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A Note on Convergence of a Sequence and its Applications to Geometry of Banach Spaces

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Abstract

The purpose of this note is to point out several obscure places in the results of Ahmed and Zeyada [J. Math. Anal. Appl. 274 (2002) 458-465]. In order to rectify and improve the results of Ahmed and Zeyada, we introduce the concepts of locally quasi-nonexpansive, biased quasi-nonexpansive and conditionally biased quasi-nonexpansive of a mapping w.r.t. a sequence in metric spaces. In the sequel, we establish some theorems on convergence of a sequence in complete metric spaces. As consequences of our main result, we obtain some results of Ghosh and Debnath [J. Math. Anal. Appl. 207 (1997) 96-103], Kirk [Ann. Univ. Mariae Curie-Sklodowska Sec. A LI.2, 15 (1997) 167-178] and Petryshyn and Williamson [J. Math. Anal. Appl. 43 (1973) 459-497]. Some applications of our main results to geometry of Banach spaces are also discussed.

Keywords: Locally Quasi-Nonexpansive, Biased Quasi-Nonexpansive, Conditionally Biased Quasi-Nonexpansive, Drop, Super Drop

1. Introduction

In the last four decades of the last century, there have been a multitude of results on fixed points of nonexpansive and quasi-nonexpansive mappings in Banach spaces (e.g., [5-7, 9-11]).

Our aim in this note is to point out several obscure places in the results of Ahmed and Zeyada [J. Math. Anal. Appl. 274 (2002) 458-465]. In order to rectify and improve the results of Ahmed and Zeyada, we introduce the concepts of locally quasi-nonexpansive, biased quasi-nonexpansive and conditionally biased quasi-nonexpansive of a mapping w.r.t. a sequence in metric spaces.

Let X be a metric space and D a nonempty subset of X. Let T be a mapping of D into X and let $F(T)$ be the set of all fixed points of T. For a given $x_0 \in D$, the sequence of iterate $\{x_n\}$ is determined by

$$x_n = T(x_{n-1}) = T^n(x_0), n = 1, 2, 3\cdots \quad \text{(I)}$$

Let X be a normed space, $\lambda \in (0,1)$ and $\mu \in (0,1)$, the sequence of iterates $\{x_n\}$ are defined by

$$\begin{aligned} x_n &= T_\lambda(x_{n-1}) = T_\lambda^n(x_0), \\ T_\lambda &= \lambda I + (1-\lambda)T, n = 1, 2, 3\cdots \end{aligned} \quad \text{(II)}$$

$$\begin{aligned} x_n &= T_{\lambda,\mu}(x_{n-1}) = T_{\lambda,\mu}^n(x_0), \\ T_{\lambda,\mu} &= (1-\lambda)I + \lambda T[(1-\mu)I + \mu T], \\ n &= 1, 2, 3.... \end{aligned} \quad \text{(III)}$$

The iteration scheme (I) is called Teoplitz iteration and the iteration scheme (II) was introduced by Mann [12] while the iteration scheme (III) was introduced by Ishikawa [9].

The concept of quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. It was further studied by Diaz and Metcalf [5] and Doston [6,7] for mappings in Banach spaces. Recently, this concept was given by Kirk [10] in metric spaces as follows:

Definition 1.1. The mapping T is said to be quasi-nonexpansive if for each $x \in D$ and for every $p \in F(T)$, $d(T(x), p) \le d(x, p)$. A mapping T is conditionally quasi-nonexpansive if it is quasi-nonexpansive whenever $F(T) \ne \varnothing$.

We now introduce the following definition:

Definition 1.2. The mapping T is said to be locally quasi-nonexpansive at $p \in F(T)$ if for each $x \in D$, $d(T(x), p) \le d(x, p)$.

Obviously, quasi-nonexpansive locally quasi-nonexpansive at each $p \in F(T)$ but the reverse implication may not be true. To this end, we observe the following example.

Example 1.1. Let $X = [0,1)$ and $D = \left[0, \frac{3}{4}\right)$ be endowed with the Euclidean metric d. Define the mapping $T : D \to X$ by $T(x) = \frac{3}{2}x^2$ for each $x \in D$. Then we observe that $F(T) = \left\{0, \frac{2}{3}\right\}$, for all $x \in D$ and $p = 0 \in F(T)$, we have that

$$d(T(x), p) = \left|\frac{3}{2}x^2 - 0\right| \le |x - 0| = d(x, p),$$

i.e., T is locally quasi-nonexpansive at $p = 0 \in F(T)$. However, one can easily see that T is not locally quasi-nonexpansive at $p = \frac{2}{3} \in F(T)$. Indeed, for all $x \in \left(0, \frac{2}{3}\right)$ and $p = \frac{2}{3} \in F(T)$ we have

$$d(T(x), p) = \left|\frac{3}{2}x^2 - \frac{2}{3}\right| > \left|x - \frac{2}{3}\right| = d(x, p).$$

Hence we conclude that T is not quasi-nonexpansive, although it is locally quasi-nonexpansive at $p = 0 \in F(T)$.

The concept of asymptotic regularity was formally introduced by Browder and Petryshyn [3] for mappings in Hilbert spaces. Recently, it was defined by Kirk [11] in metric spaces as follows:

Definition 1.3. The mapping T is said to be asymptotically regular if $\lim_{n\to\infty} d\left(T^n(x), T^{n+1}(x)\right) = 0$ for each $x \in D$.

2. Main Results

Let **N** denote the set of all positive integers and $\omega = \mathbf{N} \cup \{0\}$ Ahmed and Zeyada [1] introduce-ed the following:

Definition 2.1. The mapping T is said to be quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ if for all $n \in \omega$ and for each $p \in F(T)$, $d(x_{n+1}, p) \le d(x_n, p)$.

