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New Types of Q-Integral Inequalities

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Abstract

Several new q-integral inequalities are presented. Some of them are new, One concerning double integrals, and others are generalizations of results of Miao and Qi [1]. A new key lemma is proved as well.

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1. Introduction

For $0 < q < 1$ the q-analog of the derivative, denoted by D_q is defined by (see [2])

$$D_q f(x) = \frac{f(x) - f(qx)}{x - qx}, \quad x \neq 0. \tag{1}$$

Whenever $f'(0)$ exists, $D_q f(0) = f'(0)$, and as $q \to 1^-$, the q-derivative reduces to the usual derivative.

The q-analog of integration from 0 to a is given by (see [3])

$$\int_0^a f(x) \mathrm{d}_q x = a(1-q) \sum_{k=0}^{\infty} f(aq^k) q^k, \tag{2}$$

provided the sum converges absolutely. On a general interval $[a,b]$ the q-integral is defined by (see [4])

$$\int_a^b f(x) \mathrm{d}_q x = \int_0^b f(x) \mathrm{d}_q x - \int_0^a f(x) \mathrm{d}_q x. \tag{3}$$

The q-Jackson integral and the q-derivative are related by the fundamental theorem of quantum calculus, which can be stated as follows (see [4, p. 73]) :

If F is an anti q-derivative of the function f, namely $D_q F = f$, continuous at $x = a$, then

$$\int_a^b f(x) \mathrm{d}_q x = F(b) - F(a). \tag{4}$$

For any function f, we have

$$D_q \int_a^x f(t) \mathrm{d}_q t = f(x) \tag{5}$$

For $b > 0$ and $a = bq^n$, $n \in N$, we denote

$$[a,b]_q = \left\{bq^k : 0 \leq k \leq n\right\} \text{ and } (a,b]_q = \left[aq^{-1}, b\right]_q. \tag{6}$$

It is not difficult to check the following

$$D_q(f(x)g(x)) = f(x) D_q g(x) + g(qx) D_q f(x) \tag{7}$$

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) D_q f(x) - f(x) D_q g(x)}{g(x) g(qx)} \tag{8}$$

In [5] the following result was proved

Theorem 1.1. *Let f be a function defined on* $[a,b]_q$ *satisfying*

$f(a) \geq 0$ and $D_q f(x) \geq (t-2)(x-a)^{t-3}$ for $x \in (a,b]_q$ and $t \geq 3$.

Then

$$\int_a^b f^t(x) \mathrm{d}_q x \geq \left(\int_a^b f(qx) \mathrm{d}_q x\right)^{t-1} \tag{9}$$

and in [1], the following results were proved

Theorem 1.2. *If* $f(x)$ *is a non-negative and increasing function on* $[a,b]_q$ *and satisfies*

$$(\alpha-1) f^{\alpha-2}(qx) D_q f(x) \geq \beta(\beta-1) f^{\beta-1}(x)(x-a)^{\beta-2} \tag{10}$$

for $\alpha \geq 1$ *and* $\beta \geq 1$ *then*

$$\int_a^b f^{\alpha}(x) \mathrm{d}_q x \geq \left(\int_a^b f(x) \mathrm{d}_q x\right)^{\beta}. \tag{11}$$

Theorem 1.3. *If* $f(x)$ *is a non-negative and increasing function on* $\left[bq^{n+m}, b\right]$ *and satisfies*

$$(\alpha-1) D_q f(x) \geq \beta(\beta-1) f^{\beta-\alpha+1}(q^m x)(x-a)^{\beta-2} \tag{12}$$

on $[a,b]_q$ *and for* $\alpha, \beta \geq 1$ *then*

$$\int_a^b f^{\alpha}(x)\mathrm{d}_q x \geq \left(\int_a^b f\left(q^m x\right)\mathrm{d}_q x\right)^{\beta}. \tag{13}$$

Theorem 1.4. *If* $f(x)$ *is a non-negative function on* $[0,b]_q$ *and satisfies*

$$\int_x^b f^{\beta}(t)\mathrm{d}_q t \geq \int_x^b t^{\beta}\,\mathrm{d}_q t \tag{14}$$

for $x \in [0,b]_q$ *and* $\beta > 0$ *then the inequality*

$$\int_x^b f^{\beta+\alpha}(t)\mathrm{d}_q t \geq \int_x^b t^{\alpha} f^{\beta}(t)\mathrm{d}_q t \tag{15}$$

holds for all positive numbers α *and* β.

