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The Pell Equation $x^2 - Dy^2 = \pm k^2$

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Abstract

Let $D \neq 1$ be a positive non-square integer and $k \geq 2$ be any fixed integer. Extending the work of A. Tekcan, here we obtain some formulas for the integer solutions of the Pell equation $x^2 - Dy^2 = \pm k^2$.

Keywords: Pell's Equation, Solutions of Pell's Equation

1. Introduction

The equation $x^2 - Dy^2 = N$, with given integers D and N and unknowns x and y, is called Pell's equation. If D is negative, it can have only a finite number of solutions. If D is a perfect square, say $D = a^2$, the equation reduces to $(x - ay)(x + ay) = N$ and again there is only a finite number of solutions. The most interesting case of the equation arises when $D \neq 1$ be a positive non-square.

Although J. Pell contributed very little to the analysis of the equation, it bears his name because of a mistake by Euler.

Pell's equation $x^2 - Dy^2 = 1$ was solved by Lagrange in terms of simple continued fractions. Lagrange was the first to prove that $x^2 - Dy^2 = 1$ has infinitly many solutions in integers if $D \neq 1$ is a fixed positive non-square integer. If the lenght of the periode of $\sqrt{D}$ is 1, all positive solutions are given by $x = P_{2vk-1}$ and $y = Q_{2vk-1}$ if k is odd, and by $x = P_{vk-1}$ and $y = Q_{vk-1}$ if k is even, where $v = 1, 2, \cdots$ and $\frac{P_n}{Q_n}$ denotes the nth convergent of the continued fraction expansion of $\sqrt{D}$ Incidentally, $x = P_{(2v-1)(k-1)}$ and $y = Q_{(2v-1)(k-1)}$, $v = 1, 2, \cdots$, are the positive solutions of $x^2 - Dy^2 = -1$ provided that 1 is odd.

There is no solution of $x^2 - Dy^2 = \pm 1$ other than $x_v, y_v : v = 1, 2, \cdots$ given by $\left(x_1 + \sqrt{D} y_1\right)^v = x_v + \sqrt{D} y_v$, where x_1, y_1 is the least positive solution called the fundamental solution, which there are different method for finding it. The reader can find many references in the subject in the book [7].

For completeness we recall that there are many papers in which are considered different types of Pell's equation. Many authors such as Tekcan [1], Kaplan and Williams [2], Matthews [3], Mollin, Poorten and Williams [4], Stevenhagen [5] and the others consider eome specific Pell equations and their integer solutions. A. Tekcan in [1], considered the equation $x^2 - Dy^2 = \pm 4$, and he obtained some formulas for its integer solutions. He mentioned two conjecture which was proved by A. S. Shabani [6]. In this paper we extend the work of A. Tekcan by considering the Pell equation $x^2 - Dy^2 = \pm k^2$ when $D \neq 1$ be a positive non-square and $k \geq 2$, we obtain some formulas for its integer solutions.

2. The Pell Equation $x^2 - Dy^2 = k^2$

In this section, we consider the solutions of Pell's equation $x^2 - Dy^2 = k^2$ when $k \geq 2$.

Theorem 2.1 *Let* (x_1, y_1) *be the fundamental solution of the Pell equation* $x^2 - Dy^2 = k^2$*, and let*

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1}$$

for $n \geq 1$. Then the integer solutions of the Pell equation $x^2 - Dy^2 = k^2$ are (x_n, y_n), where

$$(x_n, y_n) = \left(\frac{u_n}{k^{n-1}}, \frac{v_n}{k^{n-1}} \right) \tag{2}$$

Proof. We prove the theorem using the method of mathematical induction. For $n = 1$, we have from (1), $(u_1, v_1) = (x_1, y_1)$ which is the fundamental solution of $x^2 - Dy^2 = k^2$. Now, we assume that the Pell equation $x^2 - Dy^2 = k^2$ is satisfied for (x_{n-1}, y_{n-1}), *i.e.*

$$x_{n-1}^2 - Dy_{n-1}^2 = \frac{u_{n-1}^2 - Dv_{n-1}^2}{k^{2n-4}} = k^2 \quad (3)$$

and we show that it holds for (x_n, y_n).

