eta\left(Y\right)\eta_i\left(\phi_\nu AX\right)+g\left(AX,Y\right)\eta_\nu\left(\phi\xi_i\right)+g\left(\phi_\nu\phi Y,\nabla_X\xi_i\right)\right)\eta\left(\xi_i\right)\xi_\nu$$
$$\left.+g\left(Y,\phi\phi_\nu\xi_i\right)\left(\left(\eta\left(\nabla_X\xi_i\right)+\eta_i\left(\phi AX\right)\right)\xi_\nu+\eta\left(\xi_i\right)\nabla_X\xi_\nu\right)\right\}$$
$$-\sum_{\nu=1}^{3}\left\{\left(-q_{\nu+1}\left(X\right)\eta_i\left(\phi_{\nu+2}\phi Y\right)+q_{\nu+2}\left(X\right)\eta_i\left(\phi_{\nu+1}\phi Y\right)+\eta_\nu\left(\phi Y\right)\eta_i\left(AX\right)-g\left(AX,\phi Y\right)\eta_\nu\left(\xi_i\right)\right.\right.$$
$$\left.+\eta\left(Y\right)\eta_i\left(\phi_\nu AX\right)+g\left(AX,Y\right)\eta\left(\phi_\nu\xi_i\right)+g\left(\phi_\nu\phi Y,\nabla_X\xi_i\right)\right)\phi_\nu\phi\xi_i$$
$$+\eta_i\left(\phi_\nu\phi Y\right)\left(-q_{\nu+1}\left(X\right)\phi_{\nu+2}\phi\xi+q_{\nu+2}\left(X\right)\phi_{\nu+1}\phi\xi_i+\eta_\nu\left(\phi\xi_i\right)AX-g\left(AX,\phi\xi_i\right)\xi_\nu\right.$$
$$\left.\left.+\eta\left(\xi_i\right)\phi_\nu AX-\eta_i\left(AX\right)\phi_\nu\xi+\phi_\nu\phi\nabla_X\xi_i\right)\right\}$$
$$-\left\{\left(\eta\left(\nabla_X\xi_i\right)+\eta_i\left(\nabla_X\xi\right)\right)\right\}\phi_i\phi Y$$
$$+\eta\left(\xi_i\right)\left(-q_{i+1}\left(X\right)\phi_{i+2}\phi Y+q_{i+2}\left(X\right)\phi_{i+1}\phi Y+\eta_i\left(\phi Y\right)AX-g\left(AX,\phi Y\right)\xi_i+\eta\left(Y\right)\phi_i AX-g\left(AX,Y\right)\phi_i\xi\right)$$

$$
\begin{aligned}
&+g(Y,\nabla_X\xi)\phi_i\phi\xi_i+\eta(Y)\big(-q_{i+1}(X)\phi_{i+2}\phi\xi_i+q_{i+2}(X)\phi_{i+1}\phi\xi_i\\
&\qquad -g(AX,\phi\xi_i)\xi_i+\phi_i\nabla_X\xi_i+\eta(\xi_i)\phi_iAX-\eta_i(AX)\phi_i\xi+\phi_i\phi\nabla_X\xi_i\big)\\
&+\big(\eta_i(\nabla_XA\xi_i)+g(A\xi_i,\nabla_X\xi_i)\big)AY+\eta_i(A\xi_i)(\nabla_XA)Y\\
&-\big\{\big(\eta_i((\nabla_XA)Y)+g(AY,\nabla_X\xi_i)\big)A\xi_i+\eta_i(AY)\nabla_XA\xi\big\}
\end{aligned}
\tag{3.2}
$$

for any X,Y tangent to M.

