

On Bicomplex Representation Methods and Applications of Matrices over Quaternionic Division Algebra*

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Abstract

In this paper, a series of bicomplex representation methods of quaternion division algebra is introduced. We present a new multiplication concept of quaternion matrices, a new determinant concept, a new inverse concept of quaternion matrix and a new similar matrix concept. Under the new concept system, many quaternion algebra problems can be transformed into complex algebra problems to express and study. These concepts can perfect the theory of [J.L. Wu, A new representation theory and some methods on quaternion division algebra, JP Journal of Algebra, 2009, 14(2): 121-140] and unify the complex algebra and quaternion division algebra.

Keywords: Quaternion Determinant, Product of Quaternion Matrix, Inverse of Quaternion Matrix, Similar Quaternion Matrix, Application, Solution

1. Introduction

In recent years, the algebra problems over quaternion division algebra have drawn the attention of mathematics and physics researchers [1-12]. Quaternion algebra theory is getting more and more important. In many fields of applied science, such as physics, figure and pattern recognition, spacecraft attitude control, 3-D animation, people start to make use of quaternion algebra theory to solve some actual problems. Therefore, it encourages people to do further research [13-17] on quaternion algebra theory and its applications.

The main obstacle in the study of quaternion algebra is the non-commutative multiplication of quaternion. Many important conclusions over real and complex fields are different from ones over quaternion division algebra, such as determinant, the trace of matrix multiplication and solutions of quaternion equation. From the conclusions on quaternion division algebra, we find it to lack for general concepts, such as the definition of quaternion matrix determinant. There are different definitions which are given in [1,3,4,6,11,18] since Dieudonne firstly introduced the quaternion determinant in 1943. In addition, the inverse of quaternion matrix has not been well de-

finied so far, because it depends on other algebra concepts. In the study of quaternion division algebra, people always expect to get some relations between quaternion division algebra and real algebra or complex algebra. However, some conclusions on real or complex fields are correct but not on quaternion division algebra. It makes us to consider establishing other algebra concept system over quaternion division algebra to unify the complex algebra and quaternion division algebra.

Recently, Wu in [19] used real representation methods to express quaternion matrices and established some new concepts over quaternion division algebra. From these definitions, we can see that they can convert quaternion division algebra problems into real algebra problems to reduce the complexity and abstraction which exist in all kinds of definitions given in [1,3,6,10,11,20]. However, as Wu in [19] mentioned, these concept system is not suitable for complex algebra.

In this paper, based on the bicomplex form of quaternion matrix, we present some new concepts to quaternion division algebra. These new concepts can perfect the theory of Wu in [19] and unify the complex algebra and quaternion division algebra.

This paper is organized as follows. In Section 2, we introduce a complex representation method of quaternion matrices and explore the relation between quaternion matrices and complex matrices. In Section 3, we present

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a series of new concepts over quaternion division algebra and study their properties. In section 4, we establish some important theorems to illustrate the applications and effectiveness of the new concept system.

Let C denote the complex field, H denote the quaternion set, $C^{m \times n}$ denote the set of $m \times n$ complex matrices, $H^{m \times n}$ denote the set of $m \times n$ quaternion matrices and A^T denote the transpose matrix of A .

2. The Bicomplex Representation Methods of Quaternion Matrices and the Relation between Quaternion Matrices and Complex Matrices

For any quaternion matrix $A \in H^{m \times n}$, A can be uniquely represented as

$$A = A_0 + A_1 j, \quad (2.1)$$

where $A_s \in C^{m \times n}$ ($s = 0, 1$), $A_1 j$ means to multiply each entries of A_1 by j from right hand side.

For above reasons, we can establish a mapping relation between quaternion matrices and complex matrices as follows:

$$f : A \in H^{m \times n} \mapsto (A_0, A_1), \quad (2.2)$$

where $A_s \in C^{m \times n}$ ($s = 0, 1$).

The set of $m \times n$ quaternion matrices is written as A and the set of image of A is written as A_{img} .

Theorem 2.1. Let $f : A \in H^{m \times n} \mapsto (A_0, A_1)$, ($A_s \in C^{m \times n}$ ($s = 0, 1$)). Then the mapping f is a bijective mapping from A to A_{img} .

Proof. For any entry (A_0, A_1) in A_{img} , there exists the corresponding quaternion matrix $A = A_0 + A_1 j$ in A , therefore f is a surjection from A to A_{img} . Simultaneously, since any quaternion matrix in A can be uniquely represented as the form (2.1), so f is an injection from A to A_{img} . Thus f is a bijective mapping from A to A_{img} .