Indeed, by (1), it is easy to prove that

$$\begin{cases} u_n = x_1 u_{n-1} + Dy_1 v_{n-1} \\ v_n = y_1 u_{n-1} + x_1 v_{n-1} \end{cases}. \quad (4)$$

Hence,

$$\begin{aligned} x_n^2 - Dy_n^2 &= \frac{u_n^2 - Dv_n^2}{k^{2n-2}} \\ &= \frac{(x_1 u_{n-1} + Dy_1 v_{n-1})^2 - D(y_1 u_{n-1} + x_1 v_{n-1})^2}{k^{2n-2}} \\ &= \frac{x_1^2 u_{n-1}^2 + 2x_1 u_{n-1} Dy_1 v_{n-1} + D^2 y_1^2 v_{n-1}^2}{k^{2n-2}} \\ &\quad - \frac{D\left(y_1^2 u_{n-1}^2 + 2y_1 u_{n-1} x_1 v_{n-1} + x_1^2 v_{n-1}^2\right)}{k^{2n-2}} \\ &= \frac{x_1^2\left(u_{n-1}^2 - Dv_{n-1}^2\right) - Dy_1^2\left(u_{n-1}^2 - Dv_{n-1}^2\right)}{k^{2n-2}} \\ &= \left(x_1^2 - Dy_1^2\right)\frac{\left(u_{n-1}^2 - Dv_{n-1}^2\right)}{k^{2n-2}} \end{aligned}$$

Applying (3), it is easily seen that

$$u_{n-1}^2 - Dv_{n-1}^2 = k^{2n-4} k^2 = k^{2n-2}.$$

hence we conclude that

$$x_n^2 - Dy_n^2 = \left(x_1^2 - Dy_1^2\right) = k^2.$$

Therefore (x_n, y_n) is also a solution of the Pell equation $x^2 - Dy^2 = k^2$. Since n is arbitrary, we get all integer solutions of the Pell equation $x^2 - Dy^2 = k^2$.

Corollary 2.2 *Let* (x_1, x_2) *is the fundamental solution of the Pell equation* $x^2 - Dy^2 = k^2$*, then*

$$x_n = \frac{x_1 x_{n-1} + Dy_1 y_{n-1}}{k}, \quad y_n = \frac{y_1 x_{n-1} + x_1 y_{n-1}}{k} \quad (5)$$

and

$$\begin{vmatrix} x_n & x_{n-1} \\ x_{n-1} & y_{n-1} \end{vmatrix} = -ky_1. \quad (6)$$

Proof. By (1), we have $u_n = x_1 u_{n-1} + Dy_1 v_{n-1}$ and $v_n = y_1 u_{n-1} + x_1 v_{n-1}$ by (2), we have $u_n = k^{n-1} x_n$ and $v_n = k^{n-1} y_n$. We get