Lemma 1.5[5]. *Let* $p \geq 1$ *and* $g(x)$ *be a nonnegative, monotonic function on* $[a,b]_q$*.Then*

$$pg^{p-1}(qx)D_q g(x) \leq D_q\left(g^p(x)\right) \leq pg^{p-1}(x)D_q g(x). \tag{16}$$

Remark 1. It may be mentioned that the function g should be non-decreasing, which is not stated. As well if g is non-increasing, (16) reverses. If g non-decreasing and p is such that $0 < p < 1$, then it is not difficult to show that (14) reverses.

2. Results

We start with the following key lemmas

Lemma 2.1. *Let* $\phi,\ f \geq 0,$ *and both non-decreasing functions,* φ *is differentiable, f defined on* $[a,b]_q$. Then

$$\varphi' o f(qx)D_q f(x) \leq D_q \phi o f(x) \leq \phi' o f(x)D_q f(x), \tag{17}$$

If f is non-increasing, (15) *reverses.*

Proof. We have

$$\phi o f(x) - \phi o f(qx) = \phi\left(f(x)\right) - \phi\left(f(qx)\right)$$
$$= \int_{f(qx)}^{f(x)} \phi'(t)\mathrm{d}t \leq \phi'\left(f(x)\right)\int_{f(qx)}^{f(x)} \mathrm{d}t$$
$$= \phi'\left(f(x)\right)\left(f(x) - f(qx)\right)$$

therefore

$$D_q \phi o f(x) = \frac{\phi o f(x) - \phi o f(qx)}{(1-q)x}$$
$$\leq \phi'\left(f(x)\right)\frac{f(x) - f(qx)}{(1-q)x} = \phi' o f(x)D_q f(x)$$

The rest is also similar.

Probably the following lemma is useful in some cases.

Lemma 2.2. *Let* $\phi,\ f \geq 0,$ *and both non-decreasing functions, f defined on* $[a,b]_q$ *Define*

$$D_q(\phi, f) = \frac{\phi o f(x) - \phi o f(qx)}{f(x) - f(qx)}. \tag{18}$$

Then

$$D_q(\phi, f)D_q f(x) \leq (\phi' o f)(x)D_q f(x) \tag{19}$$

Proof. We have,

$$D_q \phi\left(f(x)\right) = \frac{\phi\left(f(x)\right) - \phi\left(f(qx)\right)}{(1-q)x}$$
$$= \frac{\phi o f(x) - \phi o f(qx)}{f(x) - f(qx)}\frac{f(x) - f(qx)}{(1-q)x} = D_q(\phi, f)D_q f(x)$$

By (17),

$$D_q(\phi, f)D_q f(x) = D_q \phi o f(x) \leq (\phi' o f)(x)D_q f(x).$$

All the rest are similar.

