Advances in Pure Mathematics, **2011, 1, 59-62**
doi:10.4236/apm.2011.13013 Published Online May 2011 (http://www.scirp.org/journal/apm)

An Arbitrary (Fractional) Orders Differential Equation with Internal Nonlocal and Integral Conditions

Ahmed El-Sayed[1], E. O. Bin-Taher[2]
[1]*Faculty of Science, Alexandria University, Alexandria, Egypt*
[2]*Faculty of Science, Hadhramout University of Science and Technology, Hadhramout, Yemen*
E-mail: *amasayed@hotmail.com, ebtsamsam@yahoo.com*
Received February 28, 2011; *revised March* 25, 2011; *accepted March* 30, 2011

Abstract

In this paper we study the existence of solution for the differential equation of arbitrary (fractional) orders $\frac{\mathrm{d}x}{\mathrm{d}t} = f\left(t, D^{\alpha}x\right), t \in (0,1)$, with the general form of internal nonlocal condition $\sum_{k=1}^{m} a_k\, x(\tau_k) = \beta \sum_{j=1}^{p} b_j\, x(\eta_j)$, $\tau_k \in (a,c) \subseteq (0,1), \eta_j \in (d,b) \subseteq (0,1), c \le d$. The problem with nonlocal integral condition will be studied.

Keywords: Internal Nonlocal Problem, Integral Condition, Fractional Calculus, Existence of Solution, Caratheodory Theorem

1. Introduction

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to ([1-10]), and references therein.

In this work we study the existence of at least one solution for the nonlocal problem of the arbitrary (fractional) order differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f\left(t, D^{\alpha}x(t)\right), \quad t \in (0,1) \text{ and } \alpha \in (0,1] \quad (1)$$

with the general nonlocal condition

$$\sum_{k=1}^{m} a_k\, x(\tau_k) = \beta \sum_{j=1}^{p} b_j\, x(\eta_j), \quad (2)$$

where $\tau_k \in (a,c) \subseteq (0,1), \eta_j \in (d,b) \subseteq (0,1), c \le d$ and $\beta \ge 0$ is parameter.

As an application, we deduce the existence of solution for the nonlocal problem of the differential (1) with the integral condition

$$\int_a^c x(s)\,\mathrm{d}s = \beta \int_d^b x(s)\,\mathrm{d}s. \quad (3)$$

It must be noticed that the following nonlocal and integral conditions are special cases of our nonlocal and integral conditions

$$x(\tau) = \beta\, x(\eta), \tau \in (a,c) \text{ and } \eta \in (d,b), \quad (4)$$

$$\sum_{k=1}^{m} a_k x(\tau_k) = \beta\, x(\eta), \quad \tau_k \in (a,c) \text{ and } \eta \in (d,b), \quad (5)$$

$$\sum_{k=1}^{m} a_k x(\tau_k) = 0, \tau_k \in (a,c), \quad (6)$$

$$\int_a^c x(s)\,\mathrm{d}s = \beta x(\eta), \ \eta \in (d,b), \quad (7)$$

and

$$\int_a^c x(s)\,\mathrm{d}s = 0, \ (a,c). \quad (8)$$

2. Preliminaries

Let $L^1(I)$ denotes the class of Lebesgue integrable functions on the interval $I = [a,b]$, with the norm $\|u\|_{L^1} = \int_I |u(t)|\,\mathrm{d}t$ and $C(I)$ denotes the class of continuous functions on the interval I, with the norm $\|u\| = \sup_{t\in I}|u(t)|$ and $\Gamma(.)$ denotes the gamma function.

Definition 2.1 The fractional-order integral of the function $f \in L^1[a,b]$ of order $\beta \in R^{+}$ is defined by (see [11])

$$I_a^{\beta} f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\,\mathrm{d}s.$$

Definition 2.2 The Caputo fractional-order derivative of order $\alpha \in (0,1]$ of the absolutely continuous function $f(t)$ is defined by (see [11] and [12])

$$D_a^{\alpha} f(t) = I_a^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} f(t).$$

Definition 2.3 The function $f:[0,1]\times R\to R$ is called L^1 – Caratheodory if

1) $t \to f(t,x)$ is measurable for each $x \in R$,

2) $x \to f(t,x)$ is continuous for almost all $t \in [0,1]$,

3) there exists $m \in L^1([0,1], D), D \subset R$ such that $|f| \le m$.

