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Inequalities for the Polar Derivative of a Polynomial

Gulshan Singh[1], Wali Mohammad Shah[2], Yash Paul[1]
[1]*Bharathiar University Coimbatore, Tamil Nadu, India*
[2]*Department of Mathematics, Kashmir University, Srinagar, India*
E-mail: {*gulshansingh1*, *wmshah*, *yashpaul*2011}@*rediffmail.com*
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Abstract

If $P(z):=\sum_{j=0}^{n}a_j z^j$ is a polynomial of degree n, having all its zeros in $|z|\le K$, $K\ge 1$, then it was provied by Aziz and Rather [2] that for every real or complex number α with $|\alpha|\ge K$, $Max_{|z|=1}|D_\alpha P(z)|\ge \frac{n(|\alpha|-K)}{(K^n+1)}Max_{|z|=1}|P(z)|$. In this paper, we sharpen above result for the polynomials $P(z)$ of degree $n>3$.

Keywords: Polynomial, Inequality, Polar Derivative

1. Introduction

Let $P(z):=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n and $P'(z)$ its derivative, then

$$Max_{|z|=1}|P'(z)|\le nMax_{|z|=1}|P(z)| \tag{1}$$

Inequality (1) is a famous result due to Bernstein and is best possible with equality holding for the polynomial $P(z)=\lambda z^n$, where λ is a complex number.

If we restricted ourselves to a class of polynomial having no zeros in $|z|<1$, then the above inequality can be sharpened. In fact, Erdös conjectured and later Lax [6] proved that if $P(z)\ne 0$ in $|z|<1$, then

$$Max_{|z|=1}|P'(z)|\le \frac{n}{2}Max_{|z|=1}|P(z)| \tag{2}$$

On the other hand, it was proved by Turán [10] that if $P(z)$ is a polynomial of degree n having all its zeros in $|z|\le 1$, then

$$Max_{|z|=1}|P'(z)|\ge \frac{n}{2}Max_{|z|=1}|P(z)| \tag{3}$$

The inequalities (2) and (3) are also best possible and become equality for polynomials which have all zeros on $|z|=1$.

For the class of polynomials having all the zeros in $|z|\le K$, Malik [7] (See also Govil [5]) proved that if $P(z)$ is a polynomial of degree n having all zeros lie in $|z|\le K$, then

$$Max_{|z|=1}|P'(z)|\ge \frac{n}{1+K}Max_{|z|=1}|P(z)|,\ \text{if } K\le 1, \tag{4}$$

where as Govil [5] showed that

$$Max_{|z|=1}|P'(z)|\ge \frac{n}{1+K^n}Max_{|z|=1}|P(z)|,\ \text{if } K\ge 1 \tag{5}$$

Both the inequalities are best possible, with equality in (4) holding for $P(z)=(z+K)^n$ and in (5) the equality holds for the polynomial $P(z)=(z^n+K^n)$.

Let $D_\alpha P(z)$ denote the polar derivative of the polynomial $P(z)$ of degree n with respect to α, then

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha}=P'(z).$$

Aziz and Rather [2] extended (5) to the polar derivative of a polynomial and proved the following:

Theorem 1: *If the polynomial* $P(z):=\sum_{j=0}^{n}a_j z^j$ *has all its zeros in* $|z|\le K$, $K\ge 1$, *then for every real or complex number* α *with* $|\alpha|\ge K$,

$$Max_{|z|=1}|D_\alpha P(z)|\ge \frac{n(|\alpha|-K)}{(K^n+1)}Max_{|z|=1}|P(z)| \tag{6}$$

In this paper, we prove the following result which is a refinement as well as generalization of Theorem 1.

Theorem 2: *Let* $P(z):=\sum_{j=0}^{n}a_j z^j$, $a_n a_0 \neq 0$ *be a polynomial of degree* $n>3$, *having all its zeros in* $|z|\le K$, $K\ge 1$, *then for every real or complex number* α *with* $|\alpha|\ge K$,

$$Max_{|z|=1}\left|D_\alpha P(z)\right|\ge \frac{n(|\alpha|-K)}{(K^n+1)}\left\{Max_{|z|=1}|P(z)|+Min_{|z|=K}|P(z)|+\frac{2|a_{n-1}|}{(n+1)}\left[\frac{(K^n-1)}{n}-(K-1)\right]\right.$$
$$\left.+2|a_{n-2}|\left[\left\{\frac{(K^n-1)-n(K-1)}{n(n-1)}\right\}-\left\{\frac{(K^{n-2}-1)-(n-2)(K-1)}{(n-2)(n-3)}\right\}\right]\right\},\quad \text{if } n>3. \tag{7}$$

Remark 1: For $K=1$, Theorem 2 provides a refinement of a theorem proved by Shah [9].

