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New Results on Oscillation of even Order Neutral Differential Equations with Deviating Arguments

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Abstract

In this paper, we point out some small mistakes in [6] and revise them, we obtain some new oscillation results for certain even order neutral differential equations with deviating arguments. Our results extend and improve many known oscillation criteria because the article just generalizes Meng and Xu's results.

Keywords: Oscillation, Neutral Differential Equation, Deviating Argument

1. Introduction

Oscillation of some even order differential equations have been studied by many authors. For instance, see [1-7] and the references therein. We deal with the oscillatory behavior of the even order neutral differential equations with deviating arguments of the form

$$\left[x(t)+\sum_{i=1}^{m}p_i(t)x(\tau_i(t))\right]^{(n)}+\sum_{j=1}^{l}q_j(t)f_j\left(x(\sigma_j(t))\right)=0,\quad t\geq t_0 \tag{1}$$

where $n\geq 2$ is even, throughout this paper, it is assumed that:

(A_1) $p_i,q_j\in C([t_0,\infty),R^+)$, $f_j\in C(R,R)$, $uf_j(u)>0$ for $u\neq 0$ and $f_j(u)$ is non-decreasing on R, $i=1,2,\cdots,m$, $j=1,2,\cdots,l$;

(A_2) $\tau_i\in C([t_0,\infty),R^+)$, $t_i(t)\leq t$ and $\lim_{t\to\infty}\tau_i(t)=\infty$, $i=1,2,\cdots,m$;

(A_3) $\sigma_j\in C^1([t,\infty),R^+)$, $\sigma_j(t)\leq t$, $\lim_{t\to\infty}\sigma_j(t)=\infty$ and $\sigma_j{}'(t)\geq 0$, $j=1,2,\cdots,l$;

(A_4) There exists a constant $\mathrm{M}>0$ such that $f_j(x)\operatorname{sgn}(x)\geq \mathrm{M}|x|$ for $x\neq 0, j=1,2,\cdots,l$;

(A_5) $\sum_{i=1}^{m}p_i(t)\leq p, p\in(0,1)$ and there exists a function $q(t)\in C([t_0,\infty),R^+)$ such that, $q(t)\leq\min\{q_j(t): j=1,2,\cdots,l\}$.

By a solution of Equation (1) we mean a function $x(t)$ which has the property that $x(t)+\sum_{i=1}^{m}p_i(t)x(\tau_i(t))\in C^n([t_x,\infty),R)$ for some $t_x\geq t_0$ and satisfies Equation (1) on $[t_x,\infty)$. We restrict our attention to those solutions $x(t)$ of Equation (1) which exist on some half-line $[t_x,\infty)$ with $\sup\{|x(t)|:t\geq T\}\neq 0$ for any $T\geq t_x$. A nontrivial solution of Equation (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory. Equation (1) is said to be oscillatory if all of it's nontrivial solutions are oscillatory.

Recently, Meng and Xu [6] studied Equation (1) and obtained some sufficient conditions for oscillation of the Equation (1), we list the main results of [6] as follows.

Following Philos [5], we say that a function $H=H(t,s)$ belongs to a function class W, denotes by $H\in W$, if $H\in C(D,R^+)$, where $D=\{(t,s):t\geq s\geq t_0\}$, which satisfies: ($H_1$) $H(t,t)=0$ and $H(t,s)>0$ for $t_0<s<t<\infty$; (H_2) H has a continuous non-positive partial derivative $\frac{\partial H}{\partial S}$ satisfying the condition:

$$\frac{\partial H(t,s)}{\partial S}=h(t,s)-H(t,s)\frac{k'(s)}{k(s)}$$

for some $h\in L_{loc}(D,R)$, $k\in C^1([t_0,\infty),(0,\infty))$ is a non-decreasing function.

Theorem A ([6, Theorem 2.1]).