$$u_n = x_1 u_{n-1} + Dy_1 v_{n-1},$$

then,

$$k^{n-1} x_n = x_1 k^{k-2} x_{n-1} + Dy_1 k^{n-2} y_{n-1}$$

witch gives

$$x_n = \frac{x_1 x_{n-1} + Dy_1 y_{n-1}}{k}.$$

In the other hand, we have

$$v_n = y_1 u_{n-1} + x_1 v_{n-1},$$

so

$$k^{n-1} y_n = y_1 k^{k-2} x_{n-1} + x_1 k^{n-2} y_{n-1},$$

witch implies

$$y_n = \frac{y_1 x_{n-1} + x_1 y_{n-1}}{k}.$$

and hence

$$\begin{aligned} \begin{vmatrix} x_n & x_{n-1} \\ y_n & y_{n-1} \end{vmatrix} &= x_n y_{n-1} - y_n x_{n-1} \\ &= \frac{x_1 x_{n-1} + Dy_1 y_{n-1}}{k} y_{n-1} - \frac{y_1 x_{n-1} + x_1 y_{n-1}}{k} x_{n-1} \\ &= \frac{x_1 x_{n-1} y_{n-1} + Dy_1 y_{n-1}^2 - y_1 x_{n-1}^2 - x_1 x_{n-1} y_{n-1}}{k} \\ &= \frac{-y_1(x_{n-1}^2 - Dy_{n-1}^2)}{k} \\ &= -ky_1. \end{aligned}$$

Theorem 2.3 *Let* (x_1, y_1) *be the fundamental solution of the Pell equation* $x^2 - Dy^2 = k^2$*, then* (x_n, y_n) *satisfy the following recurrence relations*

$$\begin{cases} x_n = \left(\frac{2}{k} x_1 - 1\right)\left(x_{n-1} + x_{n-2}\right) - x_{n-3} \\ y_n = \left(\frac{2}{k} x_1 - 1\right)\left(y_{n-1} + y_{n-2}\right) - y_{n-3} \end{cases} \quad (7)$$

for $n \geq 4$.

Proof. The proof will be by induction on n. Using (5), we have

$$\begin{aligned} x_2 &= \frac{x_1^2 + Dy_1^2}{k} = \frac{x_1^2 + x_1^2 - k^2}{k} = \frac{2}{k} x_1^2 - k \\ y_2 &= \frac{2}{k} x_1 y_1 \end{aligned} \quad (8)$$

Using (5) and (8), we get

$$\begin{aligned} x_3 &= \frac{x_1 x_2 + Dy_1 y_2}{k} = \frac{x_1\left(\frac{2}{k} x_1^2 - k\right) + Dy_1^2 \frac{2}{k} x_1}{k} \\ &= \frac{x_1\left[\frac{2}{k}\left(x_1^2 + Dy_1^2\right) - k\right]}{k} = \frac{x_1\left[\frac{2}{k}\left(2x_1^2 - k^2\right) - \frac{2}{k}\frac{k^2}{2}\right]}{k} \\ &= \frac{x_1\left[\frac{2}{k}\left(2x_1^2 - \frac{3k^2}{2}\right)\right]}{k} = x_1\left(\frac{4}{k^2} x_1^2 - 3\right) \end{aligned}$$

$$y_3 = \frac{y_1 x_2 + x_1 y_2}{k} = \frac{y_1\left(\frac{2}{k} x_1^2 - k\right) + \frac{2}{k} x_1^2 y_1}{k} = y_1\left(\frac{4}{k^2} x_1^2 - 1\right) \quad (9)$$

Then by (5) and (9), we find x_4 and y_4.

$$x_4 = \frac{x_1x_3 + Dy_1y_3}{k}$$
$$= \frac{x_1^2\left(\frac{4}{k^2}x_1^2 - 3\right) + Dy_1^2\left(\frac{4}{k^2}x_1^2 - 1\right)}{k}$$
$$= \frac{x_1^2\left(\frac{4}{k^2}x_1^2 - 3\right) + (x_1^2 - k^2)\left(\frac{4}{k^2}x_1^2 - 1\right)}{k}$$
$$= \frac{8}{k^3}x_1^4 - \frac{8}{k}x_1^2 + k$$
$$y_4 = \frac{y_1x_3 + x_1y_3}{k}$$
$$= \frac{y_1x_1\left(\frac{4}{k^2}x_1^2 - 3\right) + x_1y_1\left(\frac{4}{k^2}x_1^2 - 1\right)}{k}$$
$$= x_1y_1\left(\frac{8}{k^3}x_1^2 - \frac{4}{k}\right)$$

So, we obtained

$$\begin{cases} x_4 = \frac{8}{k^3}x_1^4 - \frac{8}{k}x_1^2 + k \\ y_4 = x_1y_1\left(\frac{8}{k^3}x_1^2 - \frac{4}{k}\right) \end{cases} \tag{10}$$