The following lemma was quoted by Ahmed and Zeyada [1] without proof.

Lemma A. If T is quasi-nonexpansive, then T is quasi-nonexpansive w.r.t. a sequence $\{T^n x_0\}$ (respectively, $\{T_\lambda^n x_0\}, \{T_{\lambda,\mu}^n x_0\}$) for each $x_0 \in D$.

Remark 2.1. We notice that the above lemma is valid if $\{T^n x_0\} \in D$ for each $n \in \omega$ and a given $x_0 \in D$ (or D is T-invarient). So the correct version of Lemma A should be read as follows:

Lemma 2.1. If T is quasi-nonexpansive and for a given $x_0 \in D$ and each $n \in \omega$, $\{T^n x_0\} \in D$, then T is quasi-nonexpansive w.r.t. a sequence $\{T^n x_0\}$ (respectively, $\{T_\lambda^n x_0\}, \{T_{\lambda,\mu}^n x_0\}$) for each $x_0 \in D$.

Further, they claimed that the reverse implication in Lemma A may not be true in their Example 2.1. We again notice that there are several obscure places in this example. We now quote Example 2.1 of Ahmed and Zeyada [1] in the following:

Example A. Let $X = [0,1)$ and $D = \left[0, \frac{4}{5}\right)$ be endowed with the Euclidean metric **d** We define the mapping $T : D \to X$ by $T(x) = 2x^2$ for each $x \in D$. For a given $x_0 = \frac{1}{4} \in D$ we have

$$d\left(T^{n+1}(x_0), p\right) = \left|\left(\frac{1}{2}\right)^{2^{n+1}+1} - 0\right| \le \left|\left(\frac{1}{2}\right)^{2^n+1} - 0\right| = d\left(T^n(x_0), p\right)$$

where $T^n(1/4) = (1/2)^{2^n+1} \in D\, \forall n \in \mathbf{N} \cup \{0\}$ and $\mathrm{F}(\mathrm{T}) = \{0\}$, *i.e.*, T is quasi-nonexpansive w.r.t. a sequence $T^n(1/4)$ Furthermore, the map T is quasi-nonexpansive w.r.t. a sequence $\{T_{1/2}^n(1/2)\}$ and $\{T_{1/2,1/2}^n(1/2)\}$. They found that T is neither conditionally quasi-nonexpansive nor quasi-nonexpansive, for $x = \frac{3}{4} \in D$ and $p = 0 \in F(T), d(3/4, 0) > d(3/4, 0)$ and D is not closed.

Remark 2.2. We notice that the following claims made in Example A were false:

1) $T : D \to X$ is a mapping. In fact,

$$T(D) = \left[0, \frac{32}{25}\right) \supset [0,1) = X.$$

2) $\mathrm{F}(\mathrm{T}) = \{0\}$, In fact, $F(T) = \left\{0, \frac{1}{2}\right\}$.

3) T is quasi-nonexpansive w.r.t. a sequence $\{T^n(1/4)\}$.

4) T is quasi-nonexpansive w.r.t. a sequence $\{T_{1/2}^n(1/2)\}$ and $\{T_{1/2,1/2}^n(1/2)\}$.

However, (i) can be rectified by taking X as $\left[0, \frac{32}{25}\right)$ or any superset of $\left[0, \frac{32}{25}\right)$ in $[0, \infty)$ Even if this correction is made we find that the remaining statements 2) - 4) will remain false. Consequently, the claim of Ahmed and Zeyada [1] that the reverse implication in Lemma 2.1 may not be true seems false.

We now introduce the following definition.

Definition 2.2. The mapping T is said to be locally quasi-nonexpansive at $p \in F(T)$ w.r.t. a sequence $\{x_n\}$

if for all $n \in \omega$, $d(x_{n+1}, p) \le d(x_n, p)$.

Obviously, locally quasi-nonexpansiveness at $p \in F(T) \Rightarrow$ locally quasi-nonexpansiveness at $p \in F(T)$ w.r.t. a sequence $\{x_n\}$.

We now state the following lemma without proof.

Lemma 2.2. If T is quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ then T is locally quasi-nonexpansive at each $p \in F(T)$ w.r.t. the sequence $\{x_n\}$.

The reverse implication in Lemma 2.2 may not be true as shown in the following example:

Example 2.1. Let $X = [0,1)$ and $D = \left[0, \frac{2}{3}\right)$ be endowed with the Euclidean metric d Define the map- ping $T: D \to X$ by $T(x) = 2x^2$ for each $x \in D$ Then we observe that $F(T) = \left\{0, \frac{1}{2}\right\}$. For a given $x_0 = \frac{1}{4} \in D$ and $p = 0 \in F(T)$ we have that

$$d\left(T^{n+1}(x_0), p\right) = \left|\left(\frac{1}{2}\right)^{2^{n+1}+1} - 0\right| < \left|\left(\frac{1}{2}\right)^{2^n+1} - 0\right| \qquad (*)$$
$$= d\left(T^n(x_0), p\right)$$

where $T^n\left(\frac{1}{4}\right) = \left(\frac{1}{2}\right)^{2^n+1} \in D$ *i.e.*, T is locally quasi-nonexpansive at $p = 0 \in F(T)$ w.r.t. a sequence $\left\{T^n\left(\frac{1}{4}\right)\right\}$ However, one can easily see that T is not locally quasi-nonexpansive at $p = \frac{1}{2} \in F(T)$ w.r.t. the sequence $\left\{T^n\left(\frac{1}{4}\right)\right\}$. Indeed, we have

$$d\left(T^{n+1}(x_0), p\right) = \left|\left(\frac{1}{2}\right)^{2^{n+1}+1} - \frac{1}{2}\right| > \left|\left(\frac{1}{2}\right)^{2^{n+1}+1} - \frac{1}{2}\right| \qquad (**)$$
$$= d\left(T^n(x_0), p\right)$$

for all $n \in \omega$ Consequently, T is neither quasi-nonexpansive nor quasi-nonexpansive w.r.t. the sequence $\left\{T^n\left(\frac{1}{4}\right)\right\}$.