Assume that (A_1) - (A_5) hold, let the functions H,h,k satisfy (H_1) and (H_2), suppose

$$\limsup_{t\to\infty}\left[\lambda C_1F(t,r)-\frac{1}{4\lambda C_2}G(t,r)\right]=\infty \tag{2}$$

holds for every $r \ge t_0, C_1 > 0, C_2 > 0$, where

$$F(t,r) = \frac{1}{H(t,r)}\int_r^t H(t,s)k(s)\sum_{j=1}^{l} q_j(s)\mathrm{d}s,$$

$$G(t,r) = \frac{1}{H(t,r)}\int_r^t \frac{k(s)h^2(t,s)}{H(t,s)\sigma_j^{n-2}(s)\sigma_j'(s)}\mathrm{d}s$$

and $\lambda = 1 - p$, then every solution of Equation (1) is oscillatory.

Theorem B ([6, Theorem 2.2]).

Assume that (A_1)-(A_5) hold, and H, h, k are the same as in Theorem A, suppose that

$$\inf_{s \ge t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\} > 0 \tag{3}$$

and

$$\limsup_{t\to\infty} G(t,t_0) < \infty \tag{4}$$

If there exists a function $m \in C([t_0,\infty), R)$ such that for all $t \ge T \ge t_0$.

$$\liminf_{t\to\infty}\left[\lambda C_1 F(t,T) - \frac{1}{4\lambda C_2}G(t,T)\right] \ge m(T) \tag{5}$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\sigma_j^{n-2}(s)\sigma_j'(s)m_+^2(s)}{k(s)}\mathrm{d}s = \infty,\ j = 1,2,\cdots,l \tag{6}$$

where $m_+(t) = \max\{m(t), 0\}$, then every solution of Equation (1) is oscillatory.

In Theorem A and B, function $G(t,r)$ should be $G_j(t,r)$, so each of the condition (2), (4), (5) and (6) has as many as l conditions. Meanwhile, the Riccati function $\omega(t)$ is not well-defined and there exist some small errors in the proof of the theorems. The purpose of this paper is further to strengthen oscillation results obtained for Equation (1) by Meng and Xu [6]. In our paper, we redefine the functions $F(t,r), G(t,r), \omega(t)$ and provide some new oscillation criteria for oscillation of Equation (1).

2. Main Results

In the sequel, we need the following lemmas:

Lemma 2.1 ([1]).

Let $x(t)$ be a n times differentiable function on $[t_0,\infty)$ of one sign, $x^{(n)}(t) \ne 0$ on $[t_0,\infty)$ which satisfies $x^{(n)}(t)x(t) \le 0$. Then:

(I_1) There exists a $t_1 \ge t_0$ such that $x^{(i)}(t), i = 1,2,\cdots,n-1$ are of one sign on $[t_1,\infty)$;

(I_2) There exists a number $h \in \{1,3,5,\cdots,n-1\}$ when n is even, or $h \in \{2,4,6,\cdots,n-1\}$ when n is odd, such that $x(t)x^{(i)}(t) > 0$ for $i = 0,1,\cdots,h$, $t \ge t_1$; $(-1)^{n+i+1}x(t)x^{(i)}(t) > 0$ for $i = h+1, h+2,\cdots,n$, $t \ge t_1$.

Lemma 2.2 ([1]).

If $x(t)$ is as in Lemma 2.1 and $x^{(n-1)}(t)x^{(n)}(t) \le 0$ for $t \ge t_0$, then for every $\lambda (0 < \lambda < 1)$, there exists a constant $N > 0$, such that $\left|x(\lambda(t))\right| \ge N t^{n-1}\left|X^{(n-1)}(t)\right|$ for all large t.

Lemma 2.3([7]).

Suppose that $x(t)$ is an eventually positive solution of Equation (1), let $z(t) = x(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t))$, then there exists a number $t_1 \ge t_0$ such that $z(t) > 0$, $z'(t) > 0, z^{(n-1)}(t) > 0$ and $z^{(n)}(t) \le 0, t \ge t_1$.