Theorem 2.3. *Let* $\phi, \varphi, f, g \geq 0, \varphi, g$ *are both non-decreasing and defined on* $[a,b]_q$, $\varphi o g(a) = 0$. *If*

$$(\phi o f)(x) \geq (\varphi' o g)(x)D_q g(x), \tag{20}$$

then

$$\int_a^x (\phi o f)(t)\mathrm{d}_q t \geq (\varphi o g)(x). \tag{21}$$

If g is non-increasing and

$$(\phi o f)(x) \geq (\varphi' o g)(qx)D_q g(x), \tag{22}$$

satisfies, then (21) *reverses.*

Proof. Set

$$F(x) = \int_a^x (\phi o f)(t)\mathrm{d}_q t - (\varphi o g)(x).$$

We have, by lemma 2.1,

$$D_q F(x) = D_q\left(\int_a^x (\phi o f)(t)\mathrm{d}_q t\right) - D_q(\varphi o g)(x)$$
$$\geq (\phi o f)(x) - (\varphi' o g)(x)D_q g(x) \geq 0.$$

Therefore, F is non-decreasing, which implies $F(x) \geq F(a) = 0.$

The result follows.

Corollary 2.4. *Let* $f(x)$ *be a nonnegative and increasing function on* $[a,b]_q$ *such that* $f(a) = 0$. *Let* $\alpha > \gamma > 0, \alpha \geq 1,\ \beta \geq 2$. *If*

$$\begin{aligned}&(\alpha-\gamma)f^{\alpha-\gamma-1}(qx)D_q f(x)\\ &\geq \beta(\beta-1)f^{\gamma(\beta-1)}(x)(x-a)^{\beta-2},\end{aligned} \tag{23}$$

is satisfied, then

$$\int_a^b f^{\alpha}(x)\mathrm{d}_q x \geq \left(\int_a^b f^{\gamma}(x)\mathrm{d}_q x\right)^{\beta}. \tag{24}$$

Furthermore, if

$$\begin{aligned}&(\alpha-\gamma) f^{\alpha-\gamma-1}(qx) D_q f(x)\\ &\geq \beta(\beta-1) f^{\gamma(\beta-1)}(qx)(x-a)^{\beta-2}\end{aligned} \tag{25}$$

is satisfied, then

$$\int_a^b f^{\alpha}(x)\mathrm{d}_q x \geq \left(\int_a^b f^{\gamma}(qx)\mathrm{d}_q x\right)^{\beta} \tag{26}$$

Proof. For $x\in[a,b]_q$ let

$$\phi(x)=x^{\alpha},\ \varphi(x)=x^{\beta},\ g(x)=\int_a^x f^{\gamma}(x)\mathrm{d}_q x$$

then, we have, via lemma 1.5,

$$\begin{aligned}&(\phi o f)(x)-(\varphi' o g)(x)D_q g(x)\\ &=f^{\alpha}(x)-\beta\left(\int_a^x f^{\gamma}(t)d_q t\right)^{\beta-1} f^{\gamma}(x)\\ &=f^{\gamma}(x)\left(f^{\alpha-\gamma}(x)-\beta\left(\int_a^x f^{\gamma}(t)d_q t\right)^{\beta-1}\right)=f^{\gamma}(x)h(x).\end{aligned}$$

Now,

$$\begin{aligned}D_q h(x)&=D_q f^{\alpha-\gamma}(x)-\beta D_q\left(\int_a^x f^{\gamma}(t)d_q t\right)^{\beta-1}\\ &\geq(\alpha-\gamma) f^{\alpha-\gamma-1}(qx)D_q f(x)\\ &\quad-\beta(\beta-1)\left(\int_a^x f^{\gamma}(t)d_q t\right)^{\beta-2} f^{\gamma}(x)\\ &\geq(\alpha-\gamma) f^{\alpha-\gamma-1}(qx)D_q f(x)\\ &\quad-\beta(\beta-1) f^{\gamma(\beta-1)}(x)(x-a)^{\beta-2}\\ &\geq 0.\end{aligned}$$

Therefore, $h(x)$ is non-decreasing, but $h(a)=0$, then $h(x)\geq 0$. The result follows by theorem 2.3.

The proof of the second part is similar, and therefore, it is omitted.

Remark 2. Theorem 1.2 follows from corollary 2.4, the first part, by putting $\gamma=1$.

