

Liouville Type Theorems for Lichnerowicz Equations and Ginzburg-Landau Equation: Survey

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Abstract

In this survey paper, we firstly review some existence aspects of Lichnerowicz equation and Ginzburg-Landau equations. We then discuss the uniform bounds for both equations in R^n . In the last part of this report, we consider the Liouville type theorems for Lichnerowicz equation and Ginzburg-Landau equations in R^n via two approaches from the use of maximum principle and the monotonicity formula.

Keywords: Liouville Theorems, Ancient Solution, Ginzburg-Landau Equation, Lichenrowicz Equation

1. Introduction

This article is based on the lecture given at March 3rd, 2011 in the international conference “Recent Advances in Nonlinear Partial Differential Equations: Part I” held at the Chinese University of Hong Kong.

The initial-value problem of general relativity consists in the resolution of a coupled system of three linear equations and a quasilinear equation which determines the conformality factor, on an initial Riemannian manifold (M, g) . In the case where there are sources, the quasi-linear equation can be written as

$$8\Delta_g u - Ru + Vu^{-7} + Qu^{-3} + \tau u^5 = 0 \quad (1.1)$$

where $u > 0$ is the unknown on the Riemannian manifold (M, g) , R is the scalar curvature, V , Q , τ are functions derived from the Ricci curvature of (M, g) . (1.1) is called the Lichnerowicz equation on (M, g) . Let $2^* = 2n/(n-2)_+$ and S be the best Sobolev constant.

One interesting result derived from mountain pass lemma is below.

Theorem 1.1 Assume that (M^n, g) is compact, $n \geq 3$. Consider the following Lichnerowicz equation

$$-\Delta u + u = Bu^{2^*-1} + Au^{-2^*-1}$$

with $A > 0$ and $\max_M B > 0$. Assume that there is a positive function $\phi > 0$ and a constant $C(n) > 0$ such

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that

$$|\phi|_{H^1}^{2^*} \int_M A \phi^{-2^*} \leq \frac{C(n)}{(S \max_M |B|)^{n-1}}, \int_M B \phi^{2^*} > 0$$

Then there is a positive solution to Lichnerowicz equation.

Hebey-Pacard-Pollack [11] have applied the mountain pass lemma to the perturbation functionals to get positive approximation solutions and have proved the convergence of a subsequence to a positive solution.

The Ginzburg-Landau (GL) model is proposed in 50's in the context of super-conductivity theory and its energy density is

$$e(u) = \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (1 - |u|^2)^2.$$

Here $u: R^n \rightarrow R^k$. The stationary E-L equation for GL model is

$$\Delta u = u(|u|^2 - 1).$$

Yanyan Li and Z. C. Han (1995) have also studied p-laplacian GL models. Other models with term

$$(1 - |u|^2)^{2m}, \quad m \geq 1, \text{ is also of interest.}$$

The Lichnerowicz equation and the Ginzburg-Landau equations are important models in mathematical physics. The existence of solutions to both can be obtained via variational methods, the monotone method (also called sub-super solution method or barrier method), and per-

turbation methods.

There are huge literatures about G-L models [2,5, 10,14,15,20], but there is not much works about the Lichnerowicz equation [4,12,18]. For the existence results of the Lichnerowicz equation, one may see the works [7], [8], and [11]. For the existence results of the Ginzburg-Landau equation, one may find more references from the work [22].

Our main topic for both equations is about the Liouville type results, which are closely related to compactness theorems of the solution spaces. The heat flow method to both equations will be an interesting topic for studying.

2. Classical Liouville Type Theorems and Keller-Osserman Theory

The famous Liouville theory says that any non-negative harmonic function is constant. One can prove this by Harnack inequality or differential gradient Harnack estimate.

People may extend this to non-negative p-harmonic function or to non-negative solution to the elliptic equation

$$Lu = \sum_{i,j} a_{ij}u_{ij} = 0, \text{ in } R^n.$$

We now recall the famous Keller-Osserman theory obtained around 1957. Given a domain Ω . Consider the differential inequality

$$\Delta u \geq f(u), \text{ in } \Omega \subset R^n$$

where $f(t)$ is a positive, continuous, and monotone increasing function for $t \geq t_0$ satisfying the Osgood condition

$$\int_0^\infty \left(\int_0^t f(s) ds \right)^{-1/2} dt < \infty.$$

Then any twice continuously differentiable function u can not satisfy $\Delta u > 0$ on the whole space and $\Delta u \geq f(u)$ outside of some ball.