Theorem 2.1

Assume that (A_1) - (A_5) hold, let the functions H, h, k satisfy (H_1) and (H_2), suppose

$$\limsup_{t\to\infty}\left[\lambda M F(t,r) - \frac{\beta}{4\lambda N}G(t,r)\right] = \infty \tag{7}$$

holds for every $r \ge t_0$ and for some $\beta \ge 1$, where

$$F(t,r) = \frac{1}{H(t,r)}\int_r^t H(t,s)k(s)q(s)\mathrm{d}s$$

$$G(t,r) = \frac{1}{H(t,r)}\int_r^t \frac{k(s)h^2(t,s)}{H(t,s)\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}\mathrm{d}s$$

and $\lambda = 1 - p$, then every solution of Equation (1) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of Equation (1) and that $x(t)$ is even- tually positive (when $x(t)$ is eventually negative, the proof is similar).

Let $z(t)$ be defined as in Lemma 2.3, then following the proof of Theorem 2.1 in [6], without loss of generality, assume there exists a $t_1 \ge t_0$ such that

$$x(t) > 0, z(t) > 0, z'(t) > 0,$$
$$z^{(n-1)}(t) > 0, z(\sigma_j(t)) > z(\lambda\sigma_j(t)) > 0$$

$z'(\lambda\sigma_j(t)) \ge N\sigma_j^{n-2}(t)z^{(n-1)}(t)$ (by lemma 2.2) and $z^{(n)}(t) \le -\lambda M\sum_{j=1}^{l} q_j(t)z(\sigma_j(t))$ for all $t \ge t_1$.

Let

$$\omega(t) = k(t)\frac{z^{(n-1)}(t)}{\sum_{j=1}^{l} z(\lambda\sigma_j(t))} \quad \text{(not as [6])},$$

then we have

$$\omega'(t) \le -\lambda M k(t) q(t) + \frac{k'(t)}{k(t)}\omega(t) - \frac{\lambda N \sum_{j=1}^{l} \sigma_j^{n-2}(t)\sigma_j'(t)}{k(t)\omega^2(t)}, \quad t \ge t_1, \quad \text{(not as [6])}.$$

Multiplying the above equation, with t replaced by s, by $H(t,s)$ and integrating it from T to t, for all $t \ge T \ge t_1$, for some $\beta \ge 1$, we obtain

$$\lambda M \int_T^t H(t,s)k(s)q(s)\mathrm{d}s \le H(t,T)\omega(T) + \int_T^t h(t,s)\omega(s)\mathrm{d}s - \int_T^t \lambda N \frac{\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}{k(s)} H(t,s)\omega^2(s)\mathrm{d}s$$

$$= H(t,T)\omega(T) + \frac{\beta}{4}\int_T^t \frac{k(s)h^2(t,s)}{\lambda N H(t,s)\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}\mathrm{d}s - \frac{\beta}{\beta-1}\int_T^t \frac{\lambda N H(t,s)\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}{k(s)}\omega^2(s)\mathrm{d}s$$

$$-\int_T^t \left\{ \frac{\sqrt{\lambda N H(t,s)\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}}{\sqrt{\beta k(s)}}\omega(s) - \frac{\sqrt{\beta k(s)}}{\sqrt{4\lambda N H(t,s)\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}}h(t,s) \right\}^2 \mathrm{d}s$$

$$\le H(t,T)\omega(T) + \frac{\beta}{4}\int_T^t \frac{k(s)h^2(t,s)}{\lambda N H(t,s)\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}\mathrm{d}s$$

Hence, we have

$$\lambda M F(t,T) - \frac{\beta}{4\lambda N}G(t,T) \le \omega(T)$$

for all $t \ge T \ge t_1$, this gives

$$\limsup_{t\to\infty}\left[\lambda M F(t,r) - \frac{\beta}{4\lambda N}G(t,r)\right] < \infty$$

which contradicts (7). This completes the proof of the Theorem.

The assumption (7) in Theorem 2.1 can fail, consequently, Theorem 2.1 does not apply. The following results provide some essentially new oscillation criteria for Equation (1).

