Advances in Pure Mathematics, 2011, 1, 63-66
doi:10.4236/apm.2011.13014 Published Online May 2011 (http://www.scirp.org/journal/apm)

Several Ideas on Some Integral Inequalities

Waadallah T. Sulaiman
Department of Computer Engineering, College of Engineering, University of Mosul, Mosul, Iraq
E-mail: *waadsulaiman@hotmail.com*
Received March 4, 2011; *revised March* 14, 2011; *accepted March* 20, 2011

Abstract

Several new integral inequalities are presented via new ideas.

Keywords: Integral Inequality

1. Introduction

In [1] the following result was proved

Theorem 1.1. *If* $f \geq 0$ *is continuous function on* [0,1] *such that*

$$\int_x^1 f(t)\,\mathrm{d}t \geq \int_x^1 t\,\mathrm{d}t,\quad \forall x \in [0,1] \tag{1}$$

then

$$\int_0^1 f^{\alpha+1}(x)\,\mathrm{d}x \geq \int_0^1 x^{\alpha} f(x)\,\mathrm{d}x,\quad \forall \alpha > 0 \tag{2}$$

and the following question was posed.

If f satisfies the above assumption, under what additional assumption can one claim that

$$\int_0^1 f^{\alpha+\beta}(x)\,\mathrm{d}x \geq \int_0^1 x^{\alpha} f^{\beta}(x)\,\mathrm{d}x,\ \forall \alpha,\beta > 0 \tag{3}$$

The following result as well, was achieved in [2]

Theorem 1.2 *If* $f \geq 0$ *is a continuous function on* [0,b] *satisfying*

$$\int_x^b f^{\alpha}(t)\,\mathrm{d}t \geq \int_x^b t^{\alpha}\,\mathrm{d}t,\quad b > 0,\quad \forall x \in [0,b] \tag{4}$$

then

$$\int_0^b f^{\alpha+\beta}(x)\,\mathrm{d}x \geq \int_0^b x^{\alpha} f^{\beta}(x)\,\mathrm{d}x,\quad \forall \beta > 0. \tag{5}$$

In their roles, Zabadan [3] and Hoang [4] generalized the previous results by introducing the following theorems respectively.

Theorem 1.3 *Suppose* $f, g \in L^1[a,b]$, $f, g \geq 0$, g *is non-decreasing. If*

$$\int_x^b f(t)\,\mathrm{d}t \geq \int_x^b g(t)\,\mathrm{d}t,\quad \forall x \in [a,b], \tag{6}$$

then

$$\int_a^b f^{\alpha+\beta}(x)\,\mathrm{d}x \geq \int_a^b f^{\alpha}(x) g^{\beta}(x)\,\mathrm{d}x,\ \forall \alpha,\beta \geq 0,\ \alpha+\beta \geq 1 \tag{7}$$

Theorem 1.4 *Suppose* $f \in L^1[a,b]$, $g \in C^1[a,b]$ $f, g \geq 0$, g *is non-decreasing. If*

$$\int_x^b f(t)\,\mathrm{d}t \geq \int_x^b g(t)\,\mathrm{d}t,\ \ \forall x \in [a,b],$$

then $\forall \alpha,\beta \geq 0$, $\alpha+\beta \geq 1$, *the following integral inequality holds*

$$\int_a^b f^{\alpha+\beta}(x)\,\mathrm{d}x \geq \int_a^b f^{\alpha}(x) g^{\beta}(x)\,\mathrm{d}x.$$

The object of this paper is to prove some of the above results via simpler methods as well as to present some other new results.

2. Results

We start with the following:

Lemma 2.1 *Let* $\varphi \geq 0$ *is non-decreasing on* $[a,b]$, *If*

$$\int_x^b \phi(t)\,\mathrm{d}t \geq 0,\quad \forall x \in [a,b], \tag{8}$$

then

$$\int_a^b \varphi(x)\phi(x)\,\mathrm{d}x \geq 0. \tag{9}$$

If (8) *reverses, then* (9) *reverses.*

Proof.