As an application of above theory, J. B. Keller and R. Osserman consider the non-existence result for the Gaussian curvature equation on the plane

$$-\Delta u = K(x)e^u, \text{ in } R^2.$$

Professor. Ni, Lin, W. Ding, W. Chen and C. Li, etc., have obtained a lot interesting existence results to this problem. More results and references may be found in my work [17].

It is also interesting to study the non-existence of non-trivial non-negative solutions or energy solutions to

$$-\Delta u = K(x)|u|^{p-1}u, \text{ in } R^n.$$

where $n \geq 2$ and $p > 1$.

Another application of the Keller and Osserman theory (H. Brezis, 1984) we have that for $0 \leq v \in C^2(R^n)$ satisfying

$$\Delta v \geq v^p, p > 1, \text{ in } R^n$$

we have $v = 0$.

In fact, H. Brezis getting upper bound of solutions by using the following boundary blow-up super-solution on the ball $B_R(p)$,

$$u_R(x) = \frac{CR^\beta}{(R^2 - |x-p|^2)^\beta}$$

with $\beta = (p-1)/2$ and a suitable constant C independent of $R > 0$. Then for any fixed point $p \in R^n$, sending $R \rightarrow \infty$, we get $v = 0$.

If the Laplacian is replaced by p-Laplacian, Du and Guo (2002) can extend the Keller and Osserman theorem to this case. For more, one may see the works of A. Farina, A. Ratto, M. Rigoli.

Parallel result to Keller and Osserman can be done for non-negative ancient solutions to the parabolic inequality:

$$(\Delta - \partial_t)v \geq v^p, p > 1, \text{ in } R^n \times (-\infty, 0).$$

See also the interesting work of J. Serrin [24] for more Liouville type theorems about elliptic and parabolic equations.

We have the following **Liouville type Theorem**.

Theorem 2.1 (L. Ma, 2010 [16]) *Let $u > 0$ in (1.1) with $M = R^n$, $Q = 0$, $V = 1$, $\tau = -1$. Then $u = 1$. For positive solutions to the general equation*

$$\Delta u = u^q - u^{-q-2},$$

on R^n with $q > 1$, we have the same result.

However, H. Brezis [3] proves that for $q \in (0,1]$, the same result is not true, but we always have $u \geq 1$.

Similar result is also true for M being a complete Riemannian manifold (M, g) with its Ricci curvature bounded from blow.

Here is the argument of Theorem 2.1. Let $f(s) = u^q - u^{-q-2}$ for $q > 0$. For any fixed $x \in R^n$ and $\varepsilon > 0$, consider the new function

$$u_\varepsilon(y) = u(y) + \varepsilon|y-x|^2.$$

Note that $u(y) \leq u_\varepsilon(y) \rightarrow \infty$ as $|y| \rightarrow \infty$. Then the minimum of it can be achieved at some point z . Then,

$$u_\varepsilon(z) \leq u_\varepsilon(x) = u(x),$$

which implies that $u(z) \leq u(x)$.

Using the monotonicity of f , i.e. $f' \geq 0$, we have

$$f(u(z)) \leq f(u(x))$$

At this point z , we have $(D^2u(z) + 2\varepsilon\delta) \geq 0$,
 $0 \leq \Delta u_\varepsilon(z) = \Delta u(z) + 2n\varepsilon = f(u(z)) + 2n\varepsilon$.

Then we have

$$0 \leq f(u(z)) + 2n\varepsilon \leq f(u(x)) + 2n\varepsilon \rightarrow f(u(x))$$

as $\varepsilon \rightarrow 0$.

Recall that

$$f(u) = u^q - u^{-p}$$

or some $q > 0$ and $p > 0$. Then we have $u \geq 1$.

Assume now that $q > 1$. $v = u - 1 \geq 0$. Then

$$\Delta v = f(v+1) \geq v^q.$$

Using the Keller-Osserman theory we then conclude that

$$v = 0, \text{ i.e. } u = 1.$$

This completes the argument of Theorem 2.1.

We now give some remark.

Set $f(u) = u^3 - u$ for $u \geq 0$, which is the special case of Ginzburg-Landau equation. We then conclude from the argument above that $u = 1$.

In general, we know the below.

Theorem 2.2 (Du, Ma [9], 2001) Assume that $u \in C^2(R^n, R)$ such that

$$\Delta u = u^3 - u$$

on R^n . Then we have $|u| \leq 1$.

In the papers of Du and Ma (2001-2003), more general logistic equations have been studied. It is there that we are interested in a problem related to Di Giorgi conjecture, which has been completely solved by Del Pino, J. Wei, etc., Savin, C. Gui, Ambrosio and Cabre, etc.

Interestingly, Du and Guo can obtain the below.