We will write $\xi=\eta(X_0)X_0+\eta(\xi_1)\xi_1$, for a unit $X_0\in D$, where we suppose $\eta(X_0)\eta(\xi_1)\neq 0$. Then we have $g(\phi_\nu\phi\xi_1,\xi_1)=0$, $\nu=1,2,3$. Notice this is true even if $\xi\in D$. Thus the covariant derivative of R_{ξ_1} is given by Equation (3.3), for any X,Y tangent to M. From this expression we have:

Lemma 3.1 *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ such that D or $D^\perp$-component of the Reeb vector field is A-invariant. If $(\nabla_X R_{\xi_i})Y=(\nabla_Y R_{\xi_i})X$, $i=1,2,3$, for any $X,Y\in TM$, then $\xi\in D$ or $\xi\in D^\perp$.*

Proof: As we suppose $A\xi=\alpha\xi$ and have written $\xi=\eta(X_0)X_0+\eta(\xi_1)\xi_1$ with $\eta(X_0)$ and $\eta(\xi_1)$ non null, where $X_0\in D$ is unit, as $\phi\xi=0$ we get $\phi X_0=-\eta(\xi_1)\phi_1X_0$. Moreover, $AX_0=\alpha X_0$.

Taking $X=X_0$ in Proposition 2.1 we have

$$
\begin{aligned}
&-\alpha\eta(\xi_1)A\phi_1X_0-\alpha^2\eta(\xi_1)\phi_1X_0+2\alpha\eta(\xi_1)\phi_1X_0\\
&=4\eta^2(X_0)\eta(\xi_1)\phi_1X_0
\end{aligned}.
$$

From this, if $\alpha=0$ we obtain $\eta(X_0)\eta(\xi_1)=0$, giving us the result. Thus we suppose $\alpha\neq 0$. Therefore

