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Normality of Meromorphic Functions Family and Shared Set by One-way

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Abstract

We studied the normality criterion for families of meromorphic functions which related to One-way sharing set, and obtain two normal criterions, which improve the previous results.

Keywords: Meromorphic Function, Normality Criterion, Shared Values, Shared Set by One-way

1. Introduction

For Shared values, Schwick proved the following result [1]:

Theorem A *Let F be a family of meromorphic functions in the domain D,* a_1, a_2 *and* a_3 *be three finite complex numbers. If for every*

$$f \in F, \overline{E}_{f'}(a_i) = \overline{E}_f(a_i)(i = 1,2,3)$$

then F is normal in D.

In 2000, Pang Xue-cheng and Zalcman generalized the Schwick's result [2]:

Theorem B *Let F be meromorphic functions family in the domain D, and* a_1, a_2 *be two complex number. If for every*

$$f \in F, \overline{E}_{f'}(a_i) = \overline{E}_f(a_i)(i = 1,2)$$

then F is normal in D.

Definition *For* a, b *are two distinct complex values, we have set* $S = \{a, b\}$ *and*

$$\overline{E}_f(S) = \overline{E}_f(a,b) = \{z : (f(z) - a)(f(z) - b) = 0, z \in D\}$$

If $\overline{E}_f(S) = \overline{E}_g(S)$, *we call that f and g share S in D; If* $\overline{E}_f(S) \subseteq \overline{E}_g(S)$, *we call that f and g share S by One-way in D.*

For shared set, W. H. Zhang obtained important results [3]:

Theorem C *Let F be a family of meromorphic functions in the unit disc* Δ, *a and b be two distinct nonzero complex value,* $S = \{a, b\}$, *If for every* $f \in F$, *all of whose zeros is multiple,* $\overline{E}_{f'}(S) = \overline{E}_f(S)$, *then F is normal on* Δ.

W. H. Zhang continued considering the relation between normality and the shared set, and proved the next result [4]:

Theorem D *Let* F *be meromorphic functions family in the unit disk* Δ, *a and b be two distinct nonzero complex values. If for every* $f \in F$, *all of whose zeros is multiplicity* $k+1$ *at least* (*k is a positive integer*), $\overline{E}_{f^{(k)}}(S) = \overline{E}_f(S)$, *then F is normal in* Δ.

For shared set by One-way, Lv Feng-jiao got following theorem in [5]:

Theorem E *Let F be a family of meromorphic function in the unit disk* Δ, *a and b is two distinct nonzero complex values,* k *is positive integer,* $S = \{a, b\}$. *If for every* $f \in F$, *all of whose zeros have multiplicity* $k+1$ *at least,* $\overline{E}_{f^{(k)}}(S) \subseteq \overline{E}_f(S)$, *then F is normal in* Δ.

In 2007, Pang Xue-cheng proved the following important results in [6]:

Theorem F *Let F be meromorphic functions family in D,* $S = \{a_1, a_2, a_3\}$. *If for every* $f \in F$ $\overline{E}_{f'}(S) = \overline{E}_f(S)$, *then F is normal on D.*

To promote the results of Pang Xue-cheng, we continue to discuss about normality theorem of meromorphic functions families concerning shared set and shared set by one-way, and obtain our main results as follow.

Theorem 1 *Let* F *be meromorphic functions families in D,* $S = \{a_1, a_2, a_3\}, a_4 \neq a_i$ (*i* = 1, 2, 3).

If for every $f \in F$, $\overline{E}_{f'}(S) \subset \overline{E}_f(S)$, *and* $f' = a_4$ *whenever* $f = a_4$, *then F is normal on D.*

Theorem 2 *Let* F *be meromorphic functions families in D,* $S = \{a_1, a_2\}, a_3 \in C$. *If for every* $f \in F$, $\overline{E}_{f'}(S') = \overline{E}_f(S)$, *and* $f' = a_3$, *whenever* $f = a_3$, *then*

F is normal on D.