Theorem 2.2

Assume that (A_1)-(A_5) hold, the functions H, h, k, F and G be the same as in Theorem 2.1, suppose that

$$\inf_{s \ge t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\} > 0 \quad (8)$$

If there exists a function $m \in C([t_0,\infty), R)$ such that for all $t \ge T \ge t_0$ and for some $\beta > 1$,

$$\limsup_{t\to\infty}\left[\lambda M F(t,T) - \frac{\beta}{4\lambda N}G(t,T)\right] \ge m(T) \quad (9)$$

and

$$\limsup_{t\to\infty}\int_{t_0}^t \frac{\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)m_+^2(s)}{k(s)}\mathrm{d}s = \infty \quad (10)$$

where $m_+(t) = \max\{m(t), 0\}$. Then every solution of Equation (1) is oscillatory.

Proof. Assume to the contrary that (1) is non-oscillatory. Following the proof of Theorem 2.1, without loss of generality, assume for all $t \ge T \ge t_0$ and for some $\beta > 1$, we obtain

$$\lambda M \int_T^t H(t,s)k(s)q(s)\mathrm{d}s \le H(t,T)\omega(T)$$

$$+\frac{\beta}{4}\int_T^t \frac{k(s)h^2(t,s)}{\lambda N H(t,s)\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}\mathrm{d}s$$

$$-\frac{\beta}{\beta-1}\int_T^t \frac{\lambda N H(t,s)\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}{k(s)}\omega^2(s)\mathrm{d}s$$

So, we get

$$\lambda M F(t,T) - \frac{\beta}{4\lambda N}G(t,T) \le \omega(T) - \frac{\beta-1}{\beta}\lambda N B(t,T)$$

where

$$B(t,r) = \frac{1}{H(t,r)}\int_r^t \frac{H(t,s)\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}{k(s)}\omega^2(s)\mathrm{d}s, \quad r \ge t_0$$

then

$$\limsup_{t\to\infty}\left[\lambda MF(t,T)-\frac{\beta}{4\lambda N}G(t,T)\right]$$
$$\le \omega(T)-\frac{\beta-1}{\beta}\lambda N\liminf_{t\to\infty}B(t,T)$$

For all $T\ge t_0$ and for any $\beta>1$, by (9) we have

$$\omega(T)\ge m(T)+\frac{\beta-1}{\beta}\lambda N\liminf_{t\to\infty}B(t,T)$$

So

$$\omega(T)\ge m(T) \qquad (11)$$

and especially

$$\liminf_{t\to\infty}B(t,t_0)\le\frac{\beta}{(\beta-1)\lambda N}\left[\omega(t_0)-m(t_0)\right]<\infty \qquad (12)$$

Now, we claim that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}{k(s)}\omega^2(s)\mathrm{d}s<\infty \qquad (13)$$

Suppose to the contrary that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}{k(s)}\omega^2(s)\mathrm{d}s=\infty \qquad (14)$$

By (8), there is a positive constant ξ satisfying

$$\inf_{s\ge t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\}\ge\xi>0 \qquad (15)$$

Let η be any arbitrary positive number, from (14) there exists a $t_1>t_0$ such that,

$$\int_{t_0}^{t}\frac{\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}{k(s)}\omega^2(s)\mathrm{d}s\ge\frac{\eta}{\xi}\quad \text{for all } t>t_1$$

Then, for $t>t_1$, we have

$$B(t,t_0)$$
$$=\frac{1}{H(t,t_0)}\int_{t_1}^{t}H(t,s)d\left\{\int_{t_0}^{s}\frac{\sum_{j=1}^{l}\sigma_j^{n-2}(v)\sigma_j'(v)}{k(v)}\omega^2(v)\mathrm{d}v\right\}$$
$$\ge\frac{1}{H(t,t_0)}\int_{t_1}^{t}H(t,s)\left\{\int_{t_0}^{s}\frac{\sum_{j=1}^{l}\sigma_j^{n-2}(v)\sigma_j'(v)}{k(v)}\omega^2(v)\mathrm{d}v\right\}\mathrm{d}s$$
$$\ge\frac{\eta}{\xi}\frac{H(t,t_1)}{H(t,t_0)}$$