Now we state Caratheodory Theorem ([13]).

Theorem 2.1 Let $f[0,1]\times R \to R$ be L^1 – Caratheodory, then the initial-value problem

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t)), \quad \text{for a.e. } t > 0, \quad \text{and} \quad x(0) = x_o \quad (9)$$

has at least one absolutely continuous solution $x \in AC[0,T]$.

Here we generalize Caratheodory theorem for the nonlocal problem (1) - (2).

3. Main Results

Consider firstly the fractional-order integral equation

$$y(t) = I^{1-\alpha} f(t, y(t)), \quad (10)$$

Definition 3.1 The function y is called a solution of the fractional-order integral Equation (10), if $y \in C[0,1]$ and satisfies (10).

Theorem 3.1 Let $f:[0,1]\times R \to R$ be L^1 – Cara theodory. Then there exists at least one solution of the fractional-order integral Equation (10).

Proof. Let $M = \mathrm{Max}\{I_a^{\beta} m(t) : t \in (0,1), a \ge 0 \text{ and } \beta \in (0,1)\}$, then

$$\left|I_a^{\beta} f(t, y(t))\right| \le \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left|f(s, y(s))\right| \mathrm{d}s$$

$$\le \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} m(s)\,\mathrm{d}s \le M, \ a \ge 0.$$

Define the sequence $\{y_n(t)\}$ by

$$y_{n+1}(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y_n(s))\,\mathrm{d}s, \ t \in [0,1]$$

which can be written in the operator form

$$y_{n+1}(t) = I^{1-\alpha-\beta} I^{\beta} f((t), y_n(t)).$$

Then

$$|y_{n+1}(t)| \le I^{1-\alpha-\beta} \left|I^{\beta} f(t, y_n(t))\right| \le M \int_0^t \frac{(t-s)^{-\alpha-\beta}}{\Gamma(1-\alpha-\beta)} \mathrm{d}s$$

$$\le M \frac{(t)^{1-\alpha-\beta}}{\Gamma(2-\alpha-\beta)} \le \frac{M}{\Gamma(2-\alpha-\beta)}$$

For $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, then

$$y_{n+1}(t_2) - y_{n+1}(t_1) = \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y_n(s))\,\mathrm{d}s$$

$$-\int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y_n(s))\,\mathrm{d}s$$

$$= \int_0^{t_1} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y_n(s))\,\mathrm{d}s$$

$$+\int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y_n(s))\,\mathrm{d}s$$

$$-\int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y_n(s))\,\mathrm{d}s$$

$$\le \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y_n(s))\,\mathrm{d}s$$

$$+\int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y_n(s))\,\mathrm{d}s$$

$$-\int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, y_n(s))\,\mathrm{d}s.$$

Therefore

$$|y_{n+1}(t_2) - y_{n+1}(t_1)| \le \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} m(s)\,\mathrm{d}s$$

$$\le \int_{t_1}^{t_2} \frac{(t_2-\theta)^{-\alpha}}{\Gamma(1-\alpha)} m(\theta)\,\mathrm{d}\theta \le M \int_{t_1}^{t_2} \frac{(t_2-\theta)^{-\alpha-\beta}}{\Gamma(1-\alpha-\beta)} \mathrm{d}\theta$$

$$\le M \frac{(t_2-t_1)^{1-\alpha-\beta}}{\Gamma(2-\alpha-\beta)}.$$

Hence $|t_2 - t_1| < \delta \Rightarrow |y_{n+1}(t_2) - y_{n+1}(t_1)| < \varepsilon(\delta)$ and $\{y_n(t)\}$ is a sequence of equi-continuous and uniformly bounded functions. By Arzela-Ascoli Theorem, ([14] and [15]) there exists a subsequence $\{y_{n_k}(t)\}$ of continuous functions which converges uniformly to a continuous function y as $k \to \infty$.

Now we show that this limit function is the required solution.