2. Lemmas

Lemma 1 [7] *Let* F *be meromorphic functions families in the unit disk* Δ*, all of whose zeros have multiplicity* k *at least, and* $A>0$*. If for every* $f\in F$, $|f'(z)|\le A$ *whenever* $f(z)=0$*. If* F *is not normal in* Δ*, then for every* $0\le\alpha\le 1$*, there exists*

1) *a positive number* $r, 0<r<1$,
2) *complex sequence* $z_n, |z_n|<r$,
3) *Functions sequence* $f_n\in F$,
4) *and positive sequence* $\rho_n\to 0^+$,

such that $g_n(\zeta)=\rho_n^{-\alpha}f_n(z_n+\rho_n\zeta)$ *converges locally and uniformly to a noncontant meromorphic function* $g(\zeta)$*, and* $g^{\#}(\zeta)\le g^{\#}(0)=kA+1$*. Where*

$$g^{\#}(\zeta)=\frac{|g'(\zeta)|}{1+|g(\zeta)|^2}.$$

Lemma 2 [8] *Let* f *be meromorphic function with finite order on the open plane C, and* a_1,a_2,a_3 *be three finite complex values. If* $f(z)$ *have only finite zero, and*

$$f(z)=0\Leftrightarrow f'(z)\in S=\{a_1,a_2,a_3\}$$

then f *is a rational function.*

3. Proof of Theorem 1

Suppose that F be not normal in Δ, then by Lemma 1 we have that there exists

$$f_n\in F, z_n\in\Delta \text{ and } \rho_n\to 0^+,$$

such that $g_n(\xi)=\rho_n^{-1}\{f_n(z_n+\rho_n\xi)-a_4\}\to g(\xi)$ converges locally and uniformly to a noncontant meromorphic function $g(\zeta)$. We claim that the following conclusions hold.

1^0 $g(\xi)=0\Rightarrow g'(\xi)=a_4$;

2^0 $g^{\#}(\xi)\le g^{\#}(0)=|a_4|+1$;

3^0 $g'(\xi)\notin S$;

It is not difficult to prove claims $1^0, 2^0$, in what follow, we complete the proof of the claim 3^0. Suppose that there exists $\xi_0\in C$ such that $g'(\xi_0)=a_i$. Obviously, $g'(\xi)\not\equiv a_i$, in fact, if $g(\xi)=a_i\xi+c_0$, it is a contradictions for 1^0. Thus from Hurwitz Theorem, we know that there exists a point sequence $\xi_n\to\xi_0$, such that $g_n'(\xi_n)-a_i=0$, for sufficiently large *n*, that is

$$f_n'(z_n+\rho_n\xi_n)=a_i.$$

Obviously, $g_n(\xi_n)=\rho_n^{-1}(a_i-a_4)\to\infty$, as $n\to\infty$. Thus $g(\xi_0)=\infty$, this is a contradiction. Hence, claim 3^0 holds.

From claim 3^0 we have that

$$g'(\xi)\ne a_i\,(i=1,2,3)$$

So $g'(\xi)$ is identical in nonconstant. Again because claim 1^0, we know $g(\xi)=a_4(\xi-\xi_0)$ and

$$g^{\#}(0)=\frac{|a_4|}{1+|a_4\xi_0|^2}=\begin{cases}|a_4| & |\xi_0|<1\\ \frac{1}{2} & |\xi_0|\ge 1\end{cases}.$$

Clearly, this is a contradictions for claim 2^0. Therefore, *F* is normal in *D*. The proof of Theorem 1 is completed.

4. Proof of Theorem 2

Suppose that F is not normal in Δ, by Lemma 1 there exists $f_n\in F, z_n\in\Delta$ and $\rho_n\to 0^+$, such that $g_n(\xi)=f_n(z_n+\rho_n\xi)\to g(\xi)$ converges locally and uniformly to a noncontant meromorphic function $g(\xi)$ with finite orders, there $g^{\#}(\xi)\le g^{\#}(0)$.

We asserts that $g(\xi)\in S\Rightarrow g'(\xi)=0$.

In fact, suppose that there exists $\xi_0\in C$, such that $g(\xi_0)\in S$, thus there exists $a_i\,(i=1,2)$ such that $g(\xi_0)=a_i$.