We now introduce the following:

Definition 2.3. The mapping $T: D \to X$ is said to be biased quasi-nonexpansive (b.q.n) w.r.t. a sequence $\{x_n\} \subset X$ if for all $n \in \omega$ and at each $p \in \text{cond}(F(T))$,

$$d(x_{n+1}, p) \le d(x_n, p)$$

where

$$\text{cond}(F(T)) = \left\{p \in F(T) : \limsup_{n\to\infty} d(x_n, p) \le \liminf_{n\to\infty} d(x_n, F(T))\right\}$$

A mapping T is conditionally biased quasi-nonexpansive (c.b.q.n) w.r.t. a sequence $\{x_n\}$ if $\text{cond}(F(T)) \ne \varnothing$.

Remark 2.3. We observe that the following implications are obvious:

(a) Conditional biased quasi-nonexpansiveness w.r.t. a sequence $\{x_n\} \Rightarrow$ biased quasi-nonexpansiveness w.r.t. a sequence $\{x_n\}$ but the reverse implication may not be true (Indeed, any mapping $T: D \to X$ for which $\text{cond}(F(T)) \ne \varnothing$ is a biased quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ but not conditionally biased quasi-nonexpansive w.r.t. a sequence $\{x_n\}$. However, under certain conditions a biased quasi-nonexpansive map w.r.t. a sequence $\{x_n\}$ may be a conditional biased quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ (see Lemma 2.6 below).

(b) If T is conditionally biased quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ and $\text{cond}(F(T)) = F(T) \ne \varnothing$ then T is locally quasi-nonexpansive at each $p \in F(T)$ w.r.t. a sequence $\{x_n\}$.

(c) If T is biased quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ and $\varnothing \ne \text{cond}(F(T))$ Ö $F(T)$ then T is locally quasi-nonexpansive at each $p \in \text{cond}(F(T))$ w.r.t. a sequence $\{x_n\}$.

(d) Quasi-nonexpansivenes $\Rightarrow$ locally quasi-nonexpansiveness at $p \in F(T) \Rightarrow$ locally quasi-nonexpansiveness at $p \in F(T)$ w.r.t. a sequence $\{x_n\}$.

In Example 2.1 above, we observe that

1) for $p = 0 \in F(T)$, we have

$$\limsup_{n\to\infty} d(x_n, p) = \limsup_{n\to\infty} \left|\left(\frac{1}{2}\right)^{2^n+1} - 0\right| = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{2^n+1} = 0$$

2) for $p = \frac{1}{2} \in F(T)$, we have

$$\limsup_{n\to\infty} d(x_n, p) = \limsup_{n\to\infty} \left|\left(\frac{1}{2}\right)^{2^n+1} - \frac{1}{2}\right| = \lim_{n\to\infty} \left|\left(\frac{1}{2}\right)^{2^n+1} - \frac{1}{2}\right| = \frac{1}{2}$$

$$\liminf_{n\to\infty} d(x_n, F(T)) = \liminf_{n\to\infty} \left(\frac{1}{2}\right)^{2^n+1} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{2^n+1} = 0$$

Here $\text{cond}(F(T)) \neq \{0\}$ and in view of (*) and (**), it is evident that T is conditionally biased quasi-nonexpansive (c.b.q.n.) w.r.t. a sequence $\left\{T^n\left(\frac{1}{4}\right)\right\}$ and hence it is biased quasi-nonexpansive (b.q.n.) w.r.t. a sequence $\left\{T^n\left(\frac{1}{4}\right)\right\}$.

We now show in the following example that $\text{cond}(F(T))$ need not be a singleton set.

Example 2.2. *Let* $X = [0,2]$ *and* $D = [0,1) \cup (1,2]$ *be endowed with the Euclidean metric* **d** *Define the mapping* $T : D \to X$ *by* $Tx = +\sqrt{x}$ *for* $x \in [0,1) \cup (1,2)$ *and* $T(x) = 2$ *for* $x = 2$. *Clearly,* $\text{F(T)} = \{0,2\}$ *Consider the sequence* $\{x_n\} \equiv \{1\}$ *in* X *then we observe that*

1) for $p = 0 \in F(T)$, we have

$$\limsup_{n\to\infty} d(x_n, p) = \limsup_{n\to\infty} |1-0| = \lim_{n\to\infty} 1 = 1;$$

2) for $p = 2 \in F(T)$, we have

$$\limsup_{n\to\infty} d(x_n, p) = \limsup_{n\to\infty} |1-2| = \lim_{n\to\infty} 1 = 1;$$

and

$$\liminf_{n\to\infty} d(x_n, F(T)) = \lim_{n\to\infty} 1 = 1.$$

Thus we have $\text{cond}(\text{F}(T)) = \{0,2\}$ and it is evident that T is conditionally biased quasi nonexpansive (c.b.q.n.) w.r.t. the sequence $\{x_n\} \equiv \{1\}$ in X, and hence it is biased quasi-nonexpansive (b.q.n.) w.r.t. the sequence $\{x_n\} \equiv \{1\}$ in X.