The proof is complete.

Theorem 2.2. Bijection $f : A \leftrightarrow (A_0, A_1)$, $A_s \in C^{m \times n}$ ($s = 0, 1$) is an isomorphism mapping from A to A_{img} .

By the concept of isomorphism mapping, this theorem is easy to prove and we omit it here. \square

We shall mention that Theorem 2.2 is the foundation of this article, because isomorphism vector spaces have the same properties.

3. The Bicomplex Matrix Concept System over Quaternion Division Algebra

According to the complex representation of quaternion

matrices above, a series of new definitions of quaternion division algebra which are helpful to discuss the algebra problems on quaternion division algebra can be given as follows.

Definition 3.1. The matrix $\tilde{E} = E + E j$ is said to be a $n \times n$ unit quaternion matrix if E is a $n \times n$ unit matrix over complex field. In particular, if $n = 1$, then $\tilde{E} = E + E j = 1 + 1 j$ is said to be a unit quaternion written as a_u .

Definition 3.2. Let and $B = B_0 + B_1 j \in H^{n \times t}$ be given. The operator $A * B = A_0 B_0 + A_1 B_1 j$ (where $A_0 B_0, A_1 B_1$ are both the multiplications of complex matrices) is called the *-product of quaternion matrices A and B . In particular, if $m = n = t = 1$, then we can derive the *-product of quaternions.

Note: when $A \in C^{m \times n}$, $B \in C^{n \times t}$, then $A * B = AB$.

Under the Definition 3.1 and Definition 3.2, we give some relative properties.

For any matrix $A, B \in H^{n \times n}$, we have:

- 1) $\tilde{E} * A = A * \tilde{E} = A$, where \tilde{E} is a $n \times n$ unit quaternion matrix;
- 2) $A + B = B + A$;
- 3) $(A + B) * C = A * C + B * C$;
- 4) $(A * B)^T = B^T * A^T$;
- 5) $Tr(A * B) = Tr(B * A)$.

Similarly, we establish a new definition as follows.

Definition 3.3. Let $X \in H^{n \times 1}$ and $a \in H$ be given. Then $a * X = X * a = a_0 X_0 + a_1 X_1 j$ is called the *-product of quaternion and quaternion vector, where $X = X_0 + X_1 j$, $X_0 \in C^{n \times 1}$, $X_1 \in C^{n \times 1}$, $a = a_0 + a_1 j$, $a_0 \in C, a_1 \in C$.

Now, we introduce the following concept to quaternion division algebra.

Definition 3.4. For any quaternion matrix $A \in H^{n \times n}$ ($A = A_0 + A_1 j$), $\|A\| = |A_0| + |A_1| j$ is said to be the determinant of A , where $|\cdot|$ is the determinant of a complex matrix.

Note: when $A \in C^{n \times n}$, then $\|A\| = |A|$.

The Definition 3.3 is reasonable. First of all, the result of a quaternion matrix determinant under Definition 3.4 is a quaternion. Secondly, from Definition 3.4 we can see that it can convert the determinant of a quaternion matrix into that of complex matrices to reduce the complexity and abstraction. Finally, the new determinant has the same fundamental properties as that over complex field. That is, if A is a $n \times n$ quaternion matrix and $i \neq j$, then we have

$$1) \|A\| = \|A^T\|.$$

2) If quaternion matrix B is obtained from quaternion matrix A by interchanging two rows (or columns) of A , then $\|B\| = -\|A\|$.

3) If quaternion matrix A has a zero row (or column), then $\|A\| = \|A^T\| = 0$.

4) $\|k * A\| = k^{*n} \|A\|$, where $k^{*n} = \underbrace{k * k * \dots * k}_n$, $k \in H$.

5) If the j th row (column) of quaternion matrix A equal a multiple of the i th row (column) of the matrix, then $\|A\| = 0$.

6) Suppose that A , B and C are all $n \times n$ quaternion matrices. If all rows of B and C both equal the corresponding to rows (columns) of A except that the i th row (column) of A equal the sum of the i th of B and C , then $\|A\| = \|B\| + \|C\|$.

7) If quaternion matrix B is the $n \times n$ matrix resulting from adding a multiple of the i th row (or column) of matrix A to the j th row (or column) of matrix A , then $\|B\| = \|A\|$.