Remark 2: For $K>1$, and for $y>1$, $\frac{\left[(K^y-1)-y(K-1)\right]}{y(y-1)}$ and $\frac{(K^y-1)}{y}$ are both increasing functions of y and so the expressions

$$\left[\left\{\frac{(K^n-1)-n(K-1)}{n(n-1)}\right\}-\left\{\frac{(K^{n-2}-1)-(n-2)(K-1)}{(n-2)(n-3)}\right\}\right]$$

and

$$\left[\frac{(K^n-1)}{n}-(K-1)\right]$$

are always non-negative so that for polynomials of degree $n>3$, Theorem 2 is an improvement of Theorem 1.

Dividing both sides of (7) by $|\alpha|$ and letting $|\alpha|\to\infty$, we get the following:

Corollary 1: *Let* $P(z)=\sum_{j=0}^{n}a_j z^j$, $a_n a_0\neq 0$ *be a polynomial of degree* $n>3$, *having all its zeros in* $|z|\le K$, $K\ge 1$, *then*

$$Max_{|z|=1}\left|P'(z)\right|\ge \frac{n}{(K^n+1)}\left\{Max_{|z|=1}|P(z)|+Min_{|z|=K}|P(z)|+\frac{2|a_{n-1}|}{(n+1)}\left[\frac{(K^n-1)}{n}-(K-1)\right]\right.$$
$$\left.+2|a_{n-2}|\left[\left\{\frac{(K^n-1)-n(K-1)}{n(n-1)}\right\}-\left\{\frac{(K^{n-2}-1)-(n-2)(K-1)}{(n-2)(n-3)}\right\}\right]\right\},\quad \text{if } n>3. \tag{8}$$

2. Lemmas

We need the following lemmas.

Lemma 1: *Let* $P(z)$ *be a polynomial of degree n, then for* $R\ge 1$.

$$Max_{|z|=R}|P(z)|\le R^n Max_{|z|=1}|P(z)|.$$

The above lemma is a simple consequence of the maximum modulus principle [8].

Lemma 2: *If* $P(z):=\sum_{j=0}^{n}a_j z^j$, $a_n\neq 0$, *is a polynomial of degree n having all its zeros in* $|z|\le 1$, *then*

$$Max_{|z|=1}|P'(z)|\ge \frac{n}{2}\left\{Max_{|z|=1}|P(z)|+Min_{|z|=1}|P(z)|\right\}.$$

This lemma is due to Aziz and Dawood [1].

Lemma 3: *If* $P(z):=\sum_{j=0}^{n}a_j z^j$ *is a polynomial of degree n having no zeros in* $|z|\le 1$, *and* $m=Min_{|z|=1}|P(z)|$, *then for* $R\ge 1$ *and* $n>3$,

$$M(P,R)\le \frac{(R^n+1)}{2}Max_{|z|=1}|P(z)|-\frac{(R^n-1)}{2}m-\frac{2|P'(0)|}{(n+1)}\left[\frac{(R^n-1)}{n}-(R-1)\right]$$
$$-|P''(0)|\left[\left\{\frac{(R^n-1)-n(R-1)}{n(n-1)}\right\}-\left\{\frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}\right\}\right].$$

The above result is a special case of a result due to Dewan, Singh and Mir [4, Theorem 1] with $K=1$ and $\mu=1$.

Remark 3: Here we note that for the proof of this result an additional hypothesis that $P(0)\neq 0$ is required. A simple counter example in this case is $P(z)=z^n$.