However, interested reader can check that if we consider the sequence $\{x_n\}$ such that $x_n \to 1^+$ then $\text{cond}(\text{F(T)}) = \{2\}$ Further, we observe that for $p = 2 \in \text{cond}(\text{F}(T))$ and for all $n \in \omega$ we have

$$d(x_{n+1}, p) \leq d(x_n, p)$$

Thus, T is conditionally biased quasi-nonexpansive (c.b.q.n.) w.r.t. the sequence $\{x_n\}$ in X.

On the other hand, if we consider the sequence $\{x_n\}$ such that $x_n \to 1^-$ then $\text{cond}(\text{F}(T)) = \{0\}$ and T is conditionally biased quasi-nonexpansive (c.b.q.n.) w.r.t. the sequence $\{x_n\}$ in X.

Remark 2.4. Example 2.2 above also shows that $\text{cond}(\text{F}(T))$ is a closed set even though T is discontinuous at $p = 2$.

We need the following lemmas to prove our main theorem:

Lemma 2.3. Let T be locally quasinonexpansive at $p \in F(T)$ w.r.t. $\{x_n\}$ and $\lim_{n\to\infty} d(x_n, F(T)) = 0$.
Then $\{x_n\}$ is a Cauchy sequence.

Proof. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ then for any given $\varepsilon > 0$ there exists $n_1 \in$ □ such that for each $n \geq n_1$, $d(x_n, F(T)) < \frac{\varepsilon}{2}$ So, there exists $q \in F(T)$ such that for all $n \geq n_1, d(x_n, q) < \frac{\varepsilon}{2}$.

Thus, for any $m, n \geq n_1$ we have

$$d(x_m, x_n) \leq d(x_m, q) + d(x_n, q) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad q \in F(T),$$

Hence $\{x_n\}$ is a Cauchy sequence.

Lemma 2.4. Let T be conditionally biased quasi-nonexpansive w.r.t. $\{x_n\}$,and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$
Then:

1) $\{x_n\}$ converges to a point p in $\text{cond}(F(T))$ and T is locally quasi-nonexpansive at $p \in \text{cond}(\text{F}(T))$ w.r.t. $\{x_n\}$.

2) $\{x_n\}$ is a Cauchy sequence.

Proof. 1) Since T is conditionally biased quasi-nonexpansive w.r.t. $\{x_n\}$, it follows that $\text{cond}(F(T)) \neq \varnothing$. As $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ we have that $\limsup_{n\to\infty} d(x_n, p) = 0$ for some $p \in \text{cond}(\text{F}(T))$.So, we have $\lim_{n\to\infty} d(x_n, p) = 0$ for some $p \in \text{cond}(\text{F}(T))$; *i.e.*, $\{x_n\}$ converges to a point p in $\text{cond}(\text{F}(T))$ and T is locally quasi-nonexpansive at $p \in \text{cond}(\text{F}(T))$ w.r.t. $\{x_n\}$.

2) From $\lim_{n\to\infty} d(x_n, p) = 0$ it follows that for any given $\varepsilon > 0$ there exists $n_1 \in \mathbf{N}$ such that for each $n \geq n_1, d(x_n, p) < \frac{\varepsilon}{2}$. Thus, for any $m, n \geq n_1$, *we have*

$$d(x_m, x_n) \leq d(x_m, q) + d(x_n, q) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad q \in F(T),$$

Hence $\{x_n\}$ is a Cauchy sequence.

The following lemma follows easily.

Lemma 2.5. Let T be biased quasi-nonexpansive w.r.t. $\{x_n\}$, and let $\{x_n\}$ converges to a point p in $F(\text{T})$ Then:

1) $\{x_n\}$ converges to a point p in $\text{cond}(F(T))$ and T is conditionally biased quasi-nonexpansive w.r.t. $\{x_n\}$;

2) $\{x_n\}$ is a Cauchy sequence.

We now state our main theorem in the present paper.

Theorem 2.1. Let $F(T)$ be a nonempty closed set. Then

1) $\lim_{n\to\infty} d(x_n, F(T)) = 0$ if $\{x_n\}$ converges to a point p in $F(T)$;

2) $\{x_n\}$ converges to a point in $F(T)$ if

$\lim_{n\to\infty} d(x_n, F(T)) = 0$, T is locally quasi-nonexpansive at $p \in F(T)$ w.r.t. $\{x_n\}$ and X is complete.

Proof. 1) Since $F(T)$ is closed, $p \in F(T)$ and the mapping $x \mapsto d(x, F(T))$ is continuous (see [1, p. 13]), then

$$\lim_{n\to\infty} d(x_n, F(T)) = d\left(\lim_{n\to\infty} x_n, F(T)\right) = d(p, F(T)) = 0$$

2) From Lemma 2.3, $\{x_n\}$ is a Cauchy sequence. Since X is complete, then $\{x_n\}$ converges to a point, say q in X. Since $F(T)$ is closed, then

$$0 = \lim_{n\to\infty} d(x_n, F(T)) = d\left(\lim_{n\to\infty} x_n, F(T)\right) = d(p, F(T))$$

implies that $q \in F(T)$.

As consequences of Theorem 2.1, we have the following:

Corollary 2.1. Let $F(T)$ a nonempty closed set and for a given $x_0 \in D$ and each $n \in \omega, \{T^n x_0\} \in D$ Then

1) $\lim_{n\to\infty} d(T^n x_0, F(T)) = 0$ if $\{T^n x_0\}$ converges to a point p in $F(T)$;

2) $\{T^n x_0\}$ converges to a point in $F(T)$ if, $\lim_{n\to\infty} d(T^n x_0, F(T)) = 0$, T is locally qusi-nonexpansive at $p \in F(T)$ w.r.t. $\{T^n x_0\}$ and X is complete.