From Hurwitz Theorem and $g(\xi)\not\equiv a_i$, we have there exists $\xi_n\to\xi_0$ such that $g_n(\xi_n)=a_i$, that is $g_n(\xi_n)=f_n(z_n+\rho_n\xi)=a_i$ for sufficiently large *n*. Thus in contrast with conditions of Theorem, we get $f_n'(z_n+\rho_n\xi)\in S$. Obviously, $|f_n'(z_n+\rho_n\xi_n)|\le A$. So we get $g'(\xi_0)=0$.

Since $g(\xi)$ is a nonconstant entire function, without loss of generality, we assume that $g(\xi)-a_1$ have zero on *C* for a_1, and consider function sequence $G_n(\xi)$:

$$G_n(\xi)=\frac{g_n(\xi)-a_1}{\rho_n}=\frac{f_n(z_n+\rho_n\xi)-a_1}{\rho_n}$$

Obviously, $\{G_n\}$ is not normal in zero of $g(\xi)-a_1$. In fact, if ξ_0 is zero of $g(\xi)-a_1$, then $G_n(\xi_0)=0\Rightarrow f_n(z_n+\rho_n\xi_0)=0$. With conditions of Theorem, we get $|f_n'(z_n+\rho_n\xi_0)|\le A$ and $|G_n'(\xi_0)|\le A$. Therefore, $\{G_n\}$ is not normal in zero of $g(\xi)-a_1$. So there exists G_n, $\xi_n\in\Delta$ and $\eta_n\to 0^+$, such that

$$F_n(\zeta)=\eta_n{}^{-1}G_n(\xi_n+\eta_n\zeta)=\eta_n^{-1}\left[g_n(\xi_n+\eta_n\zeta)-a_1\right]\to F(\zeta)$$

converges locally and uniformly to a noncontant and meromorphic function $F(\zeta)$ with finite order, and

1^0 the number of zeros of $F(\zeta)$ is finite,

2^0 $F(\zeta)=0 \Rightarrow F'(\zeta)\in S\cup\{a_3\}$,

3^0 $F'(\zeta)\in S \Rightarrow F(\zeta)=0$,

4^0 $F(\zeta_0)=\infty \Rightarrow \left(1/F(\zeta)\right)\Big|_{\zeta=\zeta_0}=0$

In fact, suppose that ξ_0 is the zero of $g(\xi)-a_1$ with order *k*. If there exists $k+1$ distinct $\zeta_1,\zeta_2\cdots\zeta_{k+1}$ at least, such that $F(\zeta_j)=0, j=1,2,\cdots,k+1$.

By Hurwitz Theorem, it is certainly that there exist a positive integer N, such that $F_n(\zeta_{n_j})=0, j=1,2,\cdots,k+1$ as $n>N$. Thus,

$$g_n\left(\xi_n+\eta_n\zeta_{n_j}\right)-a_1=0.$$

Since $\xi_n+\eta_n\zeta_{n_j}\to\xi_0\ (n\to\infty)$, $j=1,2,\ldots,k+1$, we deduce that ξ_0 is a zero of $g(\xi)-a_1$ with $k+1$ orders, this is a contradictions for suppose. Therefore zeros numbers of $F(\zeta)$ is finite.

Suppose that ζ_0 is a zero of $F(\zeta_0)=0$. For $F(\zeta)\not\equiv 0$ and Hurwitz theorem, we know that there exists sequence $\zeta_n\to\zeta_0$, such that

$$F_n(\zeta_n)=\frac{f_n\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]-a_1}{\rho_n\eta_n}=0,$$

$$f_n\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]=a_1$$

Thus, we get $f_n'\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]\in S\cup\{a_3\}$ and subsequence $f_n\in F$ such that

$$f_n'\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]\to a_i,$$

thus $F'(\zeta_0)=\lim_{n\to\infty} f_n'\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]\in S\cup\{a_3\}$, for $a_i\in S\cup\{a_3\}$.

If there exists ζ_0 such that $F'(\zeta_0)\in S$, that is, there exists $a_i\in S$ such that $F'(\zeta_0)=a_i$. Since $F'(\zeta)\not\equiv a_i$, by Hurwitz theorem, there exists $\zeta_n\to\zeta_0$ such that

$$F_n'(\zeta_n)=f_n'\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]=a_i.$$

Hence, $f_n\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]\in S$, $F'(\zeta_0)\in S$.