By (15), there exists a $t_2\ge t_1$ such that, for all $t>t_2$, $\frac{H(t,t_1)}{H(t,t_0)}\ge\xi$, which implies $B(t,t_0)\ge\eta$ for all $t>t_2$. Since η is arbitrary, we have

$$\liminf_{t\to\infty}B(t,t_0)=\lim_{t\to\infty}B(t,t_0)=\infty$$

which contradicts (12), thus (13) holds. Then by (11) and (13) we get

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}{k(s)}m_+^2(s)\mathrm{d}s$$
$$\le\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}{k(s)}\omega^2(s)\mathrm{d}s<\infty,$$

which contradicts (10). This completes the proof.

Remark 1 Let $\beta=1$ in Theorem 2.1, Theorem 2.1 reduces to Theorem A [6]; we obtain the same result in Theorem 2.2 in which we omit the assumption (4) in Theorem B [6]. Therefore, Theorem 2.1 and 2.2 are generalizations and improvements of the results obtained in [6].

Remark 2 With an appropriate choices of the functions H,h and k, one can derive a number of oscillation criteria for Equation (1) from our theorems.

Let $k(t)\equiv1,\alpha>0$ is a constant, $H(t,s)=(t-s)^{\alpha}$, $h(t,s)=-\alpha(t,s)^{\alpha-1}$ $t\ge s\ge t_0$, and we have

$$\lim_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}=\lim_{t\to\infty}\frac{(t,s)^{\alpha}}{(t,t_0)^{\alpha}}=1\quad\text{for any } s\ge t_0.$$

Consequently, let $\alpha=2$, using Theorem 2.2, we have:

Corollary 2.1 Assume that (A_1)-(A_5) and (8) hold, suppose that there exists a function $m\in C([t_0,\infty),R)$ such that, for some $\beta>1$,

$$\limsup_{t\to\infty}\frac{1}{t^2}\int_{T}^{t}\left[\lambda M(t-s)^2q(s)-\frac{\beta}{\lambda N\sum_{j=1}^{l}\sigma_j^{n-2}(s)\sigma_j'(s)}\right]\mathrm{d}s$$
$$\ge m(T),\quad t\ge T\ge t_0 \qquad (16)$$

and (10) (with $k(s)\equiv1$) hold. Then every solution of Equation (1) is oscillatory.

Example 1 Let $t\in[4,\infty)$, consider the following second order neutral differential equation

$$\left[x(t)+p(t)x(\tau(t))\right]''+q(t)x(\sigma(t))=0 \qquad (17)$$

where $p(t) \equiv \frac{1}{2}$, $q(t) = \max\{2(1+t)\sin t, 0\}$, $f(x) = x$, $\sigma(t) = \int_4^t \frac{4}{(1+s)(2+\sin s)} \mathrm{d}s$, in this case $M = 1$, Let $N = 1, \beta = 2$, by direct calculation, we get

$$\limsup_{t\to\infty} \frac{1}{t^2}\int_T^t \left[\lambda M (t-s)^2 q(s) - \frac{\beta}{\lambda N \sigma'(s)}\right] \mathrm{d}s$$

$$\geq \limsup_{t\to\infty} \frac{1}{t^2}\int_T^t \left[(t-s)^2 (1+s)\sin s - (1+s)(2+\sin s)\right] \mathrm{d}s$$

$$\overset{\Delta}{=} m(T) = \cos T - \sin T + T\cos T - 1$$

It is easy to verify that (10) holds, therefore, Equation (17) is oscillatory by Corollary 2.1. However, we can easily find that

$$\limsup_{t\to\infty} G(t,t_0) = \limsup_{t\to\infty} \frac{1}{t^2}\int_{t_0}^t (1+s)(2+\sin s)\mathrm{d}s = \infty$$

so condition (4) in Theorem B is not satisfied, these show that Theorem B cannot be applied to Equation (17). Obviously our results are superior to the results obtained before.

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