Corollary 2.2. Let X be a normed linear space, $F(T)$ a nonempty closed set and for a given $x_0 \in D$ and each $n \in \omega$, $\{T_\lambda^n x_0\} \in D$.

(1) If the sequence $\{T_\lambda^n x_0\}$ converges to a point p in $F(T)$, then

$$\lim_{n\to\infty} d(T_\lambda^n x_0, F(T)) = 0$$

(2) If $\lim_{n\to\infty} d(T_\lambda^n x_0, F(T)) = 0$ T is locally quasi-nonexpansive at $p \in F(T)$ w.r.t. $\{T_\lambda^n x_0\}$ and X is complete, then $\{T_\lambda^n x_0\}$ converges to a point p in $F(T)$.

Corollary 2.3. Let X be a normed linear space, $F(T)$ a nonempty closed set and for a given $x_0 \in D$ and each $n \in \omega$, $\{T_{\lambda,\mu}^n x_0\} \in D$ Then

(1) $\lim_{n\to\infty} d(T_{\lambda,\mu}^n x_0, F(T)) = 0$ if the sequence $\{T_{\lambda,\mu}^n x_0\}$ converges to a point p in $F(T)$;

(2) $\{T_{\lambda,\mu}^n x_0\}$ converges to a point p in $F(T)$ if $\lim_{n\to\infty} d(T_{\lambda,\mu}^n x_0, F(T)) = 0$, T is locally quasi-nonexpansive at $p \in F(T)$ w.r.t. $\{T_{\lambda,\mu}^n x_0\}$ and X is complete.

Note that the continuity of T implies that $F(T)$ is closed but the converse need not be true. To effect this consider the following example.

Example 2.3. Let $X = [0, \infty)$ and $D = [0,1)$ be endowed with the Euclidean metric d. Define the mapping $T : D \to X$ by $T(x) = x$ if $x \in \left[0, \frac{1}{2}\right]$ and $T(x) = 3x^2$ if $x \in \left(\frac{1}{2}, 1\right)$ Obviously, $F(T) = [0, 1/2]$ is a nonempty closed but T is not continuous at $x = 1/2$.

Remark 2.5. (a) In order to support the above fact Ahmed and Zeyada [1] stated wrongly in their Example 2.2, where $X = [0,1)$, $D = [0,1/4) \cup (1/2, 5/6]$, $T(x) = x$.

If $X \in [0, 1/4)$ and $T(x) = x/2$ if $x \in (1/2, 5/6)$ that T is not continuous. In fact, we observe that in this example T is continuous.

(b) From Lemma 2.1, Examples 2.1 and 2.3, the continuity of T implies that $F(T)$ is closed but the converse may not be true; then we have that Corollaries 2.1, 2.2 and 2.3 are improvement of Theorem 1.1 in [13, p.462], Theorem 1.1′ in [13, p. 469], and Theorem 3.1 in [8, p. 98], respectively.

(c) Since every quasi-nonexpansive map w.r.t. a sequence $\{x_n\}$ is locally quasi-nonexpansive at each $p \in F(T)$ w.r.t. a sequence $\{x_n\}$, but the converse may not be true; we have that Theorem 2.1, Corollaries 2.1, 2.2 and 2.3 are improvement of corresponding Theorem 2.1, Corollary 2.1, 2.2 and 2.3 of Ahmed and Zeyada [1].

(d) By considering the closedness of $F(T)$ in lieu of the continuity of T and $T : D \to X$ instead of $T : X \to X$ we have that our Corollary 2.1 improves Proposition 1.1 of Kirk [10, p. 168].

(e) The closedness condition of D in Theorem 1.1 and 1.1′ of Petryshyn and Williamson [12, p. 462, 469] and Theorem 3.1 in [8, p. 98] is superfluous.

(f) The convexity condition of D in Theorem 1.1′ of Petryshyn and Williamson [12, p. 469] is superfluous because the author assumed in their theorem that $\{T_\lambda^n x_0\} \in D$ for each $n \in \omega$ and a given $x_0 \in D$ in condition $(1.3')$.

Theorem 2.2. Let $\mathrm{cond}(F(T))$ be a nonempty closed set. Then $\{x_n\}$ converges to a point in $\mathrm{cond}(F(T))$ if $\liminf_{n\to\infty} d(x_n, \mathrm{cond}(F(T))) = 0$, T is condionally biased quasi-nonexpansive w.r.t. $\{x_n\}$ and X is complete.

Proof. Since $\mathrm{cond}(F(T)) \subset F(T)$ we have that $\liminf_{n\to\infty} d(x_n, \mathrm{cond}(F(T))) = 0$ implies $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ Now using the technique of the proof of Theorem 2.1 the conclusion follows from Lemma 2.3.

The following results follows easily from Lemma 2.5.

Theorem 2.3. Let $F(T)$ be a nonempty closed set. Then $\{x_n\}$ converges to a point in $\mathrm{cond}(F(T))$ if $\{x_n\}$ converges to a point p in $F(T)$, T is biased quasi-nonexpansive w.r.t. $\{x_n\}$ and X is complete.