$$A\phi_1X_0=\frac{1}{\alpha}\big(4\eta^2(X_0)+\alpha^2\big)\phi_1X_0.$$

We also have

$$\phi\xi_1=\phi_1\xi=\eta(X_0)\phi_1X_0.$$

$$
\begin{aligned}
&(\nabla_XR_{\xi_1})Y=\nabla_X(R_{\xi_1}(Y))-R_{\xi_1}(\nabla_XY)\\
&=-g(Y,\nabla_X\xi_1)\xi_1-\eta_1(Y)\nabla_X\xi_1\\
&-3\big\{\{\eta(Y)\eta_1(AX)-g(AX,Y)\eta(\xi_1)+g(\phi Y,\nabla_X\xi_1)\}\phi\xi_1+\eta_1(\phi Y)\eta(\xi_1)AX-\eta_1(AX)\xi+\phi\nabla_X\xi_1\big\}\\
&-3\sum_{\nu=1}^{3}\big\{\{-q_{\nu+1}(X)\eta_1(\phi_{\nu+2}Y)+q_{\nu+2}(X)\eta_1(\phi_{\nu+1}Y)+\eta_\nu(Y)\eta_1(AX)-g(AX,Y)\eta_\nu(\xi_1)+g(\phi_\nu Y,\nabla_X\xi_1)\}\phi_\nu\xi_1\\
&\qquad+\eta_1(\phi_\nu Y)\{-q_{\nu+1}(X)\phi_{\nu+2}\xi_1+q_{\nu+2}(X)\phi_{\nu+1}\xi_1+\eta_\nu(\xi_1)AX-\eta_1(AX)\xi_\nu+\phi_\nu\nabla_X\xi_1\}\big\}\\
&+\sum_{\nu=1}^{3}\big\{\{-q_{\nu+1}(X)g(Y,\phi\phi_{\nu+2}\xi_1)+q_{\nu+2}(X)g(Y,\phi\phi_{\nu+1}\xi_1)+\eta_\nu(\phi Y)\eta_1(AX)-g(AX,\phi Y)\eta_\nu(\xi_1)\\
&\qquad+\eta(Y)\eta(\phi_\nu AX)+g(AX,Y)\eta_\nu(\phi\xi_1)+g(\phi_\nu\phi Y,\nabla_X\xi_1)\}\eta(\xi_1)\xi_\nu\\
&\qquad+g(Y,\phi\phi_\nu\xi_1)\{\eta(\nabla_X\xi_1)+\eta_1(\phi AX)\xi_\nu+\eta(\xi_1)\nabla_X\xi_\nu\}\big\}\\
&-\sum_{\nu=1}^{3}\big\{\{-q_{\nu+1}(X)\eta_1(\phi_{\nu+2}\phi Y)+q_{\nu+2}(X)\eta_1(\phi_{\nu+1}\phi Y)+\eta_\nu(\phi Y)\eta_1(AX)-g(AX,\phi Y)\eta_\nu(\xi_1)+\eta_1(\phi_\nu AX)\eta(Y)\\
&\qquad+g(AX,Y)\eta(\phi_\nu\xi_1)+g(\phi_\nu\phi Y,\nabla_X\xi_1)\}\phi_\nu\phi\xi_1\\
&\qquad+\eta_1(\phi_\nu\phi Y)\{-q_{\nu+1}(X)\phi_{\nu+2}\phi\xi_1+q_{\nu+2}(X)\phi_{\nu+1}\phi\xi_1+\eta_\nu(\phi\xi_1)AX\\
&\qquad\qquad-g(AX,\phi\xi_1)\xi_\nu+\eta(\xi_1)\phi_\nu AX-\eta_1(AX)\phi_\nu\xi+\phi_\nu\phi\nabla_X\xi_1\}\big\}\\
&-\big\{\{\eta(\nabla_X\xi_1)+\eta_1(\nabla_X\xi)\}\phi_1\phi Y\\
&\qquad+\eta(\xi_1)\{-q_2(X)\phi_3\phi Y+q_3(X)\phi_2\phi Y+\eta_1(\phi Y)AX-g(AX,\phi Y)\xi_1+\eta(Y)\phi_1AX-g(AX,Y)\phi_1\xi\}\big\}\\
&+g(Y,\nabla_X\xi)\phi_1\phi\xi_1+\eta(Y)\big(-q_2(X)\phi_3\phi\xi_1+q_3(X)\phi_2\phi\xi_1-g(AX,\phi\xi_1)\xi_1+\phi_1\nabla_X\xi_1+\eta(\xi_1)\phi_1AX-\eta_1(AX)\phi_1\xi+\phi_1\phi\nabla_X\xi_1\big)\\
&+\{\eta_1(\nabla_XA\xi_1)+g(A\xi_1,\nabla_X\xi_1)\}AY+\eta_1(A\xi_1)(\nabla_XA)Y\\
&-\big\{\{\eta_1((\nabla_XA)Y)+g(AY,\nabla_X\xi_1)\}A\xi_1+\eta_1(AY)\nabla_XA\xi_1\big\}
\end{aligned}
\tag{3.3}
$$

From (3.3) we get

$$g\left(\left(\nabla_{\xi}R_{\xi_1}\right)\xi_1,\phi_1X_0\right)=-\alpha\eta\left(X_0\right)-\alpha^2\eta\left(X_0\right)-7\alpha\eta^3\left(X_0\right)+\alpha\eta\left(X_0\right)\eta^2\left(\xi_1\right). \tag{3.4}$$

and

$$g\left(\left(\nabla_{\xi_1}R_{\xi_1}\right)\xi,\phi_1X_0\right)=-4\alpha\eta\left(X_0\right)+4\alpha\eta\left(X_0\right)\eta^2\left(\xi_1\right)-4\alpha\eta^3\left(X_0\right). \tag{3.5}$$

As we suppose that R_{ξ_1} is of Codazzi type (3.4) and (3.5) must be equal. This yields $\alpha^2\eta\left(X_0\right)=0$. As we suppose $\alpha\neq 0$ the result follows. □

With the hypothesis in Lemma 3.1, we can prove:

Lemma 3.2 *If* $\xi\in D^{\perp}$ *then* $g\left(AD,D^{\perp}\right)=0$

Proof: In this case, we can take $\xi=\xi_1$. Thus the condition of R_{ξ_1} being of Codazzi type is equivalent to R_{ξ} also being. Taking $Y=\xi$ and $X\in D$ we get

$$\left(\nabla_X R_{\xi}\right)\xi=-\phi AX-\alpha A\phi AX-2\eta_3\left(AX\right)\xi_2+2\eta_2\left(AX\right)\xi_3-\phi_1AX. \tag{3.6}$$