8) Let A and B be $n \times n$ quaternion matrices respectively. We have $\|A * B\| = \|A\| * \|B\|$.

Up to now, people still treat the inverse matrix concept of quaternion matrix as complex matrix, that is, if quaternion matrix A satisfies $A^{-1}A = E$ (where E is a real unit matrix), then people think that quaternion matrix A exists its inverse matrix A^{-1} . However, people pointedly ignore two questions. An issue is how to define the product of quaternion matrices A^{-1} and A . The other one is how to make a calculation of A^{-1} .

It indicates that the terminology of inverse matrix does not have a clear definition in quaternion algebra theory.

In the following, we shall give a new definition and specific computational method for the inverse of quaternion matrix.

Definition 3.5. Let $A = A_0 + A_1j \in H^{n \times n}$ be given (where A_0, A_1 both are complex matrices). If the inverse matrices of A_0 and A_1^{-1} both exist, then quaternion matrix A is said to be invertible and the inverse matrix is written as $A^{-} = A_0^{-} + A_1^{-}j$, where A_0^{-}, A_1^{-} denote the inverse of complex matrices A_0, A_1 respectively.

Note: when $A \in C^{n \times n}$, then $A^{-} = A^{-1}$.

The inverse of quaternion matrix under the new definition has the same fundamental properties as those under the traditional algebra system. It is easy to show the following facts by the new concept, namely, if a quaternion matrix A is invertible, then we have:

$$1) (A^{-})^{-} = A.$$

$$2) (A^{-})^k = (A^k)^{-} = (A_0^{-})^k + (A_1^{-})^k j, \text{ where}$$

$A^k = A * A * \dots * A$ is product of kA which is defined in Definition 3.2.

3) If A_1, A_2, \dots, A_m are all invertible quaternion matrices, then

$$(A_1 * A_2 * \dots * A_m)^{-} = A_m^{-} * A_{m-1}^{-} * \dots * A_1^{-}.$$

Obviously, by the new definition of inverse of quaternion matrix above, people can determine easily whether the inverse matrix of quaternion matrix exists or not and calculate the inverse matrix if possible.

Under the definition of inverse of quaternion matrix above, a new concept of similar quaternion matrices can be given as follows:

Definition 3.6. Let $A, B \in H^{n \times n}$, if there exists an invertible quaternion matrix P such that $A = P^{-} * B * P$, then A and B are said to be similar quaternion matrices written as $A \sim B$.

Note: when $A, B \in C^{n \times n}$, $A = P^{-} * B * P$ is equivalent to $A = P_0^{-1} B P_0$, where $P = P_0 + P_1j$, $P_0, P_1 \in C^{n \times n}$.

For similar quaternion matrices, we will deduce many important properties in the next section.

4. Some Applications of the Bicomplex Matrix Concept System

In this section, we establish some important theorems to illustrate the applications and effectiveness of the new concept system for the research of quaternion division algebra. The eigenvalue is an important issue in quaternion division algebra theory, so under the new concept system, we will study firstly the eigenvalues of quaternion matrix and the relation between eigenvalues of similar quaternion matrices in detail.

Before showing the application, we'll introduce firstly some concepts associated with eigenvalue.

Definition 4.1. For any matrix $A = (a_{ij}) \in H^{n \times n}$, if there exists nonzero quaternion vector $X \in H^{n \times 1}$ and a quaternion $\lambda = \lambda_0 + \lambda_1j$ (where λ_0, λ_1 are both complex numbers) such that $A * X = \lambda * X$, then λ is said to be the left eigenvalue of A , and X is the left eigenvector corresponding to λ .

For the sake of distinction, we call the left eigenvalue and the left eigenvector under Definition 4.1 the left quaternion eigenvalue and the left quaternion eigenvector respectively.

According to the new definition of quaternion matrix multiplication and $A * X = \lambda * X$, we can derive that $(\lambda * \tilde{E} - A) * X = 0$. Thus $f(\lambda) = \|\lambda * \tilde{E} - A\|$ is said to be the characteristic polynomial of A (where the operator $\|\cdot\|$ denotes the determinant of quaternion matrix under Definition 3.4).

Theorem 4.1. A $n \times n$ quaternion matrix $A = A_0 + A_1j$ (where A_0, A_1 both are complex matrices), if λ and μ are the left eigenvalues of A_0 and A_1 respectively, then $\lambda + aj$ and $b + \mu j$ ($\forall a \in C, \forall b \in C$) are the left quaternion eigenvalues of A .