Now, replacing (8) and (9) in (7), one obtains

$$x_4 = \left(\frac{2}{k}x_1 - 1\right)(x_3 + x_2) - x_1$$
$$= \left(\frac{2}{k}x_1 - 1\right)\left[x_1\left(\frac{4}{k^2}x_1^2 - 3\right) + \left(\frac{2}{k}x_1^2 - k\right)\right] - x_1$$
$$= \left(\frac{2}{k}x_1 - 1\right)\left(\frac{4}{k^2}x_1^3 - 3x_1 + \frac{2}{k}x_1^2 - k\right) - x_1$$
$$= \frac{8}{k^3}x_1^4 - \frac{8}{k}x_1^2 + k.$$

and

$$y_4 = \left(\frac{2}{k}x_1 - 1\right)(y_3 + y_2) - y_1$$
$$= \left(\frac{2}{k}x_1 - 1\right)\left[y_1\left(\frac{4}{k^2}x_1^2 - 1\right) + \frac{2}{k}x_1y_1\right] - y_1$$
$$= x_1y_1\left(\frac{8}{k^3}x_1^2 - \frac{4}{k}\right).$$

which are the same formulas as in (10). Therefore (7) holds for $n = 4$.

Now, we assume that (7) holds for $n \geq 4$ and we show that it holds for $n+1$.

Indeed, by (5) and by hypothesis we have

$$x_{n+1} = \frac{x_1x_n + Dy_1y_n}{k}$$
$$= \frac{x_1\left[\left(\frac{2}{k}x_1 - 1\right)(x_{n-1} + x_{n-2}) - x_{n-3}\right]}{k}$$
$$+ \frac{Dy_1\left[\left(\frac{2}{k}x_1 - 1\right)(y_{n-1} + y_{n-2}) - y_{n-3}\right]}{k}$$
$$= \left(\frac{2}{k}x_1 - 1\right)\left[\frac{x_1x_{n-1} + Dy_1y_{n-1}}{k} + \frac{x_1x_{n-2} + Dy_1y_{n-2}}{k}\right]$$
$$- \frac{x_1x_{n-3} + Dy_1y_{n-3}}{k}$$
$$= \left(\frac{2}{k}x_1 - 1\right)(x_n + x_{n-1}) - x_{n-2}.$$

$$y_{n+1} = \frac{y_1x_n + x_1y_n}{k}$$
$$= \frac{y_1\left[\left(\frac{2}{k}x_1 - 1\right)(x_{n-1} + x_{n-2}) - x_{n-3}\right]}{k}$$
$$+ \frac{x_1\left[\left(\frac{2}{k}x_1 - 1\right)(y_{n-1} + y_{n-2}) - y_{n-3}\right]}{k}$$
$$= \left(\frac{2}{k}x_1 - 1\right)\left[\frac{y_1x_{n-1} + x_1y_{n-1}}{k} + \frac{y_1x_{n-2} + x_1y_{n-2}}{k}\right]$$
$$- \frac{y_1x_{n-3} + x_1y_{n-3}}{k}$$
$$= \left(\frac{2}{k}x_1 - 1\right)(y_n + y_{n-1}) - y_{n-2}$$

completing the proof.

3. The Negative Pell Equation $x^2 - Dy^2 = -k^2$

Theorem 3.1 *Let* (x_1, y_1) *be the fundamental solution of the Pell equation* $x^2 - Dy^2 = -k^2$, *then the other solutions are* (x_{2n+1}, y_{2n+1}), *where*

$$(x_{2n+1}, y_{2n+1}) = \left(\frac{u_{2n+1}}{k^{2n}}, \frac{v_{2n+1}}{k^{2n}}\right), \tag{11}$$

for $n \geq 0$.