Copyright © 2011 SciRes.

$$\int_a^b \varphi'(x)\left(\int_x^b \phi(t)\,\mathrm{d}t\right)\mathrm{d}x$$
$$=\left[\varphi(x)\int_x^b \phi(t)\,\mathrm{d}t\right]_a^b - \int_a^b \varphi(x)\left(-\phi(x)\right)\mathrm{d}x$$
$$=-\varphi(a)\int_a^b \phi(t)\,\mathrm{d}t + \int_a^b \varphi(x)\phi(x)\,\mathrm{d}x.$$

Therefore

$$\int_a^b \varphi(x)\phi(x)\,\mathrm{d}x = \int_a^b \varphi'(x)\left(\int_x^b \phi(t)\,\mathrm{d}t\right)\mathrm{d}x + \varphi(a)\int_a^b \phi(t)\,\mathrm{d}t \geq 0,$$

being the sum of two non-negative terms.

Lemma 2.2. *Let* $\varphi \geq 0$ *is non-increasing, If*

$$\int_a^x \phi(t)\,\mathrm{d}t \geq 0, \ \forall x \in [a,b], \tag{10}$$

then

$$\int_a^b \varphi(x)\phi(x)\,\mathrm{d}x \geq 0. \tag{11}$$

If (10) *reverses*, *then* (11) *reverses.*

Proof.

$$\int_a^b \varphi'(x)\left(\int_a^x \phi(t)\,\mathrm{d}t\right)\mathrm{d}x = \left[\varphi(x)\int_a^x \phi(t)\,\mathrm{d}t\right]_a^b - \int_a^b \varphi(x)\phi(x)\,\mathrm{d}x$$
$$= \varphi(b)\int_a^b \phi(t)\,\mathrm{d}t - \int_a^b \varphi(x)\phi(x)\,\mathrm{d}x$$

Therefore

$$\int_a^b \varphi(x)\phi(x)\,\mathrm{d}x = -\int_a^b \varphi'(x)\left(\int_x^b \phi(t)\,\mathrm{d}t\right)\mathrm{d}x + \varphi(b)\int_a^b \phi(t)\,\mathrm{d}t \geq 0,$$

being the sum of two non-negative terms.

The following two results are similar to theorem 1.4 and proved via short simple methods:

Theorem 2.3 *Let* $f, g \geq 0$ *defined on* [a,b], f *is non-decreasing. If*

$$\int_x^b f^{\beta}(t)\,\mathrm{d}t \geq \int_x^b g^{\beta}(t)\,\mathrm{d}t, \ \forall x \in [a,b], \tag{12}$$

then

$$\int_a^b f^{\alpha+\beta}(x)\,\mathrm{d}x \geq \int_a^b f^{\alpha}(t)g^{\beta}(t)\,\mathrm{d}t, \alpha \geq 0, \tag{13}$$

Proof. The proof follows from Lemma 2.1 by putting

$$\varphi(x) = f^{\alpha}(x), \quad \phi(x) = f^{\beta}(x) - g^{\beta}(x).$$

Theorem 2.4 *Let* $f, g \geq 0$ *defined on* $[a,b]$, g *is non-decreasing. If* (12) *is satisfied with* β *replaced by* α *, then* (13) *is satisfied for* $\alpha, \beta > 0$.