On the other hand

$$\left(\nabla_{\xi}R_{\xi}\right)X=\xi\left(\alpha\right)AX+\alpha\left(\nabla_{\xi}A\right)X. \tag{3.7}$$

Therefore we have

$$-\phi AX-\alpha A\phi AX-2\eta_3\left(AX\right)\xi_2+2\eta_2\left(AX\right)\xi_3-\phi_1AX=\xi\left(\alpha\right)AX+\alpha\left(\nabla_{\xi}A\right)X$$

Taking its scalar product with ξ_2 it follows

$$-\alpha g\left(A\phi AX,\xi_2\right)-2\eta_3\left(AX\right)=\xi\left(\alpha\right)\eta_2\left(AX\right)+\alpha g\left(\left(\nabla_{\xi}A\right)X,\xi_2\right). \tag{3.8}$$

and the scalar product with ξ_3 yields

$$-\alpha g\left(A\phi AX,\xi_3\right)+2\eta_2\left(AX\right)=\xi\left(\alpha\right)\eta_3\left(AX\right)+\alpha g\left(\left(\nabla_{\xi}A\right)X,\xi_3\right) \tag{3.9}$$

Now the Codazzi equation gives

$$g\left(\left(\nabla_{\xi}A\right)X,\xi_2\right)=g\left(\left(\nabla_XA\right)\xi,\xi_2\right)=-g\left(A\phi AX,\xi_2\right)+\alpha\eta_3\left(AX\right).$$

and

$$g\left(\left(\nabla_{\xi}A\right)X,\xi_3\right)=g\left(\left(\nabla_XA\right)\xi,\xi_3\right)=-g\left(A\phi AX,\xi_3\right)-\alpha\eta_2\left(AX\right).$$

From (3.8) and (3.9) we get

$$\left(\alpha^2+2\right)\eta_3\left(AX\right)+\xi\left(\alpha\right)\eta_2\left(AX\right)=0.$$
$$-\left(\alpha^2+2\right)\eta_2\left(AX\right)+\xi\left(\alpha\right)\eta_3\left(AX\right)=0. \tag{3.10}$$

If $\xi(\alpha)=0$ we have finished. If $\xi(\alpha)\neq 0$, from (3.10) we obtain $\eta_2\left(AX\right)=-\dfrac{\alpha^2+2}{\xi\left(\alpha\right)}\eta_3\left(AX\right)$ and $\eta_3\left(AX\right)=-\dfrac{\alpha^2+2}{\xi\left(\alpha\right)}\eta_2\left(AX\right)$. Clearly, this yields $\eta_2\left(AX\right)=\eta_3\left(AX\right)=0$, finishing the proof. □

From this Lemma and Proposition 2.2, in order to finish the proof of our Theorem, we only have to see if the real hypersurfaces of either type (A) or type (B) satisfy our condition.

In the case of a real hypersurface of type (A) we get from Proposition 3 in [1], considering $\xi=\xi_1$ and taking $X=\xi_2$, $Y=\xi$, that if our condition is satisfied we should have $\left(\nabla_{\xi_2}R_{\xi}\right)\xi=\left(\nabla_{\xi}R_{\xi}\right)\xi_2$. This yields $-\alpha\left(\nabla_{\xi}A\right)\xi_2+\alpha\beta^2\xi_3+2\beta\xi_3=0.$ As $\left(\nabla_{\xi}A\right)\xi_2=\left(\alpha-\beta\right)q_3\left(\xi\right)\xi$ we have

$$-\alpha\left(\alpha-\beta\right)q_3\left(\xi\right)\xi+\beta\left(\alpha\beta+2\right)\xi_3=0. \tag{3.11}$$

From (3.11) we have $\alpha\left(\alpha-\beta\right)q_3\left(\xi\right)=0$ and $\beta\left(\alpha\beta+2\right)=0$. If $\alpha=0$, from the second equality we also obtain $\beta=0$, but $\beta=\sqrt{2}\cot\left(\sqrt{2}r\right)$ for some $r\in\left(0,\dfrac{\pi}{\sqrt{8}}\right)$. Thus this is impossible.