Proof. Since λ and μ are the left eigenvalues of A_0 and A_1 respectively, then there exist nonzero vectors

$\xi \in C^{n \times 1}$ and $\eta \in C^{n \times 1}$ such that $A_0 \xi = \lambda \xi$, $A_1 \eta = \mu \eta$. We have

$$A * \xi = (A_0 + A_1 j) * (\xi + \mathbf{0} j) = A_0 \xi = \lambda \xi = (\lambda + a j) * \xi,$$

for $\forall a \in C$.

$$\begin{aligned} A * (\eta j) &= (A_0 + A_1 j) * (\mathbf{0} + \eta j) = A_1 \eta j = \mu \eta j \\ &= (b + \mu j) * (\eta j), \end{aligned}$$

for $\forall b \in C$.

So $\lambda + a j$ and $b + \mu j$ are all the left quaternion eigenvalues of A .

The proof is complete. \square

Similarly, we introduce a new right quaternion eigenvalue concept.

Definition 4.2. For any matrix $A = (a_{ij}) \in H^{n \times n}$, if there exists nonzero quaternion vector $\in H^{1 \times n}$ and quaternion $\mu = \mu_0 + \mu_1 j$ (where μ_0, μ_1 are both complex numbers) such that

$Y * A = \mu * Y$, then μ is said to be the right quaternion eigenvalue of A , and Y is the right quaternion eigenvector corresponding to μ .

For the right eigenvalue of quaternion matrix, we have the following theorem:

Theorem 4.2. A $n \times n$ quaternion matrix $A = A_0 + A_1 j$ (where A_0, A_1 are both complex matrices), if λ and μ are the right eigenvalues of A_0 and A_1 respectively, then $\lambda + a j$ and $b + \mu j$ ($\forall a \in C, \forall b \in C$) are the right quaternion eigenvalues of A .

Proof. Since λ and μ are the right eigenvalues of A_0 and A_1 respectively, then there exist nonzero vectors $\xi \in C^{1 \times n}$ and $\eta \in C^{1 \times n}$ such that $\xi A_0 = \lambda \xi$, $\eta A_1 = \mu \eta$. We have

$$\begin{aligned} \xi * A &= (\xi + \mathbf{0} j) * (A_0 + A_1 j) = \xi A_0 \\ &= \lambda \xi = (\lambda + a j) * \xi, \text{ for } \forall a \in C \end{aligned}$$

$$\begin{aligned} (\eta j) * A &= (\mathbf{0} + \eta j) * (A_0 + A_1 j) = \eta A_1 j = \mu \eta j \\ &= (b + \mu j) * (\eta j), \text{ for } \forall b \in C \end{aligned}$$

So $\lambda + a j$ and $b + \mu j$ are the right quaternion eigenvalues of A .

The proof is complete. \square

Theorem 4.3. If the left eigenvalues of A_0 are $\lambda_1, \lambda_2, \dots, \lambda_k$ and the left eigenvalues of A_1 are $\mu_1, \mu_2, \dots, \mu_m$ (where A_0, A_1 both are complex matrices), then the left quaternion eigenvalues of matrix $A = A_0 + A_1 j$ are $\{\lambda_s + a j$ or $b + \mu_t j, \forall a \in C, \forall b \in C, s = 1, \dots, k, t = 1, \dots, m\}$.

Proof. Suppose that γ is arbitrary left quaternion eigenvalue of A , then $\exists \psi \neq \mathbf{0}$, $\psi = \psi_0 + \psi_1 j \in H^{n \times 1}$, such that $A * \psi = \gamma * \psi$, that is, $\begin{cases} A_0 \psi_0 = \gamma_0 \psi_0 \\ A_1 \psi_1 = \gamma_1 \psi_1 \end{cases}$. Since

$\psi \neq \mathbf{0}$, we know that both ψ_0 and ψ_1 are not zeroes. So there are two cases as follows:

1) When $\psi \neq \mathbf{0}$, obviously, we have

$$\gamma_0 \in \{\lambda_1, \lambda_2, \dots, \lambda_k\}.$$

So, $\gamma \in \{\lambda_i + a j, i = 1, 2, \dots, k\}$.

2) When $\psi \neq \mathbf{0}$, obviously, we have

$$\gamma_1 \in \{\mu_1, \mu_2, \dots, \mu_m\}.$$

So, $\gamma \in \{b + \mu_t j, t = 1, 2, \dots, m\}$.