3. Proof of Theorem 2

Since $P(z)$ has all its zeros in $|z| \le K$, therefore $G(z) = P(Kz)$ has all its zeros in $|z| \le 1$ and hence by applying lemma 2 to the polynomial $G(z)$, we get

$$Max_{|z|=1}|G'(z)| \ge \frac{n}{2}\left\{Max_{|z|=1}|G(z)| + Min_{|z|=1}|G(z)|\right\}. \quad (9)$$

Let $H(z) = z^n \overline{G\left(\frac{1}{\bar{z}}\right)}$. Then it can be easily verified that

$$|H'(z)| = |nG(z) - zG'(z)|, \quad \text{for } |z| = 1. \quad (10)$$

The polynomial $H(z)$ has all its zeros in $|z| \ge 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$, therefore, by result of a de Bruijn [3]

$$|H'(z)| \le |G'(z)| \quad \text{for } |z| = 1 \quad (11)$$

Now for every real or complex number α with $|\alpha| \ge K$, we have

$$\left|D_{\frac{\alpha}{K}} G(z)\right| = \left|nG(z) - zG'(z) + \frac{\alpha}{K}G'(z)\right|$$

$$\ge \left|\frac{\alpha}{K}\right||G'(z)| - |nG(z) - zG'(z)|$$

For this, we get by using (10) and (11)

$$Max_{|z|=1}\left|D_{\frac{\alpha}{K}} G(z)\right| \ge \frac{|\alpha| - K}{K} Max_{|z|=1}|G'(z)| \quad (12)$$

Using (9) in (12), we get

$$Max_{|z|=1}\left|D_{\frac{\alpha}{K}} G(z)\right| \ge \frac{(|\alpha| - K)}{K}\frac{n}{2}\left\{Max_{|z|=1}|G(z)| + Min_{|z|=1}|G(z)|\right\}.$$

Replacing $G(z)$ by $P(Kz)$, we have

$$Max_{|z|=1}\left|D_{\frac{\alpha}{K}} P(Kz)\right| \ge \frac{n(|\alpha| - K)}{2K}\left\{Max_{|z|=1}|P(Kz)| + Min_{|z|=1}|P(Kz)|\right\}.$$

This gives

$$Max_{|z|=1}\left|nP(Kz) + \left(\frac{\alpha}{K} - z\right)KP'(Kz)\right| \ge \frac{n(|\alpha| - K)}{2K}\left\{Max_{|z|=1}|P(Kz)| + Min_{|z|=1}|P(Kz)|\right\}.$$

Equivalently

$$Max_{|z|=K}|D_\alpha P(z)| \ge \frac{n(|\alpha| - K)}{2K}\left\{Max_{|z|=K}|P(z)| + Min_{|z|=K}|P(z)|\right\}. \quad (13)$$

Since the polynomial P(z) has all its zeros in $|z| \le K$, $K \ge 1$. If $Q(z) = z^n P\left(\frac{1}{z}\right)$ be the reciprocal polynomial of $P(z)$. Then the polynomial $Q\left(\frac{z}{K}\right)$ has all its zeros in $|z| \ge 1$. Hence applying lemma 3 to the polynomial $Q\left(\frac{z}{K}\right)$, $K \ge 1$, we get

$$Max_{|z|=K}\left|Q\left(\frac{z}{K}\right)\right| \le \frac{(K^n + 1)}{2}Max_{|z|=1}\left|Q\left(\frac{z}{K}\right)\right| - \frac{(K^n - 1)}{2}Min_{|z|=1}\left|Q\left(\frac{z}{K}\right)\right| - \frac{2|a_{n-1}|}{(n+1)}\left[\frac{(K^n - 1)}{n} - (K-1)\right]$$
$$- 2|a_{n-2}|\left[\left\{\frac{(K^n - 1) - n(K-1)}{n(n-1)}\right\} - \left\{\frac{(K^{n-2} - 1) - (n-2)(K-1)}{(n-2)(n-3)}\right\}\right]$$