Theorem 2.5. *Let f,g are non-negative functions on $[a,b]_q$, either f or g is non-decreasing and they satisfies*

$$\int_x^b f^{\beta}(t)\mathrm{d}_q t\geq\int_x^b g^{\beta}(t)\mathrm{d}_q t, x\in[a,b]_q, \tag{27}$$

then the inequality

$$\int_x^b f^{\alpha+\beta}(t)\mathrm{d}_q t\geq\int_x^b f^{\alpha}(t)g^{\beta}(t)\mathrm{d}_q t \tag{28}$$

holds for all positive numbers α and β.

Proof. Suppose that *f* is non-decreasing. Using the fact

$$f(b)-f(a)=\int_a^b D_q f(x)\mathrm{d}_q x$$

we have

$$\begin{aligned}&\int_b^b f^{\alpha+\beta}(x)\mathrm{d}_q x=\int_a^b f^{\beta}(x)\left(\int_a^x D_q f^{\alpha}(t)d_q t+f(a)\right)\mathrm{d}_q x\\ &=\int_a^b D_q f^{\alpha}(t)\int_t^b f^{\beta}(x)d_q x d_q t+f(a)\int_a^b f^{\beta}(x)\mathrm{d}_q x\\ &\geq\int_a^b D_q f^{\alpha}(t)\int_t^b g^{\beta}(x)d_q x d_q t+f(a)\int_a^b g^{\beta}(x)\mathrm{d}_q x\\ &=\int_a^b g^{\beta}(x)\left(\int_a^x D_q f^{\alpha}(t)d_q t+f(a)\right)\mathrm{d}_q x\\ &=\int_a^b f^{\alpha}(x)g^{\beta}(x)\mathrm{d}_q x.\end{aligned}$$

Now, suppose *g* is non-decreasing, then, we have

$$\begin{aligned}&\int_b^b f^{\beta}(x)g^{\alpha}(x)\mathrm{d}_q x\\ &=\int_a^b f^{\beta}(x)\left(\int_a^x D_q g^{\alpha}(t)\mathrm{d}_q t+g(a)\right)\mathrm{d}_q x\\ &=\int_a^b D_q g^{\alpha}(t)\int_t^b f^{\beta}(x)\mathrm{d}_q x\mathrm{d}_q t+g(a)\int_a^b f^{\beta}(x)\mathrm{d}_q x\\ &\geq\int_a^b D_q g^{\alpha}(t)\int_t^b g^{\beta}(x)\mathrm{d}_q x\mathrm{d}_q t+g(a)\int_a^b g^{\beta}(x)\mathrm{d}_q x\\ &=\int_a^b g^{\beta}(x)\left(\int_a^x D_q g^{\alpha}(t)\mathrm{d}_q t+f(a)\right)\mathrm{d}_q x\\ &=\int_a^b g^{\alpha+\beta}(x)\mathrm{d}_q x.\end{aligned} \tag{29}$$

Using the arithmetic geometric inequality yields

$$\frac{\beta}{\alpha+\beta}f^{\alpha+\beta}(x)+\frac{\alpha}{\alpha+\beta}g^{\alpha+\beta}(x)\geq f^{\beta}(x)g^{\alpha}(x).$$

Integrating the above inequality and making use of (29) gives

$$\begin{aligned}&\frac{\beta}{\alpha+\beta}\int_a^b f^{\alpha+\beta}(x)\mathrm{d}x+\frac{\alpha}{\alpha+\beta}\int_a^b g^{\alpha+\beta}(x)\mathrm{d}x\\ &\geq\int_a^b f^{\beta}(x)g^{\alpha}(x)\mathrm{d}x\geq\int_a^b g^{\alpha+\beta}(x)\mathrm{d}x.\end{aligned}$$

The result follows.