Theorem 2.4. Let X be a complete metric space and let $\operatorname{cond}(F(T))$ be a nonempty closed set. Assume that

1) T is biased quasi-nonexpansive w.r.t. $\{x_n\}$;
2) $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ or $\{x_n\}$ is a Cauchy sequence;
3) if the sequence $\{y_n\}$ satisfies $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$

then

$$\liminf_{n\to\infty} d(y_n, \operatorname{cond}(F(T))) = 0$$

or

$$\limsup_{n\to\infty} d(y_n, \operatorname{cond}(F(T))) = 0.$$

Then $\{x_n\}$ converges to a point in $\operatorname{cond}(F(T))$.

Proof. Since $\operatorname{cond}(F(T)) \neq \varnothing$ it follows from (i) that T is condionally biased quasi-nonexpansive w.r.t. $\{x_n\}$ and the sequence $\{d(x_n, \operatorname{cond}(F(T)))\}$ is monotonically decreasing and bounded from below by zero. Then $\liminf_{n\to\infty} d(x_n, \operatorname{cond}(F(T)))$ exists.

From 2) and 3), we have that

$$\liminf_{n\to\infty} d(x_n, \operatorname{cond}(F(T))) = 0$$

or

$$\limsup_{n\to\infty} d(x_n, \operatorname{cond}(F(T))) = 0.$$

Then $\lim_{n\to\infty} d(x_n, \operatorname{cond}(F(T))) = 0$ Therefore, by Theorem 2.2, the sequence $\{x_n\}$ converges to a point in $\operatorname{cond}(F(T))$.

As consequences of Theorem 2.4, we obtain the following:

Corollary 2.4. Let X be a complete metric space and let $\operatorname{cond}(F(T))$ be a nonempty closed set. Assume that

1) T is biased quasi-nonexpansive w.r.t. $\{x_n\}$;
2) T is asymptoticc regular at $x_0 \in D$ (or $\{T^n(x_0)\}$ is a Cauchy sequence);
3) if the sequence $\{y_n\}$ satisfies $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$

then

$$\liminf_{n\to\infty} d(y_n, \operatorname{cond}(F(T))) = 0$$

or

$$\limsup_{n\to\infty} d(y_n, \operatorname{cond}(F(T))) = 0.$$

Then $\{T^n(x_0)\}$ converges to a point in $\operatorname{cond}(F(T))$.

Corollary 2.5. Let X be a Banach space and let $\operatorname{cond}(F(T))$ be a nonempty closed set. Assume that

1) T is biased quasi-nonexpansive w.r.t. $\{T^n(x_0)\}$;
2) T is asymptoticc regular at $x_0 \in D$ (or $\{T^n(x_0)\}$ is a Cauchy sequence);
3) if the sequence $\{y_n\}$ satisfies $\lim_{n\to\infty} \|y_n - T_\lambda y_n\| = 0$,

then

$$\liminf_{n\to\infty} d(y_n, \operatorname{cond}(F(T))) = 0$$

or

$$\limsup_{n\to\infty} d(y_n, \operatorname{cond}(F(T))) = 0.$$

Then $\{T^n(x_0)\}$ converges to a point in $\operatorname{cond}(F(T))$.

Corollary 2.6. Let X be a Banach space and let $\operatorname{cond}(F(T))$ be a nonempty closed set. Assume that

1) T is biased quasi-nonexpansive w.r.t. $\{T^n_{\lambda,\mu}(x_0)\}$;
2) T is asymptoticc regular at $x_0 \in D$ (or $\{T^n_{\lambda,\mu}(x_0)\}$ is a Cauchy sequence);
3) if the sequence $\{y_n\}$ satisfies $\lim_{n\to\infty} \|y_n - T_{\lambda,\mu} y_n\| = 0$,

then

$$\liminf_{n\to\infty} d(y_n, \operatorname{cond}(F(T))) = 0$$

or

$$\limsup_{n\to\infty} d(y_n, \operatorname{cond}(F(T))) = 0.$$

Then $\{T^n_{\lambda,\mu}(x_0)\}$ converges to a point in $\operatorname{cond}(F(T))$.

Remark 2.6. From Lemmas 2.1 and 2.2, Examples 2.1 and 2.3, Remark 2.3, the continuity of T implies that $F(T)$ is closed but the converse may not be true; we obtain that Corollary 2.4 include Theorem 1.2 in [12, p. 464] and Theorem 3.2 in [7, p. 99] as special cases.

As another consequence of Theorem 2.1, we establish the following theorem:

Theorem 2.5. Let X be a complete metric space and let $\operatorname{cond}(F(T))$ be a nonempty closed set. Assume that

1) T is biased quasi-nonexpansive w.r.t. $\{x_n\}$;
2) for every $x \in D - \operatorname{cond}(F(T))$ there exists $p_x \in \operatorname{cond}(F(T))$ such that $d(x_{n+1}, p_x) < d(x_n, p_x)$;
3) the sequence $\{x_n\}$ contains a subsequence $\{x_{n_j}\}$ converging to $x^* \in D$.

Then $\{x_n\}$ converges to a point in $\operatorname{cond}(F(T))$.

Proof. Since $\operatorname{cond}(F(T)) \neq \varnothing$ it follows from (i) that T is condionally biased quasi-nonexpansive w.r.t. $\{x_n\}$ and the sequence $\{d(x_n, \operatorname{cond}(F(T)))\}$ is monotonically decreasing and bounded from below by zero. Then $\lim_{n\to\infty} d(x_n, \operatorname{cond}(F(T))) = d(\lim_{n\to\infty} x_n, \operatorname{cond}(F(T)))$

$= r \geq 0$ *exists.* We now apply Theorem 2.4. It suffices to show that r=0. If $\lim_{n\to\infty} x_n = x^* \in \operatorname{cond}(F(T))$ then r=0. If $x^* \notin \operatorname{cond}(F(T))$ then $x^* \in D - \operatorname{cond}(F(T))$ Thus there exists $p_{x^*} \in \operatorname{cond}(F(T))$ such that

$$d(x^*, p_{x^*}) = d\left(\lim_{n\to\infty} x_{n+1}, p_{x^*}\right) = \lim_{n\to\infty} d(x_{n+1}, p_{x^*})$$
$$< \lim_{n\to\infty} d(x^*, p_{x^*}) = d\left(\lim_{n\to\infty} x_n, p_{x^*}\right) = d(x^*, p_{x^*})$$

This is a contradiction. So, $x^* \in \operatorname{cond}(F(T))$.