If $\alpha=\beta$, from the second equality we get $\alpha^2+2=0$, having a contradiction. Thus $q_3\left(\xi\right)=0$. From the second equality we get $\alpha\beta+2=0$, with $\alpha=\sqrt{8}\cot\left(\sqrt{8}r\right)$ and $\beta=\sqrt{2}\cot\left(\sqrt{2}r\right)$ for some $r\in\left(0,\dfrac{\pi}{\sqrt{8}}\right)$. Then $\alpha\beta+2=2\cot^2\left(\sqrt{2}r\right)=0$, which is impossible and we can conclude that type (A) real hypersurfaces do not satisfy our condition.

In the case of a real hypersurface of type (B) let us suppose it satisfies our condition. From Proposition 2 in [1] it is easy to see that $g\left(\left(\nabla_{\xi}R_{\xi_1}\right)\xi_1,\phi\xi_1\right)=-4\alpha$ and $g\left(\left(\nabla_{\xi_1}R_{\xi_1}\right)\xi,\phi\xi_1\right)=-4\beta+\alpha\beta^2$. As both expressions must be equal, we obtain $\alpha=\dfrac{4\beta}{4+\beta^2}$, where now $\alpha=-2\tan(2r)$ and $\beta=2\cot(2r)$, for some $r\in\left(0,\dfrac{\pi}{4}\right)$. This yields $\tan^2\left(2r\right)=-2$, which is impossible and the proof concludes.

As a conclusion we have obtained that Jacobi operators corresponding to $D^{\perp}$-directions have the same behaviour as the normal Jacobi operator and structure Jacobi operator if we consider their covariant derivatives in the direction of any tangent vector field are null. In order to continue this research it is interesting to investigate what occurs if the covariant derivatives are taken in directions corresponding to the two distributions appearing on the real hypersurface, namely D and $D^{\perp}$. Also we can consider as a future work what happens if we deal with Lie derivatives of these Jacobi operators instead covariant derivatives.

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5. References

[1] J. Berndt and Y.-J. Suh, "Real Hypersurfaces in Complex Two-Plane Grassmannians," *Monatshefte für Mathematik*, Vol. 127, No. 1, 1999, pp. 1-14. doi:10.1007/s006050050018

[2] J. Berndt and Y.-J. Suh, "Real Hypersurfaces with Isometric Reeb Flow on Real Hypersurfaces in Complex Two-Plane Grassmannians," *Monatshefte für Mathematik*, Vol. 137, No. 2, 2002, pp. 87-98. doi:10.1007/s00605-001-0494-4

[3] I. Jeong, J. D. Pérez and Y.-J. Suh, "Real Hypersurfaces in Complex Two-Plane Grassmannians with Commuting Normal Jacobi Operator," *Acta Mathematica Hungarica*, Vol. 117, No. 3, 2007, pp. 201-217. doi:10.1007/s10474-007-6091-9

[4] I. Jeong, C. J. G. Machado, J. D. Pérez and Y.-J. Suh, "Real Hypersurfaces in Complex Two-Plane Grassmannians with $D^{\perp}$-Parallel Structure Jacobi Operator," *International Journal of Mathematics*, Vol. 22, No. 5, 2011, pp. 655-673.

[5] H.-J. Lee and Y.-J. Suh, "Real Hypersurfaces of Type B in Complex Two-Plane Grassmannians Related to the Reeb Vector," *Bulletin of the Korean Mathematical Society*, Vol. 47, No. 3, 2010, pp. 551-561. doi:10.4134/BKMS.2010.47.3.551