This in particular gives

$$Max_{|z|=1}|P(z)| \le \frac{(K^n + 1)}{2K^n}Max_{|z|=K}|P(z)| - \frac{(K^n - 1)}{2K^n}Min_{|z|=K}|P(z)| - \frac{2|a_{n-1}|}{(n+1)}\left[\frac{(K^n - 1)}{n} - (K-1)\right]$$
$$- 2|a_{n-2}|\left[\left\{\frac{(K^n - 1) - n(K-1)}{n(n-1)}\right\} - \left\{\frac{(K^{n-2} - 1) - (n-2)(K-1)}{(n-2)(n-3)}\right\}\right]$$

which is equivalent to

$$\begin{aligned} Max_{|z|=K}\left|P(z)\right| \geq & \frac{2K^n}{\left(K^n+1\right)}Max_{|z|=1}\left|P(z)\right| + \frac{\left(K^n-1\right)}{\left(K^n+1\right)}Min_{|z|=K}\left|P(z)\right| + \frac{4K^n}{\left(K^n+1\right)}\frac{|a_{n-1}|}{(n+1)}\left[\frac{\left(K^n-1\right)}{n}-(K-1)\right] \\ & + \frac{4K^n}{\left(K^n+1\right)}|a_{n-2}|\left[\left\{\frac{\left(K^n-1\right)-n(K-1)}{n(n-1)}\right\}-\left\{\frac{\left(K^{n-2}-1\right)-(n-2)(K-1)}{(n-2)(n-3)}\right\}\right] \end{aligned} \tag{14}$$

Using (14) in (13), we get

$$\begin{aligned} Max_{|z|=K}\left|D_\alpha P(z)\right| \geq & \frac{n\left(|\alpha|-K\right)}{2K}\{\frac{2K^n}{\left(K^n+1\right)}Max_{|z|=1}\left|P(z)\right| + \frac{\left(K^n-1\right)}{\left(K^n+1\right)}Min_{|z|=K}\left|P(z)\right| + \frac{4K^n}{\left(K^n+1\right)}\frac{|a_{n-1}|}{(n+1)}\left[\frac{\left(K^n-1\right)}{n}-(K-1)\right] \\ & + \frac{4K^n}{\left(K^n+1\right)}|a_{n-2}|\left[\left\{\frac{\left(K^n-1\right)-n(K-1)}{n(n-1)}\right\}-\left\{\frac{\left(K^{n-2}-1\right)-(n-2)(K-1)}{(n-2)(n-3)}\right\}\right] + Min_{|z|=K}\left|P(z)\right|\Bigg\}, \text{ if } n>3. \end{aligned}$$

Equivalently

$$\begin{aligned} Max_{|z|=K}\left|D_\alpha P(z)\right| \geq & \frac{n\left(|\alpha|-K\right)K^{n-1}}{\left(K^n+1\right)}\Bigg\{Max_{|z|=1}\left|P(z)\right| + Min_{|z|=K}\left|P(z)\right| + \frac{2|a_{n-1}|}{(n+1)}\left[\frac{\left(K^n-1\right)}{n}-(K-1)\right] \\ & + 2|a_{n-2}|\left[\left\{\frac{\left(K^n-1\right)-n(K-1)}{n(n-1)}\right\}-\left\{\frac{\left(K^{n-2}-1\right)-(n-2)(K-1)}{(n-2)(n-3)}\right\}\right]\Bigg\}, \text{ if } n>3. \end{aligned} \tag{15}$$

Since $D_\alpha P(z)$ is a polynomial of degree $n-1$ and $K \geq 1$, therefore by using Lemma 1, we get

$$Max_{|z|=K}\left|D_\alpha P(z)\right| \leq K^{n-1}Max_{|z|=1}\left|D_\alpha P(z)\right| \tag{16}$$

Combining (16) and (15) we have

$$\begin{aligned} Max_{|z|=1}\left|D_\alpha P(z)\right| \geq & \frac{n\left(|\alpha|-K\right)}{\left(K^n+1\right)}\Bigg\{Max_{|z|=1}\left|P(z)\right| + Min_{|z|=K}\left|P(z)\right| + \frac{2|a_{n-1}|}{(n+1)}\left[\frac{\left(K^n-1\right)}{n}-(K-1)\right] \\ & + 2\,|\,a_{n-2}\,|\left[\left\{\frac{\left(K^n-1\right)-n(K-1)}{n(n-1)}\right\}-\left\{\frac{\left(K^{n-2}-1\right)-(n-2)(K-1)}{(n-2)(n-3)}\right\}\right]\Bigg\}, \quad \text{if } n>3. \end{aligned}$$

This completes the proof of Theorem 2.

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5. References

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