Corollary 2.7. Let X be a complete metric space, $\operatorname{cond}(F(T))$ a nonempty closed set and for a given $x_0 \in D$ and each $n \in \omega$, $\{T^n x_0\} \in D$. Assume that

1) T is biased quasi-nonexpansive w.r.t. $\{T^n(x_0)\}$;

2) for every $x \in D - \operatorname{cond}(F(T))$ there exists $p_x \in \operatorname{cond}(F(T))$ such that

$$d(T^{n+1}(x_0), p_x) < d(T^n(x_0), p_x);$$

3) the sequence $\{T^n(x_0)\}$ contains a subsequence $\{T^{n_j}(x_0)\}$ converging to $x^* \in D$.

Then $\{T^n(x_0)\}$ converges to a point in $\operatorname{cond}(F(T))$.

Corollary 2.8. Let X be a Banach space, $\operatorname{cond}(F(T))$ a nonempty closed set and for a given $x_0 \in D$ and each $n \in \omega$, $\{T^n_{\lambda,\mu}(x_0)\} \in D$ Assume that

1) T is biased quasi-nonexpansive w.r.t. $\{T^n_{\lambda,\mu}(x_0)\}$;

2) for every $x \in D - \operatorname{cond}(F(T))$ there exists $p_x \in \operatorname{cond}(F(T))$ such that

$$\left\|T^{n+1}_{\lambda}(x_0) - p_x\right\| < \left\|T^n_{\lambda}(x_0) - p_x\right\|;$$

3) the sequence $\{T^n(x_0)\}$ contains a subsequence $\{T^{n_j}_{\lambda}(x_0)\}$ converging to $x^* \in D$.

Then $\{T^{n_j}_{\lambda}(x_0)\}$ converges to a point in $\operatorname{cond}(F(T))$.

Corollary 2.9. Let X be a Banach space, $\operatorname{cond}(F(T))$ a nonempty closed set and for a given $x_0 \in D$ and each $n \in \omega$, $\{T^n_{\lambda,\mu}(x_0)\} \in D$ Assume that

1) T is biased quasi-nonexpansive w.r.t. $\{T^n_{\lambda,\mu}(x_0)\}$;

2) for every $x \in D - \operatorname{cond}(F(T))$ there exists $p_x \in \operatorname{cond}(F(T))$ such that

$$\left\|T^{n+1}_{\lambda,\mu}(x_0) - p_x\right\| < \left\|T^n_{\lambda,\mu}(x_0) - p_x\right\|;$$

3) the sequence $\{T^n(x_0)\}$ contains a subsequence $\{T^{n_j}_{\lambda,\mu}(x_0)\}$ converging to $x^* \in D$.

Then $\{T^n_{\lambda,\mu}(x_0)\}$ converges to a point in $\operatorname{cond}(F(T))$.

Remark 2.7. From Lemmas 2.1 and 2.2, Examples 2.1 and 2.3, Remark 2.3, the continuity of T implies that $F(T)$ is closed but the converse may not be true ; we obtain that Corollary 2.7 is an improvement of Theorem 1.3 in [13, p. 466].

3. Applications to Geometry of Banach Spaces

Throughout this section, let **R** denote the set of real numbers. Let $K = K(z,r)$ be a closed ball in a Banach space X. For a sequence $\{x_n\}_{n=0}^{\infty}$ Ú K converging to X we define

$$\lim_{n\to\infty} \mathrm{D}_n = \mathrm{SD}(x,K)$$

where

$$\mathrm{D}_0 = \operatorname{conv}(\{x_0\} \cup K)$$

and

$$\mathrm{D}_{n+1} = \operatorname{conv}(\{x_n\} \cup \mathrm{D}_n) \forall n \in \omega$$

and $\mathrm{SD}(x,K)$ is called a super drop.

Clearly, for a constant sequence $\{x_n\} \equiv \{x\}$ converging to x we have $\mathrm{D}_{n+1} = \mathrm{D}_n \forall n \in \omega$ so that $\mathrm{D}(x,k) = \operatorname{conv}(\{x\} \cup K)$ and is called a drop Thus the concept of a drop is a special case of super drop It is also clear that if $y \in \mathrm{D}(x,K)$ then $\mathrm{D}(y,K) \subset \mathrm{D}(x,K)$ and if $z = 0$ then $\|y\| = \|x\|$.

Recall that a function $\varphi : X \to \mathbf{R}$ is called a *lower semicontinuous* whenever $\{x \in X : \varphi(x) \leq a\}$ is closed for each $a \in \mathbf{R}$.

Caristi [4] proved the following:

Theorem A. Let (X,d) be complete and $\varphi : X \to \mathbf{R}$ a lower semicontinuous function with a finite lower bound. Let $T : X \to X$ be any function such that $d(x,T(x)) \leq \varphi(x) - \varphi(T(x))$ for each $x \in X$. Then T has a fixed point.

We now state and prove some applications of our main results in section 2 to geometry of Banach Spaces.