Theorem 2.3 (Du, Guo, 2002) Assume that $u \in C^2(R^n, R)$ such that

$$\Delta_p u = u^3 - u, 1 < p,$$

on R^n . Then we have $|u| \leq 1$.

There is a similar De Giorgi conjecture related to the equation above (see the work of L. Caffarelli, etc., A Gradient Bound for Entire Solutions of Quasi-Linear Equations and Its Consequences, *Communications on Pure and Applied Mathematics*, Volume 47, Issue 11, November 1994, Pages: 1457-1473).

It would be interesting to study the following evolution equation

$$u_t - \Delta_p u + u^3 - u = 0, \text{ in } \Omega \times (0, T)$$

with suitable initial and boundary conditions.

The results above can be extended to nonlinear heat equations.

Set $f(u) = u^q - u^{-p}$ for some $q > 0$ and $p > 0$.

Consider ancient solutions to the following parabolic equation

$$(\Delta - \partial_t)u = f(u), u > 0, \text{ in } R^n \times (-\infty, 0). \quad (2.1)$$

For any fixed $(x, \tau) \in R^n \times (-\infty, 0)$ and $\varepsilon > 0$, consider the new function

$$u_\varepsilon(y, t) = u(y) + \varepsilon|y - x|^2 - \varepsilon^2(t - \tau), \\ (y, t) \in R^n \times (-\infty, \tau]$$

Note that $u_\varepsilon(y, t) \rightarrow \infty$ as $|y| - t \rightarrow \infty$. Then the minimum of it can be achieved at some point (z, t) .

Then at this point $(\Delta - \partial_t)u_\varepsilon \geq 0$, which implies that

$$0 \leq (\Delta - \partial_t)u(z, t) + 2n\varepsilon + \varepsilon^2 = f(u(z, t)) + 2n\varepsilon + \varepsilon^2,$$

which is less than

$$f(u(x, \tau)) + 2n\varepsilon + \varepsilon^2 \rightarrow f(u(x, \tau))$$

as $\varepsilon \rightarrow 0$. Hence $u \geq 1$ (and one may also show that $u = 1$).

Theorem 2.4 Let $u > 0$ be an ancient solution to (2.1). Then $u \geq 1$.

3. Results for Ginzburg-Landau Equations

Our uniform bound result (due to H. Brezis) is

Theorem 3.1 Any smooth solution to GL model is bounded in the sense that $|u(x)| \leq 1$.

H. Brezis uses the Kato inequality to prove Theorem 3.1. We shall report here his argument. My argument is different but it is also based on the maximum principle.

Before I give the proof, let's recall the famous Kato inequality. Assume that $u \in L^1_{loc}$ and $\Delta u \in L^1_{loc}$. Firstly we may assume that u is smooth. Note that for $\varepsilon > 0$, we have

$$\sqrt{|u|^2 + \varepsilon} = \frac{\varepsilon}{(\sqrt{|u|^2 + \varepsilon})^3} |\nabla u|^2 + \frac{u}{\sqrt{|u|^2 + \varepsilon}} \Delta u.$$

The latter is bigger than $\frac{u}{\sqrt{|u|^2 + \varepsilon}} \Delta u$.

Hence we have for any $0 \leq \phi \in C^2_0$ we have

$$\int \sqrt{|u|^2 + \varepsilon} \Delta \phi \geq \int \frac{u \phi}{\sqrt{|u|^2 + \varepsilon}} \Delta u.$$

Sending $\varepsilon \rightarrow 0$, we then get

$$\int |u| \Delta \phi \geq \int \text{sign}(u) \phi \Delta u,$$

i.e.,

$$\Delta |u| \geq \text{sign}(u) \Delta u.$$

We have the parabolic version of the Kato inequality.

Consider $u = u(x, t)$ with in addition $u_t \in L^1_{loc}$. Using

$$\partial_t \sqrt{|u|^2 + \varepsilon} = \frac{uu_t}{\sqrt{|u|^2 + \varepsilon}},$$

we have

$$(\Delta - \partial_t) \sqrt{|u|^2 + \varepsilon} \geq \frac{u\phi}{\sqrt{|u|^2 + \varepsilon}} (\Delta u - u_t).$$

Then, letting $\varepsilon \rightarrow 0$, we have

$$(\Delta - \partial_t) |u| \geq \text{sign}(u) (\Delta u - u_t).$$

With the understanding above, we have that for any non-negative ancient solution to

$$\Delta v - v_t \geq v^p, \quad p > 1, \quad \text{in } R^n \times (-\infty, 0)$$

is trivial.

The argument of this fact is almost the same as Brezis's argument (1984).

Therefore, we have the following extension of H. Brezis' theorem.