Since

$$\left|f(s, y_{n_k}(s))\right| \le m(s) \in L^1,$$

and $f(s, y_{n_k}(s))$ is continuous in the second argument, *i.e.* $f\left(s, y_{n_k}(s)\right) \to f\left(s, y(s)\right)$ as $k \to \infty$,

therefore the sequence $\left\{(t-s)^{-\alpha} f\left(s, y_{n_k}(s)\right)\right\}$, $\alpha \in (0,1)$ satisfies Lebesgue dominated convergence theorem. Hence

$$\lim_{k\to\infty}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y_{n_k}(s)\right) \mathrm{d}s$$

$$= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f\left(s, y(s)\right) \mathrm{d}s = y(t),$$

which proves the existence of at least one solution $y \in C[0,1]$ of the fractional-order functional integral Equation (10).

For the existence of solution for the nonlocal problem (1) - (2) we have the following theorem.

Theorem 3.2 Let the assumptions of Theorem 3.1 are satisfied. Then nonlocal problem (1) - (2) has at least one solution $x \in AC[0,1]$.

Proof. Consider the nonlocal problem (1) - (2).

Let $y(t) = D^\alpha x(t)$, then

$$y(t) = I^{1-\alpha} \frac{\mathrm{d}x(t)}{\mathrm{d}t}, \tag{11}$$

$$y(t) = I^{1-\alpha} f\left(t, y(t)\right) \tag{12}$$

and y is the solution of the fractional-order integral Equation (10).

Operating by I^α on both sides of Equation(11), we obtain

$$I^\alpha y(t) = I \frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t) - x(0) \Rightarrow \tag{13}$$

$$x(t) = x(0) + I^\alpha y(t). \tag{14}$$

Let $t = \tau_k$ in Equation (13), we get

$$\sum_{k=1}^m a_k x(\tau_k) = \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s + x(0)\sum_{k=1}^m a_k.$$

And let $t = \eta_j$ in Equation (13), we get

$$\sum_{j=1}^p b_j x(\eta_j) = \sum_{j=1}^p bj \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s + x(0)\sum_{j=1}^p b_j.$$

From Equation (2), we get

$$\sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s + x(0)\sum_{k=1}^m a_k$$

$$= \beta \sum_{j=1}^p b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s + x(0)\beta\sum_{j=1}^p b_j.$$

Then we get

$$x(0) = A\left(\sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s - \beta\sum_{j=1}^p b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s\right)$$

and

$$x(t) = A\left(\sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s - \beta\sum_{j=1}^p b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s\right) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s \tag{15}$$

where

$$A = \left(\beta\sum_{j=1}^p b_j - \sum_{k=1}^m a_k\right)^{-1}$$

which, by Theorem 3.1, has at least one solution $x \in AC(0,1)$.

Now, from Equation (15), we have

$$x(0) = \lim_{t\to 0^+} x(t) = A\sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s - A\beta\sum_{j=1}^p b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s$$

and

$$x(1) = \lim_{t\to 1^-} x(t) = A\sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s - A\beta\sum_{j=1}^p b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s$$

from which we deduce that Equation (15) has at least one solution $x \in AC[0,1]$.

To complete the proof, differentiating (15), we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y(t) = f\left(t, D^\alpha x(t)\right).$$

Also from (15) we can prove that the solution satisfies the nonlocal condition (2).

4. Nonlocal Integral Condition

Let $x \in AC[0,1]$. be the solution of the nonlocal problem (1) - (2).

Let $a_k = t_k - t_{k-1}$, $\tau_k \in (t_{k-1}, t_k)$, $a = t_0 < t_1 < t_2, \cdots < t_m = c$

and $b_j = s_j - s_{j-1}, \eta_j \in (s_{j-1}, s_j), d = s_0 < s_1 < s_2, \cdots < s_p = b$ then the nonlocal condition (2) will be

$$\sum_{k=1}^{m} (t_k - t_{k-1}) x(\tau_k) = \beta \sum_{j=1}^{p} (s_j - s_{j-1}) x(\eta_j).$$

From the continuity of the solution x of the nonlocal problem (1) - (2) we can obtain

$$\lim_{m\to\infty} \sum_{k=1}^{m} (t_k - t_{k-1}) x(\tau_k) = \beta \lim_{p\to\infty} \sum_{j=1}^{p} (s_j - s_{j-1}) x(\eta_j).$$

and the nonlocal condition (2) transformed to the integral one

$$\int_a^c x(s)\,\mathrm{d}s = \beta \int_d^b x(s)\,\mathrm{d}s. \qquad (16)$$

Now, we have the following Theorem

Theorem 4.1 Let the assumptions of Theorem 3.2 are satisfied. Then there exist at least one solution $x \in AC[0,1]$. of the nonlocal problem with integral condition,

$$x'(t) = f\left(t, D^{\alpha} x(t)\right),\ t \in (0,1),$$

$$\int_a^c x(s)\,\mathrm{d}s = \beta \int_d^b y(s)\,\mathrm{d}s,\ \beta(b-d) \neq (c-a).$$

Letting $\beta = 0$ in (16), the we can easily prove the following corollary .