To sum up 1), 2) and Theorem 4.1, we can draw the conclusion.

The proof is complete. \square

Theorem 4.4. If the right eigenvalues of A_0 are $\lambda_1, \lambda_2, \dots, \lambda_k$ and the right eigenvalues of A_1 are $\mu_1, \mu_2, \dots, \mu_m$ (where A_0, A_1 both are complex matrices), then the right quaternion eigenvalues of matrix $A = A_0 + A_1 j$ are $\{\lambda_s + a j$ or $b + \mu_t j, \forall a \in C, \forall b \in C, s = 1, \dots, k, t = 1, \dots, m\}$.

This proof is similar to Theorem 4.3. So we omit it here.

Theorem 4.5. Let $A \in H^{n \times n}$, then A and A^T have the same quaternion left (right) eigenvalues.

Proof. Since $A = A_0 + A_1 j$ (where $A_0 \in C^{n \times n}, A_1 \in C^{n \times n}$), then $A^T = A_0^T + A_1^T j$. We know A_i and A_i^T have the same left (right) eigenvalues ($i = 1, 2$). By Theorem 4.3 and Theorem 4.4, we can draw the conclusion.

The proof is complete. \square

Theorem 4.6. Let $A \in H^{n \times n}$ and $\mu, \lambda \in H$ be given. If $\lambda(\mu)$ is the left (right) quaternion eigenvalue of A , then $\lambda(\mu)$ is the right (left) quaternion eigenvalue of A .

Proof. Since λ is the left quaternion eigenvalue of A , then there exists nonzero vector ξ such that $A * \xi = \lambda * \xi$. Then $(A * \xi)^T = (\lambda * \xi)^T$, we can have $\xi^T * A^T = \lambda * \xi^T$. So λ is the right quaternion eigenvalue of A^T , by Theorem 4.5, we know λ is the right quaternion eigenvalue of A . The same proof to μ .

So, the proof is complete. \square

Specially, when $A \in C^{n \times n}$, if $\lambda(\mu)$ is the left (right) eigenvalue of A , then $\lambda(\mu)$ is the right (left) eigenvalue of A .

Note: By the new definition of quaternion multiplication, the left quaternion eigenvalue of a quaternion matrix is equivalent to its right quaternion eigenvalue. So they are both called quaternion eigenvalue of the quaternion matrix.

In the following, we show an important result.

Theorem 4.7. Let $A, B \in H^{n \times n}$ be given. If $A \sim B$, then A and B have the same eigenvalues.

Proof. Since $A \sim B$, there exists an invertible matrix $P \in H^{n \times n}$ such that $A = P^{-1} * B * P$, that is equivalent to $A = P_0^{-1} B P_0$ and $A_1 = P_1^{-1} B_1 P_1$ (where $A = A_0 + A_1 j$, $B = B_0 + B_1 j$, $P = P_0 + P_1 j$). We know B_s and A_s ($s = 0, 1$) have the same eigenvalues. By Theorem 4.3 and Theorem 4.4, we can draw that A and B have

the same eigenvalues.

The proof is complete. \square

Theorem 4.8 (The generalized Cayley-Hamilton theorem over quaternion division algebra). A quaternion matrix A must be the root of its characteristic polynomial $f(\lambda) = \|\lambda * \tilde{E} - A\|$.

Proof. According to Definition 3.4, we know that:

$$\begin{aligned} f(\lambda) &= f(\lambda_0 + \lambda_1 j) = \|\lambda * \tilde{E} - A\| \\ &= \|(\lambda_0 E + \lambda_1 E j) - (A_0 + A_1 j)\| \\ &= |\lambda_0 E - A_0| + |\lambda_1 E - A_1| j = g(\lambda_0) + h(\lambda_1) j \end{aligned}$$

where $g(\lambda_0) = |\lambda_0 E - A_0|$, $h(\lambda_1) = |\lambda_1 E - A_1|$.

According to the Cayley-Hamilton theorem on complex field, we know $g(A_0) = \mathbf{0}$, $h(A_1) = \mathbf{0}$. So,

$f(A) = g(A_0) + h(A_1) = \mathbf{0}$. It indicates that quaternion matrix A must be the root of its characteristic polynomial $f(\lambda)$.

So, the proof is complete. \square

Theorem 4.9. Let (where $A = A_0 + A_1 j \in \mathbf{H}^{n \times n}$, $A_0, A_1 \in \mathbf{C}^{n \times n}$) be given. A is a diagonalizable matrix if and only if both A_0 and A_1 are diagonalizable matrices.