Proof. By putting $\varphi(x) = g^{\beta}(x)$, $\phi(x) = f^{\alpha}(x) - g^{\alpha}(x)$ in Lemma 2.1, we get

$$\int_a^b f^{\alpha}(x)g^{\beta}(x)\,\mathrm{d}x \geq \int_a^b g^{\alpha+\beta}(x)\,\mathrm{d}x. \tag{14}$$

Now making use of the arithmetic-geometric inequality, we have for $\alpha, \beta > 0$.

$$\frac{\alpha}{\alpha+\beta} f^{\alpha+\beta}(x) + \frac{\beta}{\alpha+\beta} g^{\alpha+\beta}(x) \geq f^{\alpha}(x)g^{\beta}(x),$$

The above inequality, via (14) implies

$$\alpha\int_a^b f^{\alpha+\beta}(x)\,\mathrm{d}x$$
$$\geq (\alpha+\beta)\int_a^b f^{\alpha}(x)g^{\beta}(x)\,\mathrm{d}x - \beta\int_a^b g^{\alpha+\beta}(x)\,\mathrm{d}x$$
$$\geq (\alpha+\beta)\int_a^b f^{\alpha}(x)g^{\beta}(x)\,\mathrm{d}x - \beta\int_a^b f^{\alpha}(x)g^{\beta}(x)\,\mathrm{d}x$$
$$= \alpha\int_a^b f^{\alpha}(x)g^{\beta}(x)\,\mathrm{d}x.$$

The theorem follows.

The coming two results are new:

Theorem 2.5 *Let* $f, g \geq 0$ *defined on* [a,b], f *is non-decreasing. If*

$$\int_x^b f^{-\beta}(t)\,\mathrm{d}t \geq \int_x^b g^{-\beta}(t)\,\mathrm{d}t, \tag{15}$$

then

$$\int_a^b f^{\alpha-\beta}(x)\,\mathrm{d}x \geq \int_a^b f^{\alpha}(t)g^{-\beta}(t)\,\mathrm{d}t, \ \alpha, \beta \geq 0, \tag{16}$$

Proof. The proof follows from lemma 2.1 by putting

$$\varphi(x) = f^{\alpha}(x), \quad \phi(x) = f^{-\beta}(x) - g^{-\beta}(x).$$

Theorem 2.6 *Let* $f, g \geq 0$ *defined on* $[a,b]$, f *is non-increasing. If*

$$\int_a^x f^{\beta}(t)\,\mathrm{d}t \geq \int_a^x g^{\beta}(t)\,\mathrm{d}t, \tag{17}$$

then

$$\int_a^b f^{\alpha+\beta}(x)\,\mathrm{d}x \geq \int_a^b f^{\alpha}(t)\, g^{\beta}(t)\,\mathrm{d}t, \ \alpha \geq 0, \tag{18}$$

Proof. The proof follows from lemma 2.2 by putting

$$\varphi(x) = f^{\alpha}(x), \quad \phi(x) = f^{\beta}(x) - g^{\beta}(x).$$

Copyright © 2011 SciRes.

Theorem 2.7 [4] *Suppose* $f, g \geq 0$, g *is non-decreasing. If*

$$\int_x^b f(t)\,\mathrm{d}t \geq \int_x^b g(t)\,\mathrm{d}t \tag{19}$$

then $\forall \alpha \geq 1$, $\beta \geq 0$, *the following integral inequality holds*

$$\int_a^b f^{\alpha+\beta}(x)\,\mathrm{d}x \geq \int_a^b f^{\alpha}(x) g^{\beta}(x)\,\mathrm{d}x.$$

Proof. On putting $\varphi(x) = g^{\alpha-1}(x)$, $\phi(x) = f(x) - g(x)$ in lemma 2.1, we obtain

$$\int_a^b f(x) g^{\alpha-1}(x)\,\mathrm{d}x \geq \int_a^b g^{\alpha}(x)\,\mathrm{d}x. \tag{20}$$

By the AG inequality, we have, for $\alpha \geq 1$,

$$\frac{1}{\alpha} f^{\alpha}(x) + \frac{\alpha-1}{\alpha} g^{\alpha}(x) \geq f(x) g^{\alpha-1}(x).$$