Proof. We prove the theorem using the method of mathematical induction. For $n = 0$, we have from (11), $(u_1, v_1) = (x_1, y_1)$ which is the fundamental solution of $x^2 - Dy^2 = -k^2$. Now, we assume that the Pell equation $x^2 - Dy^2 = -k^2$ is satisfied for $n \geq 0$. So, (x_{2n+1}, y_{2n+1}), *i.e.*

$$x_{2n+1}^2 - Dy_{2n+1}^2 = \frac{u_{2n+1}^2 - Dv_{2n+1}^2}{k^{4n}} = -k^2 \quad (12)$$

and we show that it holds for $n+1$.

Indeed, by (1), it is easily to seen that

$$\begin{pmatrix} u_{2n+3} \\ v_{2n+3} \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n+3} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2} \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} x_1^2 + Dy_1^2 & 2Dx_1y_1 \\ 2x_1y_1 & x_1^2 + Dy_1^2 \end{pmatrix} \begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (13)$$
$$= \begin{pmatrix} \left(x_1^2 + Dy_1^2\right) u_{2n+1} + 2Dx_1y_1v_{2n+1} \\ 2x_1y_1u_{2n+1} + \left(x_1^2 + Dy_1^2\right) v_{2n+1} \end{pmatrix}$$

Hence, by (*), we have $(x_{2n+2})^4 - D(y_{2n+2})^4 = -k^4$

Therefore $\left(x_{2(n+1)+1}, y_{2(n+1)+1}\right) = \left(x_{2n+3}, y_{2n+3}\right)$ is also a solution of the Pell equation $x^2 - Dy^2 = -k^2$. Since n is arbitrary, we get all integer solutions of the Pell equation $x^2 - Dy^2 = -k^2$.

Corollary 3.2 *Let* (x_1, x_2) *is the fundamental solution of the Pell equation* $x^2 - Dy^2 = -k^2$, *then*

$$x_{2n+1} = \frac{(x_1^2 + Dy_1^2)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{k^2},$$
$$y_{2n+1} = \frac{2x_1y_1x_{2n-1} + (x_1^2 + Dy_1^2)y_{2n-1}}{k^2} \quad (14)$$

and

$$\begin{vmatrix} x_{2n+1} & x_{2n-1} \\ y_{2n+1} & y_{2n-1} \end{vmatrix} = 2x_1y_1. \quad (15)$$

Proof. Using (1), we have $u_{2n+1} = \left(x_1^2 + Dy_1^2\right) u_{2n-1} + 2Dx_1y_1v_{2n-1}$ and $v_{2n+1} = 2x_1y_1u_{2n-1} + \left(x_1^2 + Dy_1^2\right) v_{2n-1}$. By (11), we have $u_{2n+1} = k^{2n}x_{2n+1}$ and $v_{2n+1} = k^{2n}y_{2n+1}$. We get

$$u_{2n+1} = \left(x_1^2 + Dy_1^2\right) u_{2n-1} + 2Dx_1y_1v_{2n-1}$$

then,

$$k^{2n}x_{2n+1} = \left(x_1^2 + Dy_1^2\right) k^{2n-2}x_{2n-1} + 2Dx_1y_1k^{2n-2}y_{2n-1}$$

witch gives

$$x_{2n+1} = \frac{\left(x_1^2 + Dy_1^2\right) x_{2n-1} + 2Dx_1y_1y_{2n-1}}{k^2}.$$

In the other hand, we have

$$v_{2n+1} = 2x_1y_1u_{2n-1} + \left(x_1^2 + Dy_1^2\right) v_{2n-1},$$

so

$$k^{2n}y_{2n+1} = 2x_1y_1k^{2n-2}x_{2n-1} + \left(x_1^2 + Dy_1^2\right) k^{2n-2}y_{2n-1},$$

witch implies

$$y_{2n+1} = \frac{2x_1y_1x_{2n-1} + \left(x_1^2 + Dy_1^2\right) y_{2n-1}}{k^2}.$$