Proof. A is diagonalizable matrix, that is, there exists an invertible quaternion matrix P such that $A = P^{-1} * \Delta * P$. It is equivalent to $A_0 = P_0^{-1} \Delta_0 P_0$ and $A_1 = P_1^{-1} \Delta_1 P_1$ (where $\Delta = \Delta_0 + \Delta_1 j$ is diagonal matrix). So, A is diagonalizable matrix if and only if both A_0 and A_1 are diagonalizable matrices.

The proof is complete. \square

Corollary 4.9. Let $A = A_0 + A_1 j \in \mathbf{H}^{n \times n}$ (where $A_0, A_1 \in \mathbf{C}^{n \times n}$) be given. If A_0 and A_1 both have n different eigenvalues, then A is diagonalizable matrix.

Corollary 4.9'. Let $A = A_0 + A_1 j \in \mathbf{H}^{n \times n}$ (where $A_0, A_1 \in \mathbf{C}^{n \times n}$) be given. Quaternion matrix A is diagonalizable matrix if and only if and A_1 both have n linearly independent eigenvectors.

Corollary 4.9''. Let $A = A_0 + A_1 j \in \mathbf{H}^{n \times n}$ (where $A_0, A_1 \in \mathbf{C}^{n \times n}$) be given. Quaternion matrix A is diagonalizable matrix if and only if the geometric multiplicity of A_0 and A_1 both equal their algebraic multiplicity respectively.

In Section 3, we have given the new definition of the inverse of quaternion matrix, but that of quaternion is not defined. In fact, a quaternion can be treated as a 1×1 matrix. So we can define the inverse of quaternion as follows:

Definition 4.3. For any quaternion $a = a_0 + a_1 j$, if neither of a_0 and a_1 are zeroes, then $a^{-1} = a_0^{-1} + a_1^{-1} j$ is said to be the inverse of a , where $a_s^{-1} (s = 0, 1)$ is the reciprocal of a_s .

It is easy to verify the following facts. For any

$a, b \in \mathbf{H}$, we have:

- 1) $a_u * a = a * a_u = a$;
- 2) $a + b = b + a$;
- 3) $(a + b) * c = a * c + b * c$;
- 4) $a^n = (a_0)^n + (a_1)^n j$;
- 5) If $a = a_0 + a_1 j$ has the inverse a^{-1} , then $a * a^{-1} = a_u$.

In addition, we discover that there are some special phenomena about the roots of quaternion polynomial under the new definition of quaternion multiplication.

Definition 4.4. The polynomial which has the form as follows: $a_0 * x^{*0} + a_1 * x^{*(n-1)} + \dots + a_{n-1} * x^{*1} + a_n * x^{*0}$ is said to be quaternion polynomial with complex coefficients (where $a_i, i = 0, 1, \dots, n$ are all complex numbers, $x = x_0 + x_1 j$, x^{*0} is the $*$ -product of i quaternion x and x^{*0} is unit quaternion).

Theorem 4.10. Let $f(x)$ be a quaternion polynomial with complex coefficients. Then $f(x)$ has infinite quaternion roots.

Proof. By Fundamental Theorem of algebra, $f(x)$ exists at least one complex root x_0 , then for any given complex number x_1 , obviously, $x_0 + x_1 j$ is the root of $f(x)$.

The proof is complete. \square

Theorem 4.11. Let $f(x)$ be a quaternion polynomial with complex coefficients and $A = A_0 + A_1 j \in \mathbf{H}^{n \times n}$ be a given quaternion matrix (where, both A_0 and A_1 are complex matrices). If λ is the eigenvalue of A_0 , then $f(\lambda)$ is the eigenvalue of $f(A)$.

Proof. According to the new definition of quaternion multiplication, we can easily obtain $f(A) = f(A_0)$. Since λ is the eigenvalue of A_0 , so $f(\lambda)$ is the eigenvalue of $f(A_0)$.

The proof is complete. \square

Under the new concept system, we can also solve the problems of existence and uniqueness of the solutions to the quaternion system of linear equations $A * X = b$, where operator ' $*$ ' denotes the new multiplication of quaternion matrices.