The above inequality, via (19) implies for $\alpha \geq 1$,

$$\begin{aligned}
&\int_a^b f^{\alpha}(x)\,\mathrm{d}x - \int_a^b g^{\alpha}(x)\,\mathrm{d}x \\
&\geq \alpha\left(\int_a^b f(x) g^{\alpha-1}(x)\,\mathrm{d}x - \int_a^b g^{\alpha}(x)\,\mathrm{d}x\right) \qquad (21)\\
&\geq \alpha\left(\int_a^b g^{\alpha}(x)\,\mathrm{d}x - \int_a^b g^{\alpha}(x)\,\mathrm{d}x\right) = 0.
\end{aligned}$$

By the AG inequality again,

$$\frac{\alpha}{\alpha+\beta} f^{\alpha+\beta}(x) + \frac{\beta}{\alpha+\beta} g^{\alpha+\beta}(x) \geq f^{\alpha}(x) g^{\beta}(x),$$

which implies

$$\begin{aligned}
\int_a^b f^{\alpha+\beta}(x)\,\mathrm{d}x &= \frac{\alpha}{\alpha+\beta}\int_a^b f^{\alpha+\beta}(x)\,\mathrm{d}x + \frac{\beta}{\alpha+\beta}\int_a^b f^{\alpha+\beta}(x)\,\mathrm{d}x \\
&\geq \frac{\alpha}{\alpha+\beta}\int_a^b f^{\alpha+\beta}(x)\,\mathrm{d}x + \frac{\beta}{\alpha+\beta}\int_a^b g^{\alpha+\beta}(x)\,\mathrm{d}x \\
&\geq \int_a^b f^{\alpha}(x) g^{\beta}(x)\,\mathrm{d}x.
\end{aligned}$$

All of the following results are new:

Theorem 2.8 *Let* $f, g \geq 0$, f *is non-decreasing such that* (15) *is satisfied, then*

$$\int_a^b f^{-\beta}(x) g^{\alpha}(x)\,\mathrm{d}x \geq \int_a^b g^{\alpha-\beta}(x)\,\mathrm{d}x, \tag{22}$$

for all $\alpha, \beta \geq 0$.

Proof. As $\left(f^{\alpha}(x) - g^{\alpha}(x)\right)\left(f^{-\beta}(x) - g^{-\beta}(x)\right) \leq 0$, then

$$\begin{aligned}
&\int_a^b g^{\alpha}(x)\left(f^{-\beta}(x) - g^{-\beta}(x)\right)\mathrm{d}x \\
&\geq \int_a^b f^{\alpha}(x)\left(f^{-\beta}(x) - g^{-\beta}(x)\right)\mathrm{d}x.
\end{aligned}$$

But by Lemma 2.1 by putting $\varphi(x) = f^{\alpha}(x)$, $\phi(x) = f^{-\beta}(x) - g^{-\beta}(x)$, we obtain

$$\int_a^b f^{\alpha}(x)\left(f^{-\beta}(x) - g^{-\beta}(x)\right)\mathrm{d}x \geq 0,$$

then, the result follows:

Theorem 2.9 *Suppose* $f, g \geq 0, g$ *is non-decreasing. If* (19) *satisfied, then the following integral inequality holds*

$$\int_a^b f^{\alpha-\beta}(x)\,\mathrm{d}x \leq \int_a^b f^{\alpha}(x) g^{-\beta}(x)\,\mathrm{d}x,\ \alpha, \beta > 0, \alpha - \beta \geq 1. \tag{23}$$

Proof. By the AG inequality,

$$f^{\alpha}(x) g^{-\beta}(x) \geq \frac{\alpha}{\alpha-\beta} f^{\alpha-\beta}(x) - \frac{\beta}{\alpha-\beta} g^{\alpha-\beta}(x).$$

Integrating, we have, via (21),

$$\begin{aligned}
&\frac{\alpha}{\alpha-\beta}\int_a^b f^{\alpha-\beta}(x)\,\mathrm{d}x \\
&\leq \int_a^b f^{\alpha}(x) g^{-\beta}(x)\,\mathrm{d}x + \frac{\beta}{\alpha-\beta}\int_a^b g^{\alpha-\beta}(x) \\
&\leq \int_a^b f^{\alpha}(x) g^{-\beta}(x)\,\mathrm{d}x + \frac{\beta}{\alpha-\beta}\int_a^b f^{\alpha-\beta}(x)
\end{aligned}$$

which implies (23).