If there exists N such that

$$f_n\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]\neq a_1 \quad \text{for} \quad n>N,$$

we get

$$F(\zeta_0)=\lim_{n\to\infty}\frac{f_n\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]-a_1}{\rho_n\eta_n}=\infty$$

This contradicts $F'(\zeta_0)=a_i$. Thus exists subsequence f_n, such that $f_n\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]=a_1$ for every n.

Therefore,

$$F(\zeta_0)=\lim_{n\to\infty}\frac{f_n\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]-a_1}{\rho_n\eta_n}=0$$

Now we prove that $F(\zeta_0)=\infty \Rightarrow \left(\frac{1}{F(\zeta)}\right)'\Bigg|_{\zeta=\zeta_0}=0$.

Since

$$\frac{1}{F_n(\zeta)}-\frac{\eta_n}{a_3-a_1}=\frac{\eta_n}{G_n\left(\xi_n+\eta_n\zeta\right)}-\frac{\eta_n}{a_3-a_1}$$

$$=\frac{\eta_n}{g_n(\xi_n+\eta_n\zeta)-a_1}-\frac{\eta_n}{a_3-a_1}\to 0$$

there exists $\zeta_n\to\zeta_0$, such that $\frac{1}{F_n(\zeta_n)}-\frac{\eta_n}{a_3-a_1}=0$, we get

$$f_n\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]=a_3,$$

thus

$$f_n'\left[z_n+\rho_n\left(\xi_n+\eta_n\zeta_n\right)\right]=a_3,$$

that is, $F_n'(\zeta_n)=a_3$. Therefore,

$$\left(\frac{1}{F(\zeta)}\right)'\Bigg|_{\zeta=\zeta_0}=-\frac{F'(\zeta)}{F^2(\zeta)}\Bigg|_{\zeta=\zeta_0}=\lim_{\zeta\to\zeta_0}\left[-\frac{F_n'(\zeta_n)}{F_n^2(\zeta_n)}\right]=0.$$

So far, we give complete proofs of all assertion. Next we will complete the proof of theorem 2 using assertion $1^0\sim 4^0$.

By Lemma 2 and assertion 2^0, we get that $F(\zeta)$ is a rational function. Again by assertion 4^0, it is clear that the pole of *F* be multiple. If G_n is not normal at ξ_0, thus ξ_0 be zero of $g(\xi)-a_1$. By the isolation of zero, we have that G_n are holomorphic functions at ξ_0 for sufficiently large *n*. We get that $F_n(\zeta)=\eta_n^{-1}G_n(\xi_n+\eta_n\zeta)$ are holomorphic functions in $|\zeta|<R$ for sufficiently large R, thus $F(\zeta)$ be nonconstant holomorphic functions in C. Therefore $F(\zeta)$ be a polynomial. Let it, s order is $p\,(p>0)$. Thus,

$$T(r,F')=(p-1)(\ln r),$$

$$N(r,F)=p(\ln r) \quad \text{and} \quad S(r,F')=O(1)$$

Therefore, $2(p-1)(\ln r)\le p(\ln r)+O(1), r\to\infty$. We get $0<p\le 2$ easily.

If $p=1$, thus $F(\zeta)=c_0\zeta+c_1\ (c_0\neq 0)$, by 2^0 and 3^0, we find that there exists a_i for every ζ, such that $F'(\zeta)=a_i$. Therefore ζ be an zero of $F(\zeta)$. But $F(\zeta)$ have only a zero, this is a contradiction.

If $p=2$, thus

$$F(\zeta)=c_0(\zeta-\zeta_0)(\zeta-\zeta_1)(c_0\neq 0,\zeta_0\neq\zeta_1)$$

As a result, $F'(\zeta)=c_0(2\zeta-\zeta_0-\zeta_1)$. Obviously zeros of $F'(\zeta)-a_i$ are $(a_i+c_0\zeta_0+c_0\zeta_1)/(2c_0)$. Hence we get that $F(\zeta)$ have three zeros, this still is a contradiction from

$$F(\zeta)=c_0(\zeta-\zeta_0)(\zeta-\zeta_1)(c_0\neq 0,\zeta_0\neq\zeta_1)$$

and the proof of theorem 2 is completed.

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