As we known, for any $A \in \mathbf{H}^{m \times n}$, A can be represented uniquely as $A = A_0 + A_1 j$, where $A_s (s = 0, 1)$ are $n \times n$ complex matrices. Let

$X = (x_{10} + x_{11} j, x_{20} + x_{21} j, \dots, x_{n0} + x_{n1} j)^T$ and $b = (b_{10} + b_{11} j, b_{20} + b_{21} j, \dots, b_{n0} + b_{n1} j)^T$ be $n \times 1$ quaternion vectors, then the following theorems are valid.

Theorem 4.12. Let $A = A_0 + A_1 j \in \mathbf{H}^{n \times n}$ be given and $X = X_0 + X_1 j$ be $n \times 1$ quaternion vector. If $\text{rank}(A_s) = r_s$ and the fundamental system of solutions to the system of homogeneous linear equations $A_s X_s = \mathbf{0}$ is $\eta_{i1}, \eta_{i2}, \dots, \eta_{i(n-r_s)} (s = 0, 1)$ respectively, then any solution to the quaternion system of homogeneous linear

equations $A * X = \mathbf{0}$ can be expressed as follows:

$$X = \left(c_{01}\eta_{01} + c_{02}\eta_{02} + \cdots + c_{0(n-r_0)}\eta_{0(n-r_0)} \right) \\ + \left(c_{11}\eta_{11} + c_{12}\eta_{12} + \cdots + c_{1(n-r_1)}\eta_{1(n-r_1)} \right) j$$

where $c_{st_s} \in C$, $t_s = 0, \dots, n - r_s$, $s = 0, 1$.

Proof. By the new definition of quaternion matrix multiplication, the quaternion system of homogeneous linear equations $A * X = \mathbf{0}$ is equivalent to the system

of homogeneous linear equations $\begin{cases} A_0 X_0 = \mathbf{0} \\ A_1 X_1 = \mathbf{0} \end{cases}$. Since any

solution to the system of homogeneous linear equations $A_s X_s = \mathbf{0}$ can be expressed as

$$X_s = \left(c_{s1}\eta_{s1} + c_{s2}\eta_{s2} + \cdots + c_{s(n-r_s)}\eta_{s(n-r_s)} \right) \\ \text{(where } c_{st_s} \in C, t_s = 1, \dots, n - r_s, s = 0, 1)$$

and the solutions of the quaternion system of homogeneous linear equations $A * X = \mathbf{0}$ are $X = X_0 + X_1 j$. So we can draw the conclusion.

So, the proof is complete. \square

Corollary 4.12. Let $A = A_0 + A_1 j$ be a given quaternion matrix (where $A_s \in C^{m \times n}$, $s = 0, 1$).

If $\text{rank}(A_0) = \text{rank}(A_1) = n$, then the quaternion system of homogeneous linear equations $A * X = \mathbf{0}$ has unique solution $X = \mathbf{0} = \{0, 0, \dots, 0\}^T$.

Corollary 4.12'. Let $A = A_0 + A_1 j$ be a given quaternion matrix (where $A_s \in C^{m \times n}$, $s = 0, 1$). If $\text{rank}(A_0) < n$ and $\text{rank}(A_1) = n$, then the quaternion system of homogeneous linear equations $A * X = \mathbf{0}$ only exists complex solutions.

Theorem 4.13. Let $A = A_0 + A_1 j$ be a given quaternion matrix, $X = X_0 + X_1 j$ and $b = b_0 + b_1 j$ be quaternion vectors (where $A_s \in C^{m \times n}$, $X_s \in C^{n \times 1}$,

$$b_s = (b_{s1}, b_{s2}, \dots, b_{sn})^T, b_{st} \in C, s = 0, 1, t = 1, 2, \dots, n).$$

If there is at least one $s_0 \in \{0, 1\}$ such that $\text{rank}(A_{s_0}) < \text{rank}(A_{s_0} : b_{s_0})$, then the quaternion system of linear equations $A * X = b$ has no solution.

Proof. By the new definition of quaternion matrix multiplication, the quaternion system of linear equations $A * X = b$ is equivalent to the system of linear equations $\begin{cases} A_0 X_0 = b_0 \\ A_1 X_1 = b_1 \end{cases}$, since $\text{rank}(A_{s_0}) < \text{rank}(A_{s_0} : b_{s_0})$, the

system of linear equations $\begin{cases} A_0 X_0 = b_0 \\ A_1 X_1 = b_1 \end{cases}$ have no solution,

that is, the quaternion system of linear equations $A * X = b$ has no solution.