Remark 3. Theorem 1.4 follows from theorem 2.5 by putting $a=0,\ g(x)=x.$

Corollary 2.6. *Let $f\geq 0$. If*

$$\frac{\sin f(x)}{f(x)} \geq \cos\left(\int_a^x f(t)\,\mathrm{d}_q t\right),\ x \in [a, \pi/2]_q \quad (30)$$

then

$$\int_a^x \sin(f(t))\,\mathrm{d}_q t \geq \sin\left(\int_a^x f(t)\,\mathrm{d}_q t\right) \quad (31)$$

Proof. The proof follows from theorem 2.3 by putting

$$\phi(x) = \varphi(x) = \sin x, g(x) = \int_a^x f(t)\,\mathrm{d}_q t,$$

as follows

$$(\phi o f)(x) - (\varphi' o g)(x) D_q g(x)$$
$$= \sin(f(x)) - \cos\left(\int_a^x f(t)\,d_q t\right) f(x) \geq 0.$$

The following result concerning similar inequality but for double integrals.

Theorem 2.7. *Let* $f \geq 0$ *non-decreasing in both* x *and* y, $f(a,y) = 0$, $\alpha > \beta\gamma$, $\beta \geq 2$, $\gamma > 0$. *If*

$$f^{\alpha-\beta\gamma-1}(qx,y) D_{q,x} f(x,y) \geq \frac{\beta(\beta-1)}{(\alpha-\beta\gamma)}(x-a)^{\beta-2}(y-a)^{\beta}, x, y \in [a,b]_q, \quad (32)$$

then

$$\int_a^x\int_y^b f^{\alpha}(u,v)\,\mathrm{d}_q u\,\mathrm{d}_q v \geq \left(\int_a^x\int_a^y f^{\gamma}(u,v)\,\mathrm{d}_q u\,\mathrm{d}_q v\right)^{\beta} \quad (33)$$

Proof. Set

$$F(x,y) = \int_a^x\int_y^b f^{\alpha}(u,v)\,\mathrm{d}_q u\,\mathrm{d}_q v - \left(\int_a^x\int_a^y f^{\gamma}(u,v)\,\mathrm{d}_q u\,\mathrm{d}_q v\right)^{\beta}.$$

We have via lemma 2.1 and by keeping y fixed,

$$D_{q,x}F(x,y) = D_{q,x}\int_a^x\int_y^b f^{\alpha}(u,v)\,\mathrm{d}_q u\,\mathrm{d}_q v - D_{q,x}\left(\int_a^x\int_a^y f^{\gamma}(u,v)\,\mathrm{d}_q u\,\mathrm{d}_q v\right)^{\beta}$$

$$D_{q,x}F(x,y) \geq \int_y^b f^{\alpha}(x,v)\,\mathrm{d}_q v - \beta\left(\int_a^x\int_a^y f^{\gamma}(u,v)\,\mathrm{d}_q u\,\mathrm{d}_q v\right)^{\beta-1}\int_a^y f^{\gamma}(x,v)\,\mathrm{d}_q v$$

$$\geq f^{\alpha}(x,y)(b-y) - \beta f^{\gamma(\beta-1)}(x,y)(x-a)^{\beta-1}(y-a)^{\beta-1} f^{\gamma}(x,y)(y-a)$$
$$= f^{\beta\gamma}(x,y)\left((b-y) f^{\alpha-\beta\gamma}(x,y) - \beta(x-a)^{\beta-1}(y-a)^{\beta}\right)$$
$$= f^{\beta\gamma}(x,y) k(x).$$

Now,

$$D_{q,x}k(x) \geq (\alpha-\beta\gamma)(b-y) f^{\alpha-\beta\gamma-1}(qx,y) D_q f(x,y) - \beta(\beta-1)(x-a)^{\beta-2}(y-a)^{\beta} \geq 0.$$

Therefore, k is non-decreasing, as $k(a) = 0$ then $k(x) \geq 0$ which implies $D_{q,x}F(x,y) \geq 0$. that is $F(x,y)$ is non-decreasing in x. But $F(a,y) = 0$, then $F(x,y) \geq 0$.

4. References

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