and hence

$$\begin{vmatrix} x_{2n+1} & x_{n-1} \\ y_{2n+1} & y_{n-1} \end{vmatrix} = x_{2n+1}y_{2n-1} - y_{2n+1}x_{2n-1}$$
$$= \frac{\left(x_1^2 + Dy_1^2\right) x_{2n-1} + 2Dx_1y_1y_{2n-1}}{k^2} y_{2n-1} - \frac{2x_1y_1x_{2n-1} + \left(x_1^2 + Dy_1^2\right) y_{2n-1}}{k^2} x_{2n-1}$$
$$= 2x_1y_1 \frac{Dy_{2n-1}^2 - x_{2n-1}^2}{k^2}$$
$$= 2x_1y_1 \frac{-\left(-k^2\right)}{k^2}$$
$$= 2x_1y_1.$$

$$(*) \quad x_{2n+3}^2 - Dy_{2n+3}^2 = \frac{u_{2n+3}^2 - Dv_{2n+3}^2}{k^{4n+4}} = \frac{\left(\left(x_1^2 + Dy_1^2\right) u_{2n+1} + 2Dx_1y_1v_{2n+1}\right)^2}{k^{4n+4}} - \frac{D\left(2x_1y_1u_{2n+1} + \left(x_1^2 + Dy_1^2\right) v_{2n+1}\right)^2}{k^{4n+4}}$$
$$= \frac{\left(x_1^2 + Dy_1^2\right)^2 u_{2n+1}^2 + 4Dx_1y_1\left(x_1^2 + Dy_1^2\right) u_{2n+1}v_{2n+1} + 4D^2x_1^2y_1^2v_{2n+1}^2}{k^{4n+4}}$$
$$- \frac{\left(x_1^2 + Dy_1^2\right)^2 v_{2n+1}^2 + 4Dx_1y_1\left(x_1^2 + Dy_1^2\right) u_{2n+1}v_{2n+1} + 4Dx_1^2y_1^2u_{2n+1}^2}{k^{4n+4}}$$
$$= \left(\left(x_1^2 + Dy_1^2\right)^2 - 4Dx_1^2y_1^2\right) \frac{u_{2n+1}^2 - Dv_{2n+1}^2}{k^{4n+4}} = \left(x_1^2 - Dy_1^2\right)^2 \frac{u_{2n+1}^2 - Dv_{2n+1}^2}{k^{4n+4}}$$
$$= \left(-k^2\right)^2 \frac{u_{2n+1}^2 - Dv_{2n+1}^2}{k^{4n}} \frac{1}{k^4} = k^4\left(-k^2\right)\frac{1}{k^4} = -k^2$$

Theorem 3.3 *Let* (x_1, y_1) *be the fundamental solution of the Pell equation* $x^2 - Dy^2 = -k^2$, *then* (x_n, y_n) *satisfy the following recurrence relations*

$$\begin{cases} x_{2n+1} = \left(\frac{4}{k^2}x_1^2+1\right)\left(x_{2n-1}+x_{2n-3}\right)-x_{2n-5} \\ y_{2n+1} = \left(\frac{4}{k^2}x_1^2+1\right)\left(y_{2n-1}+y_{2n-3}\right)-y_{2n-5} \end{cases} \tag{16}$$

for $n \geq 3$.

Proof. The proof will be by induction on n. Using (14), we have

$$x_3 = \frac{\left(x_1^2+Dy_1^2\right)x_1+2Dx_1y_1^2}{k^2} = \frac{x_1\left(x_1^2+3Dy_1^2\right)}{k^2} = \frac{x_1\left(x_1^2+3x_1^2+3k^2\right)}{k^2} = x_1\left(\frac{4}{k^2}x_1^2+3\right) \tag{17}$$

$$y_3 = \frac{2x_1^2y_1+\left(x_1^2+Dy_1^2\right)y_1}{k^2} = \frac{y_1\left(2x_1^2+x_1^2+Dy_1^2\right)}{k^2} = \frac{y_1\left(4x_1^2+k^2\right)}{k^2} = y_1\left(\frac{4}{k^2}x_1^2+1\right) \tag{18}$$