So, the proof is complete. \square

Theorem 4.14. Let $A = A_0 + A_1 j$ be a given quaternion matrix and $X = X_0 + X_1 j$ be a given quaternion

vector (where $A_s \in C^{m \times n}$, $X_s \in C^{n \times 1}$, $s = 0, 1$). We suppose that the fundamental system of solutions to the system of linear equations $A_s X_s = \mathbf{0}$ is $\eta_{s1}, \eta_{s2}, \dots, \eta_{s(n-r_s)}$ ($s = 0, 1$) respectively and ξ_s ($s = 0, 1$) is a special solution of the system of linear equations $A_s X_s = b_s$ respectively, and $\text{rank}(A_s) = \text{rank}(A_s : b_s)$ ($s = 0, 1$), then any solution to the quaternion system of linear equations $A * X = b$ can be expressed as:

$$X = \left(\xi_0 + c_{01}\eta_{01} + c_{02}\eta_{02} + \cdots + c_{0(n-r_0)}\eta_{0(n-r_0)} \right) \\ + \left(\xi_1 + c_{11}\eta_{11} + c_{12}\eta_{12} + \cdots + c_{1(n-r_1)}\eta_{1(n-r_1)} \right) j$$

Proof. By the new definition of quaternion matrix multiplication, the quaternion system of linear equations $A * X = b$ is equivalent to the system of linear equations

$\begin{cases} A_0 X_0 = b_0 \\ A_1 X_1 = b_1 \end{cases}$. Since any solution to the system of

linear equations $A_s X_s = b_s$ can be expressed as follows:

$$X_s = \xi_s + c_{s1}\eta_{s1} + c_{s2}\eta_{s2} + \cdots + c_{s(n-r_s)}\eta_{s(n-r_s)} \text{ (where}$$

$c_{st} \in C, t = 1, 2, \dots, n - r_s, s = 0, 1$), so any solution to the system of quaternion linear equations $A * X = b$ can be expressed as:

$$X = \left(\xi_0 + c_{01}\eta_{01} + c_{02}\eta_{02} + \cdots + c_{0(n-r_0)}\eta_{0(n-r_0)} \right) \\ + \left(\xi_1 + c_{11}\eta_{11} + c_{12}\eta_{12} + \cdots + c_{1(n-r_1)}\eta_{1(n-r_1)} \right) j$$

The proof is complete. \square

Theorem 4.15. Let $A = A_0 + A_1 j$ be a given quaternion matrix, $X = X_0 + X_1 j$ and $b = b_0 + b_1 j$ be quaternion vectors (where $A_s \in C^{m \times n}$, $X_s \in C^{n \times 1}$, $b_s \in C^{n \times 1}$, ($s = 0, 1$)). If $\text{rank}(A_s) = \text{rank}(A_s : b_s) = n$ ($s = 0, 1$), then the quaternion system of linear equations $A * X = b$ exists unique solution.

Proof. By the new definition of quaternion matrix multiplication, the quaternion system of linear equations $A * X = b$ is equivalent to the system of linear equations

$\begin{cases} A_0 X_0 = b_0 \\ A_1 X_1 = b_1 \end{cases}$, and $\text{rank}(A_s) = \text{rank}(A_s : b_s) = n$, we

know the system of linear equations $\begin{cases} A_0 X_0 = b_0 \\ A_1 X_1 = b_1 \end{cases}$ have

unique solution. So the quaternion system of linear equations $A * X = b$ exists unique solution.

The proof is complete. \square

Corollary 4.15. Let $A = A_0 + A_1 j$ be a given $n \times n$ quaternion matrix and $b = b_0 + b_1 j$ be a given $n \times 1$ quaternion vector. If $\text{rank}(A_s) = \text{rank}(A_s : b_s) = n$ ($s = 0, 1$), then the solution of the quaternion system of equations $A * X = b$ is $X = A^{-1} * b$.

Corollary 4.15'. Let $A \in C^{m \times n}$ and $b = b_0 + b_1 j$ (where $b_s \in C^{n \times 1}$, $s = 0, 1$, $b_1 \neq \mathbf{0}$) be given. Then the

quaternion system of linear equations $A * X = b$ has no solution.

Corollary 4.15". Let $A \in C^{m \times n}$ and $b \in C^{n \times 1}$ be given. If $\text{rank}(A) = \text{rank}(A:b)$, then any solution to the quaternion system of linear equations $A * X = b$ can be expressed as $X = A^{-1}b + aj$, where $a \in C$.

5. References

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