Theorem 3.2 Any ancient solution $u : R^n \times (-\infty, 0) \rightarrow R^k$ to

$$\Delta u - u_t = u(|u|^2 - 1)$$

must have $|u| \leq 1$.

Going back to the parabolic version of Lichnerowicz equation on manifold with non-negative Ricci curvature, we have the following result.

Proposition 3.1 Any positive ancient solution to

$$\Delta u - u_t = u^q - u^{-p}$$

with $q > 1$ and $p > 0$ must be $u = 1$.

We now give the proof of Theorem 3.1.

Brezis's argument: Let $W = (|u|^2 - 1)_+$. Then we have

$$\Delta W \geq \Delta |u|^2 \text{sign}^+ (|u|^2 - 1).$$

Note that

$$\Delta |u|^2 = 2u\Delta u + 2|\nabla u|^2 \geq 2|u|^2 (|u|^2 - 1).$$

Then we have

$$\Delta W \geq 2|u|^2 (|u|^2 - 1) \text{sign}^+ (|u|^2 - 1) \geq 2W(W + 1) \geq 2W^2.$$

Using the Keller-Osserman theory we then have

$$W = 0, \text{ i.e. } |u|^2 \leq 1.$$

In my proof of this bound, I have used the original barrier functions used by Keller-Osserman.

It is interesting to know if one can extend the result above to p-Laplacian G-L model.

In the following, we present the monotonicity formula method to the Liouville type theorems [1,10,19].

It is our intention here to generalize the Liouville type results to a large class of solutions of a more general non-linear equations/systems

$$\Delta u - W'(u) = 0, \quad \text{in } R^n, \quad u \in C^2(R^n, R^k), \quad (3.1)$$

where W' is the gradient of the smooth function $W(u)$ on R^k and $W(u) \geq 0$.

We shall use an idea from Professor Hesheng Hu (1980) who introduced it for harmonic maps. Please see the book of Y. L. Xin [25] for results of harmonic maps.

It is also interesting to know if one can extend this kind of result to p-laplacian case.

Theorem 3.3 Assume that $W(u) \geq 0$ is a non-trivial smooth function. Let $u : R^n \rightarrow R^k$, $n \geq 2$, be a smooth solution to the Ginzburg-Landau system (3.1). Assume that there are a positive constant $R_0 > 0$ and a positive function $\Gamma(r)$ on $[R_0, \infty)$ such that

$$\lim_{R \rightarrow \infty} \int_{B_R - B_{R_0}} \Gamma(r) (|\nabla u|^2 + 2W(u)) < \infty.$$

and

$$\int_{R_0}^{\infty} \frac{\Gamma(r)}{r} dr = \infty.$$

Then u is a constant.

For any smooth mapping from the Riemannian manifold (M, g) to R^k , we define the Stress-Energy tensor (see the paper of Baird-Eells) by

$$S_u = \frac{1}{2} |\nabla u|^2 g - du \otimes du,$$

which is

$$S_{ij} = \frac{1}{2} u_j \cdot u_j g_{ik} - u_i \cdot u_k$$

in local frames (e_j) on M . By direct computation we know that

$$\text{div}(S_u) = -\langle \Delta_g u, du \rangle.$$

A consequence of this formula is that if u is harmonic, then $\text{div}(S_u) = 0$.

Let X be a smooth vector field on M . Define the tensor ∇X by

$$\nabla X(e_i, e_j) = g(\nabla_{e_i} X, e_j).$$

Then we have

$$\begin{aligned} \frac{1}{2} \text{div}(|\nabla u|^2 X) &= \text{div}(\langle du(X), u_j \rangle e_j) \\ &\quad - \langle du(X), \Delta u \rangle + \langle S_u, \nabla X \rangle \end{aligned}$$

Take any compact domain $D \subset M$ with smooth

boundary ∂D . Choose the local frame (e_j) such that $e_n = \nu$ be the unit outward normal to the boundary. Then we have

$$\begin{aligned} & \int_{\partial D} \frac{1}{2} |\nabla u|^2 \langle X, \nu \rangle - \langle du(X), u_\nu \rangle \\ &= \int_D \operatorname{div} S_u(X) + \langle S_u, \nabla X \rangle. \end{aligned} \tag{3.2}$$

Below we let $M = R^n$.

To explain the main idea of the proof, we start with the simple case when $W(u) = 0$ and $\Gamma(r) = 1$. That is, $\Delta u = 0$ and u has finite energy. We take in (3.2) $D = B_R(0)$, $X = r\partial_r$.