Using (14), (17) and (18), we get

$$x_5 = \frac{\left(x_1^2+Dy_1^2\right)x_3+2Dx_1y_1y_3}{k^2} = \frac{\left(x_1^2+Dy_1^2\right)x_1\left(\frac{4}{k^2}x_1^2+3\right)+2Dx_1y_1^2\left(\frac{4}{k^2}x_1^2+1\right)}{k^2} = \frac{4}{k^2}x_1\left(\frac{4}{k^2}x_1^4+5x_1+5\frac{k^2}{4}\right). \tag{19}$$

$$y_5 = \frac{2x_1y_1x_3+\left(x_1^2+Dy_1^2\right)y_3}{k^2} = \frac{2x_1^2y_1\left(\frac{4}{k^2}x_1^2+3\right)+\left(x_1^2+Dy_1^2\right)y_1\left(\frac{4}{k^2}x_1^2+1\right)}{k^2} = \frac{4}{k^2}y_1\left(\frac{4}{k^2}x_1^4+3x_1^2+\frac{k^2}{4}\right). \tag{20}$$

Then by (19) and (20), we find x_7 and y_7.

$$x_7 = \frac{\left(x_1^2+Dy_1^2\right)x_5+2Dx_1y_1y_5}{k^2} = \frac{\left(x_1^2+Dy_1^2\right)\left[\frac{4}{k^2}x_1\left(\frac{4}{k^2}x_1^4+5x_1+5\frac{k^2}{4}\right)\right]}{k^2} + \frac{2Dx_1y_1\left[\frac{4}{k^2}y_1\left(\frac{4}{k^2}x_1^4+3x_1^2+\frac{k^2}{4}\right)\right]}{k^2} = \frac{4}{k^2}x_1\left(\frac{16}{k^4}x_1^6+\frac{28}{k^2}x_1^4+14x_1^2+7\frac{k^2}{4}\right)$$

$$y_7 = \frac{2x_1y_1x_5+\left(x_1^2+Dy_1^2\right)y_5}{k^2} = \frac{2x_1y_1\left[\frac{4}{k^2}x_1\left(\frac{4}{k^2}x_1^4+5x_1+5\frac{k^2}{4}\right)\right]}{k^2} + \frac{\left(x_1^2+Dy_1^2\right)\left[\frac{4}{k^2}y_1\left(\frac{4}{k^2}x_1^4+3x_1^2+\frac{k^2}{4}\right)\right]}{k^2} = \frac{4}{k^2}y_1\left(\frac{16}{k^4}x_1^6+5\frac{4}{k^2}x_1^4+6x_1^2+\frac{k^2}{4}\right)$$

So, we obtained

$$\begin{cases} x_7 = \frac{4}{k^2}x_1\left(\frac{16}{k^4}x_1^6+\frac{28}{k^2}x_1^4+14x_1^2+7\frac{k^2}{4}\right) \\ y_7 = \frac{4}{k^2}y_1\left(\frac{16}{k^4}x_1^6+5\frac{4}{k^2}x_1^4+6x_1^2+\frac{k^2}{4}\right) \end{cases} \tag{21}$$

Now, replacing (17), (18), (19) and (20) in (16), one obtains

$$x_7 = \left(\frac{4}{k^2}x_1^2+1\right)\left(x_5+x_3\right)-x_1 = \left(\frac{4}{k^2}x_1^2+1\right)\left[\frac{4}{k^2}x_1\left(\frac{4}{k^2}x_1^4+5x_1+5\frac{k^2}{4}\right)+x_1\left(\frac{4}{k^2}x_1^2+3\right)\right]-x_1$$

$$= \frac{4}{k^2}x_1\left(\frac{16}{k^4}x_1^6+\frac{28}{k^2}x_1^4+14x_1^2+7\frac{k^2}{4}\right)$$