Theorem 3.1. Let C be a closed subset of a Banach space X let $z \in X - C$ and let $K = K(z,r)$ be a closed ball of radius $r < d(z,C) = R$ Let x be an arbitrary element of C let $\{x_n\}$ be a sequence in C converging to X and let $T : C \to X$ be any continuous function defined implicitly by $T(x) \in C \cap \mathrm{SD}(x,K)$ for each $x \in C$ in the sense that $T(x_n) \in C \cap \mathrm{D}_n$ for each $n \in \omega$. Then

1) $\lim_{n\to\infty} d(x_n, F(T)) = 0$ if $\{x_n\}$ converges to a point p in $F(T)$;

2) $\{x_n\}$ converges to a point in $F(T)$ if $\lim_{n\to\infty} d(x_n, F(T)) = 0$, T is locally quasi-nonexpansive at $p \in F(T)$ w.r.t. $\{x_n\}$.

Proof. Without loss of generality we may assume that $z = 0$. Let $\|x\| = \eta \geq R$ and let $X = A \cap \mathrm{SD}(x, K)$ Then it is clear that T maps X into itself. For given $y \in X$ and a sequence $\{y_n\}$ converging to y, we shall estimate $\|y - T(y)\|$ on X.

For given $y \in X$ and the corresponding sequence $\{y_n\}$ there is a sequence $\{b_n\}$ in X with $T(y_n) = tb_n + (1-t)y_n$, $0 < t < 1$ Now $\|T(y_n)\| \leq t\|b_n\| + (1-t)\|y_n\|$, we have

$$t(\|y_n\| - \|b_n\|) \leq \|y_n\| - \|T(y_n)\|$$

so because $\|y_n\| - \|b_n\| \geq R - \eta$, we find that

$$t \leq \frac{\|y_n\| - \|T(y_n)\|}{R - \eta}.$$

Thus,

$$\begin{aligned}\|y_n\| - \|T(y_n)\| &\leq t\|y_n - b_n\| \\ &\leq t(\|y_n\| + \|b_n\|) \leq (\eta + r) \\ &\leq \frac{\eta + r}{R - r}(\|y_n\| - \|T(y_n)\|)\end{aligned}$$

Define $d(x, y) = \|x - y\| \forall x, y \in X$ and $\varphi(y) = \frac{\eta + r}{R - r}\|y\|$ then X is complete as a metric space and $\varphi: X \to \mathbf{R}$ is a continuous function. So, φ is a lower-semicontinuous function. Also, the above inequality takes the form $d(y_n, T(y_n)) \leq \varphi(y_n) - \varphi(T(y_n))$. Proceeding to the limit as $n \to \infty$ we obtain $d(y, T(y)) \leq \varphi(y) - \varphi(T(y))$ for each $y \in X$. There- fore, applying the theorem of Caristi we obtain that T has a fixed point $p = p(x)$ for each $x \in C$, *i.e.*, $F(T) \neq \varnothing$. By continuity of T, $F(T)$ is closed. Hence the conclusion follows from Theorem 2.1.

Since drop is a special case of super drop, we have the following:

Corollary 3.1. Let C be a closed subset of a Banach space X let $z \in X - C$ and let $K = K(z, r)$ be a closed ball of radius $r < d(z, C) = R$ Let x be an arbitrary element of C, and let $T: C \to X$ be any (not necessarily continuous) function defined implicitly by $T(x) \in C \cap \mathrm{D}(x, K)$ for each $x \in C$. Then

(1) $\lim_{n\to\infty} d(x_n, F(T)) = 0$ if $\{x_n\}$ converges to a point p in $F(T)$;

(2) $\{x_n\}$ converges to a point in $F(T)$ if $\lim_{n\to\infty} d(x_n, F(T)) = 0$, T is locally quasi-nonexpansive at $p \in F(T)$ w.r.t. $\{x_n\}$.

We now prove the following result for biased quasi-nonexpansive mapping w.r.t. a sequence $\{x_n\}$.

Theorem 3.2. Let C be a closed subset of a Banach space X let $z \in X - C$ and let $K = K(z, r)$ be a closed ball of radius $r < d(z, C) = R$ Let x be an arbitrary element of C, $\{x_n\}$ a sequence in C converging to X, and let $T: C \to X$ be any con- tinuous function defined implicitly by $T(x) \in C \cap \mathrm{SD}(x, K)$ for each $x \in C$ in the sense that $T(x_n) \in C \cap \mathrm{D}_n$ for each $n \in \omega$. If $\{x_n\}$ converges to a point in $F(T)$, T is biased quasi-nonexpansive w.r.t. $\{x_n\}$ then $\{x_n\}$ converges to a point in $\mathrm{cond}(F(T))$.

Proof. Using Theorem 2.3. instead of Theorem 2.1 the conclusion follows on the lines of the proof technique of Theorem 3.1.

As a consequence of Theorem 3.2, we obtain the following:

Corollary 3.2. Let C be a closed subset of a Banach space X let $z \in X - C$ and let $K = K(z, r)$ be a closed ball of radius $r < d(z, C) = R$. Let x be an arbitrary element of C, and let $T: C \to X$ be any (not necessarily continuous) function defined implicitly by $T(x) \in C \cap \mathrm{D}(x, K)$ for each $x \in C$. If $\{x_n\}$ conver- ges to a point in $F(T)$, T is biased quasi-nonexpansive w.r.t. $\{x_n\}$ then $\{x_n\}$ converges to a point in $\mathrm{cond}(F(T))$.

Open Question. To what extent can the continuity hypothesis on T be muted in Theorems 3.1 and 3.2?

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