Note that $X_{ij} = \delta_{ij}$. Then we have

$$\langle S_u, \nabla X \rangle = S_{ij} X_{ij} = \operatorname{tr} S_u = \frac{n-2}{2} |\nabla u|^2,$$

$\langle X, \nu \rangle = R$ and $\langle du(X), u_\nu \rangle = R|u_r|^2$ on $\partial B_R(0)$. Hence

$$\int_{B_R(0)} \langle S_u, \nabla X \rangle = \int \frac{n-2}{2} |\nabla u|^2 \tag{3.3}$$

and by (3.2), we have

$$R \int_{\partial B_R} \frac{1}{2} |\nabla u|^2 - |u_r|^2 = \int_{B_R} \frac{n-2}{2} |\nabla u|^2.$$

Then we have the following Liouville theorem for harmonic functions.

Theorem 3.4 *Let $u : R^n \rightarrow R$ be a harmonic function with slowly energy divergence, i.e., there are a positive constant $R_0 > 0$ and a positive function $\Gamma(r)$ on $[R_0, \infty)$ such that*

$$\lim_{R \rightarrow \infty} \int_{B_R - B_{R_0}} \Gamma(r) |\nabla u|^2 < \infty.$$

and

$$\int_{R_0}^{\infty} \frac{\Gamma(r)}{r} dr = \infty.$$

Then u is a constant.

Here is the idea of proof:

Assume that u is not a constant. Then there are some positive constants $C > 0$ and $R_0 > 0$ such that

$$\int_{B_{R_0}(0)} \frac{n-2}{2} |\nabla u|^2 \geq C/2 > 0.$$

By this we know that

$$R \int_{\partial B_R} |\nabla u|^2 \geq C \tag{3.4}$$

for any $R \geq R_0$. Hence we have

$$\int_{B_R - B_{R_0}} |\nabla u|^2 \geq \int_{R_0}^R \int_{\partial B_r} |\nabla u|^2 \geq C \int_{R_0}^R \frac{dr}{r} \rightarrow \infty,$$

as $R \rightarrow \infty$, which gives a contradiction.

In fact, the proof of Theorem 3.4 goes below. Assume

u is not a constant. By (3.4) we know that

$$\begin{aligned} & \int_{B_R - B_{R_0}} \Gamma(r) |\nabla u|^2 \geq \int_{R_0}^R \Gamma(r) \int_{\partial B_r} |\nabla u|^2 \\ & \geq C \int_{R_0}^R \frac{\Gamma(r)}{r} dr \rightarrow \infty, \end{aligned}$$

as $R \rightarrow \infty$, which gives a contradiction. This completes the proof of Theorem 3.4.

We now turn to the **proof of Theorem 3.3**. Assume that $W(u)$ is non-trivial. In this case

$$\operatorname{div}(S_u) = -\langle \Delta u, du \rangle = -\langle W'(u), du \rangle$$

and

$$\operatorname{div} S_u(X) = -\langle W'(u), \nabla_X u \rangle = -\nabla_X W(u).$$

Again we take $D = B_R(0)$. Then by (3.2) we have

$$\begin{aligned} & \int_{\partial D} \frac{1}{2} |\nabla u|^2 \langle X, \nu \rangle - \langle du(X), u_\nu \rangle \\ &= \int_D (-\nabla_X W(u) + \langle S_u, \nabla X \rangle). \end{aligned} \tag{3.5}$$

Simplifying this identity we can derive the following

$$R \int_{\partial B_R} \left[\frac{1}{2} |\nabla u|^2 + W(u) \right] \geq C.$$

Then using the argument above we obtain Theorem 3.3.

We remark that the results above can be extended to complete Riemannian manifolds with bounded Ricci curvature.

We remark that for a large class of elliptic equations/systems (also for parabolic equation/system), the Liouville type theorems are equivalent to a local uniform bound of solutions. For this direction, one may see the recent works of P. Polacik, P. Souplet, P. Quittner, H. Zou, etc. However, it is open for Lichnerowicz equation on compact Riemannian manifolds (see also in [13]).

We also make a remark below. In the study of elliptic systems, one may use the Pohozaev type identity (which is a sister of monotonicity formula) and the interpolation inequalities to derive a contraction mapping property about L_p norm of the solution. From the contraction mapping property one then get the solution trivial (and the Liouville type theorem). One may see the works of Chen-Li [6], Souplet, etc. [21,23], for this kind of results for **the Lane-Emden conjecture**.

We would like to thank Professor H. Brezis, who (see also [3]) has informed me that in the statements of Theorems 1 and 2 in [16], the power $p > 1$ should be $p > 2$. Actually, we have used $p > 2$ in the proof of Theorem 1.1 in [16].

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