and

$$y_7=\left(\frac{4}{k^2}x_1^2+1\right)\left(y_5+y_3\right)-y_1=\left(\frac{4}{k^2}x_1^2+1\right)\left[\frac{4}{k^2}y_1\left(\frac{4}{k^2}x_1^4+3x_1^2+\frac{k^2}{4}\right)+y_1\left(\frac{4}{k^2}x_1^2+1\right)\right]-y_1$$
$$=\frac{4}{k^2}y_1\left(\frac{16}{k^4}x_1^6+5\frac{4}{k^2}x_1^4+6x_1^2+\frac{k^2}{4}\right)$$

which are the same formulas as in (21). Therefore (16) holds for $n=3$

Now, we assume that (16) holds for $n\geq 3$ and we show that it holds for $n+1$.

Indeed, by (14) and by hypothesis we have

$$x_{2n+3}=\frac{\left(x_1^2+Dy_1^2\right)x_{2n+1}+2Dx_1y_1y_{2n+1}}{k^2}$$
$$=\frac{\left(x_1^2+Dy_1^2\right)\left[\left(\frac{4}{k^2}x_1^2+1\right)\left(x_{2n-1}+x_{2n-3}\right)-x_{2n-5}\right]}{k^2}+\frac{2Dx_1y_1\left[\left(\frac{4}{k^2}x_1^2+1\right)\left(y_{2n-1}+y_{2n-3}\right)-y_{2n-5}\right]}{k^2}$$
$$=\left(\frac{4}{k^2}x_1^2+1\right)\frac{\left(x_1^2+Dy_1^2\right)\left(x_{2n-1}+x_{2n-3}\right)+2Dx_1y_1\left(y_{2n-1}+y_{2n-3}\right)}{k^2}-\frac{\left(x_1^2+Dy_1^2\right)x_{2n-5}+2Dx_1y_1y_{2n-5}}{k^2}$$
$$=\left(\frac{4}{k^2}x_1^2+1\right)\frac{\left(x_1^2+Dy_1^2\right)x_{2n-1}+2Dx_1y_1y_{2n-1}}{k^2}+\left(\frac{4}{k^2}x_1^2+1\right)\frac{\left(x_1^2+Dy_1^2\right)x_{2n-3}+2Dx_1y_1y_{2n-3}}{k^2}$$
$$-\frac{\left(x_1^2+Dy_1^2\right)x_{2n-5}+2Dx_1y_1y_{2n-5}}{k^2}-\frac{\left(x_1^2+Dy_1^2\right)x_{2n-5}+2Dx_1y_1y_{2n-5}}{k^2}$$

$$y_{2n+3}=\frac{2x_1y_1x_{2n+1}+\left(x_1^2+Dy_1^2\right)y_{2n+1}}{k^2}$$
$$=\frac{2x_1y_1\left[\left(\frac{4}{k^2}x_1^2+1\right)\left(x_{2n-1}+x_{2n-3}\right)-x_{2n-5}\right]}{k^2}+\frac{\left(x_1^2+Dy\right)_1^2\left[\left(\frac{4}{k^2}x_1^2+1\right)\left(x_{2n-1}+x_{2n-3}\right)-x_{2n-5}\right]}{k^2}$$
$$=\left(\frac{4}{k^2}x_1^2+1\right)\frac{2x_1y_1x_{2n-1}+\left(x_1^2+Dy_1^2\right)y_{2n-1}}{k^2}+\left(\frac{4}{k^2}x_1^2+1\right)\frac{2x_1y_1x_{2n-3}+\left(x_1^2+Dy_1^2\right)y_{2n-3}}{k^2}$$
$$-\frac{2x_1y_1x_{2n-5}+\left(x_1^2+Dy_1^2\right)y_{2n-5}}{k^2}=\left(\frac{4}{k^2}x_1^2+1\right)\left(y_{2n-1}+y_{2n-3}\right)-y_{2n-5}.$$

completing the proof.

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5. References

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