Theorem 4.2 Let the assumptions 1) - 2) are satisfied. Then the nonlocal problem

$$x'(t) = f\left(t, D^{\alpha} x(t)\right),\ t \in (0,1),$$

$$\int_a^c x(s)\,\mathrm{d}s = 0,\ (a,c) \subset (0,1)$$

has at least one solution $x \in AC[0,1]$.

5. References

[1] A. Boucherif, "First-Order Differential Inclusions with Nonlocal Initial Conditions," *Applied Mathematics Letters*, Vol. 15, No. 4, 2002, pp. 409-414. doi:10.1016/S0893-9659(01)00151-3

[2] A. Boucherif, "Nonlocal Cauchy Problems for First-Order Multivalued Differential Equations," *Electronic Journal of Differential Equations*, Vol. 2002, No. 47, 2002, pp. 1-9.

[3] A. Boucherif and R. Precup, "On the Nonlocal Initial Value Problem for First Order Differential Equations," *Fixed Point Theory*, Vol. 4, No. 2, 2003, pp. 205-212.

[4] A. Boucherif, "Semilinear Evolution Inclusions with Nonlocal Conditions," *Applied Mathematics Letters*, Vol. 22, No. 8, 2009, pp. 1145-1149. doi:10.1016/j.aml.2008.10.004

[5] M. Benchohra, E. P. Gatsori and S. K. Ntouyas, "Existence Results for Seme-Linear Integrodifferential Inclusions with Nonlocal Conditions," *Rocky Mountain Journal of Mathematics*, Vol. 34, No. 3, Fall 2004.

[6] M. Benchohra, S. Hamani and S. K. Ntouyas, "Boundary Value Problems for Differential Equations with Fractional Order and Nonlocal Conditions," *Nonlinear Analysis*: *Theory*, *Methods & Applications*, Vol. 71, No. 7-8, 2009, pp. 2391-2396. doi:10.1016/j.na.2009.01.073

[7] A. M. A. El-Sayed and Sh. A. Abd El-Salam, "On the Stability of a Fractional Order Differential Equation with Nonlocal Initial Condition," *Electronic Journal of Qualitative Theory of Differential Equations*, Vol. 2008, No. 29, 2008, pp. 1-8.

[8] A. M. A. El-Sayed and E. O. Bin-Taher, "A Nonlocal Problem of an Arbitrary (Fractional) Orders Differential Equation," *Alexandria Journal of Mathematics*, Vol. 1, No. 2, 2010, pp. 1-7.

[9] E. Gatsori, S. K. Ntouyas and Y. G. Sficas, "On a Nonlocal Cauchy Problem for Differential Inclusions," *Abstract and Applied Analysis*, Vol. 2004, No. 5, 2004, pp. 425-434.

[10] G. M. N'Guérékata, "A Cauchy Problem for Some Fractional Abstract Differential Equation with Non Local Conditions," *Nonlinear Analysis*: *Theory*, *Methods & Applications*, Vol. 70, No. 5, 2009, pp. 1873-1876. doi:10.1016/j.na.2008.02.087

[11] I. Podlubny, "Fractional Differential Equations," Academic Press, San Diego, New York and London, 1999.

[12] I. Podlubny and A. M. A. EL-Sayed, "On Two Definitions of Fractional Calculus," Preprint UEF 03-96, ISBN 80-7099-252-2, Institute of Experimental Physics, Slovak Academy of Science, 1996.

[13] R. F. Curtain and A. J. Pritchard, "Functional Analysis in Modern Applied Mathematics," Academic Press, London, 1977.

[14] K. Deimling, "Nonlinear Functional Analysis," Springer-Verlag, Berlin, 1985.

[15] J. Dugundji and A. Granas, "Fixed Point Theory," Monografie Mathematyczne, Polska Akademia Nauk, Warszawa, Vol. 1, 1982.