Theorem 2.10 *Let* $f, g \geq 0$, *and defined on* [a,b], f *is non-decreasing and g is non-increasing or conversely.* Then

$$\begin{aligned}
&\int_a^b f^{\alpha}(x) g^{\beta}(x)\,\mathrm{d}x \leq \frac{1}{b-a}\int_a^b f^{\alpha}(x)\,\mathrm{d}x \int_a^b g^{\beta}(x)\,\mathrm{d}x \\
&\leq \frac{1}{2(b-a)} \times \left(\left(\int_a^b f^{\alpha}(x)\,\mathrm{d}x\right)^2 + \left(\int_a^b g^{\beta}(x)\,\mathrm{d}x\right)^2\right)
\end{aligned} \tag{24}$$

Proof. Concerning the left inequality , we have

$$\left(f^{\alpha}(x) - f^{\alpha}(y)\right)\left(g^{\beta}(y) - g^{\beta}(x)\right) \geq 0,\ \forall \alpha, \beta > 0,$$

which implies

$$\int_a^b\int_a^b \left(f^{\alpha}(x) - f^{\alpha}(y)\right)\left(g^{\beta}(y) - g^{\beta}(x)\right)\mathrm{d}x\mathrm{d}y \geq 0,$$

and hence

Copyright © 2011 SciRes.

$$\int_a^b\int_a^b f^{\alpha}(x)g^{\beta}(y)\,\mathrm{d}x\mathrm{d}y+\int_a^b\int_a^b f^{\alpha}(y)g^{\beta}(x)\,\mathrm{d}x\mathrm{d}y$$
$$\geq \int_a^b\int_a^b f^{\alpha}(x)g^{\beta}(x)\,\mathrm{d}x\mathrm{d}y+\int_a^b\int_a^b f^{\alpha}(y)g^{\beta}(y)$$

and the above implies

$$\int_a^b\int_a^b f^{\alpha}(x)g^{\beta}(y)\,\mathrm{d}x\mathrm{d}y \geq \int_a^b\int_a^b f^{\alpha}(x)g^{\beta}(x)\,\mathrm{d}x\mathrm{d}y$$
$$=(b-a)\int_a^b f^{\alpha}(x)g^{\beta}(x)\,\mathrm{d}x$$

therefore

$$\int_a^b f^{\alpha}(x)g^{\beta}(x)\,\mathrm{d}x \leq \frac{1}{b-a}\int_a^b f^{\alpha}(x)\,\mathrm{d}x\int_a^b g^{\beta}(y)\,\mathrm{d}x$$

The right inequality follows by opening the inequality

$$\left(\int_a^b f^{\alpha}(x)\,\mathrm{d}x-\int_a^b g^{\beta}(x)\,\mathrm{d}x\right)^2 \geq 0.$$

Remark. It may be mentioned that many other new inequalities can be obtained via Lemmas 2.1 and 2.2.

3. References

[1] Q. A. Ngo, D. D. Thang, T. T. Dat and D. A. Tuan, "Notes on an Integral Inequality," *Journal of Inequalities in Pure and Applied Mathematics*, Vol. 7, No. 4, 2006, Art. 120.

[2] W. J. Liu, C. C. Li and J. W. Dong, "On an Problem Concerning an Integral Inequality," *Journal of Inequalities in Pure and Applied Mathematics*, Vol. 8, No. 3, 2007, Art. 74.

[3] G. Zabadan, "Notes on an Open Problem," *Journal of Inequalities in Pure and Applied Mathematics*, Vol. 9, No. 2, 2007, Art. 37

[4] N. S. Hoang, "Notes on an Inequality," *Journal of Inequalities in Pure and Applied Mathematics*, Vol. 9, No. 2, 2009, Art. 42.