Applied Mathematics, 2019, 10, 817-825
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Approximation of Functions by Quadratic Mapping in (β, p)-Banach Space

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How to cite this paper: Chi, X.J., Bao L.Y., and Wang, L.G. (2019) Approximation of Functions by Quadratic Mapping in (β, p)-Banach Space. *Applied Mathematics*, **10**, 817-825.
https://doi.org/10.4236/am.2019.1010058

Received: August 5, 2019
Accepted: September 24, 2019
Published: September 27, 2019

Abstract

In this paper, we study the functions with values in (β, p)-Banach spaces which can be approximated by a quadratic mapping with a given error.

Keywords

Hyers-Ulam-Rassias Stability, Quadratic Mapping, (β, p)-Banach Space

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940 concerning the stability of group homomorphisms.

Give a group $(G_1,*)$ *and a metric group* $(G_2,\cdot,d)$ *with the metric* $d(\cdot,\cdot)$. *Given* $\varepsilon>0$, *does there exist a* $\delta>0$ *such that if* $f:G_1\to G_2$ *satisfies* $d(f(x*y),f(x)\cdot f(y))<\delta$ *for all* $x,y\in G_1$, *then there is a homomorphism* $g:G_1\to G_2$ *with* $d(f(x),g(x))<\varepsilon$ *for all* $x\in G_1$?

Hyers [2] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers's Theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias has provided a lot of influence in the development of what we call generalized Hyers-Ulam-Rassias stability of functional equations. Beginning around 1980, the stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [5]-[18]).

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called the quadratic functional equation. Every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers-Ulam stability for quadratic functional equation was first proved by Skof [5] for mappings acting between a normed space and a Banach space. P. W. Cholewa [6] showed that Skof's Theorem is also valid if the normed space is replaced with an abelian group.

Now we recall some basic facts concerning (β, p)-Banach spaces. We fixed real numbers β with $0<\beta \le 1$ and p with $0<p\le 1$. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Let X be linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following conditions:

(i) $\|x\| \ge 0, \forall x \in X$; $\|x\|=0$ if and only if $x=0$;

(ii) $\|\lambda x\| = |\lambda|^{\beta}\|x\|, \forall x\in X, \beta \in K$;

(iii) There is a constant $K\ge 1$ such that $\|x+y\| \le K(\|x\|+\|y\|), \forall x,y\in X$.

The pair $(X,\|\cdot\|)$ is called a quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the module of concavity of $\|\cdot\|$. A quasi-β-Banach space is a complete quasi-β-normed space.

A quasi-β-norm $\|\cdot\|$ is called a (β,p)-norm if $\|x+y\|^p \le \|x\|^p + \|y\|^p$ for all $x\in X$. In this case, a quasi-(β,p)-Banach space is called a (β,p)-Banach space. For more details and related stability results on (β,p)-Banach spaces, we refer to [19] [20]. Recently, L. Găvruta and P. Găvruta [21] studied the approximation of functions in Banach space. In this paper, we will consider this problem in (β,p)-Banach spaces and extend previous result for quadratic functional equations.

2. Main Results

Given $0<\beta\le 1$ and $0<p\le 1$. Throughout this paper we always assume that X is a linear space, Y is a (β,p)-Banach space and $f: X\to Y$ is a mapping.

Definition 2.1. Let $f: X\to Y$ be a mapping. We say f is Φ-approximable by a quadratic map if there exists a quadratic mapping $Q: X\to Y$ such that

$$\|f(x)-Q(x)\| \le \Phi(x) \tag{1}$$

for all $x\in X$. In this case, we say that Q is the quadratic Φ-approximation of f.

The following result is our main result in this paper.

Theorem 2.2. Let $V_1=\left\{\Phi : X\to \mathbb{R}_+ : \lim_{n\to\infty} 4^{n\beta p}\Phi^p\left(\frac{1}{2^n}x\right)=0, \forall x\in X\right\}$ and suppose $\Phi\in V_1$. Then f is Φ-approximable by a quadratic map if and only if the following two condition hold:

(i) $\lim_{n\to\infty} 4^{n\beta p}\left\|f\left(\frac{1}{2^n}x+\frac{1}{2^n}y\right)+f\left(\frac{1}{2^n}x-\frac{1}{2^n}y\right)-2f\left(\frac{1}{2^n}x\right)-2f\left(\frac{1}{2^n}y\right)\right\|^p=0$, $x,y\in X$;

(ii) There exists $\Psi\in V_1$ such that

$$\left\|f\left(\frac{1}{2^n}x\right)-\frac{1}{4^n}f(x)\right\|^p \le \Psi^p\left(\frac{1}{2^n}x\right)+\frac{1}{4^{n\beta p}}\Phi^p(x), x\in X.$$

In this case, the quadratic Φ-approximation of f is unique and is given by

$$Q(x)=\lim_{n\to\infty}4^{n}f\left(\frac{1}{2^{n}}x\right)$$

for all $x\in X$.

Proof. We first assume that f is Φ-approximable by a quadratic map. Then for $x,y\in X$, we have

$$\left\|f(x+y)-Q(x+y)\right\|\le\Phi(x+y)$$

and

$$\left\|f(x-y)-Q(x-y)\right\|\le\Phi(x-y).$$

It follows that

$$\begin{aligned}&\left\|f(x+y)+f(x-y)-2f(x)-2f(y)\right\|^{p}\\&\le\left\|f(x+y)-Q(x+y)\right\|^{p}+\left\|f(x-y)-Q(x-y)\right\|^{p}\\&\quad+\left\|2f(x)-2Q(x)\right\|^{p}+\left\|2f(y)-2Q(y)\right\|^{p}\\&\le\Phi^{p}(x+y)+\Phi^{p}(x-y)+2^{\beta p}\Phi^{p}(x)+2^{\beta p}\Phi^{p}(y)\end{aligned}$$

for all $x,y\in X$. Hence

$$\begin{aligned}&4^{n\beta p}\left\|f\left(\frac{1}{2^{n}}x+\frac{1}{2^{n}}y\right)+f\left(\frac{1}{2^{n}}x-\frac{1}{2^{n}}y\right)-2f\left(\frac{1}{2^{n}}x\right)-2f\left(\frac{1}{2^{n}}y\right)\right\|^{p}\\&\le4^{n\beta p}\Phi^{p}\left(\frac{1}{2^{n}}x+\frac{1}{2^{n}}y\right)+4^{n\beta p}\Phi^{p}\left(\frac{1}{2^{n}}x-\frac{1}{2^{n}}y\right)\\&\quad+4^{n\beta p}\cdot2^{\beta p}\Phi^{p}\left(\frac{1}{2^{n}}x\right)+4^{n\beta p}\cdot2^{\beta p}\Phi^{p}\left(\frac{1}{2^{n}}y\right)\end{aligned}$$

for all $x,y\in X$. By letting $n\to\infty$, we obtain condition (i) since $\Phi\in V_1$. Since Q is quadratic, we have

$$\begin{aligned}\left\|f\left(\frac{1}{2^{n}}x\right)-\frac{1}{4^{n}}f(x)\right\|^{p}&\le\left\|f\left(\frac{1}{2^{n}}x\right)-Q\left(\frac{1}{2^{n}}x\right)\right\|^{p}+\left\|\frac{1}{4^{n}}Q(x)-\frac{1}{4^{n}}f(x)\right\|^{p}\\&\le\Phi^{p}\left(\frac{1}{2^{n}}x\right)+\frac{1}{4^{n\beta p}}\Phi^{p}(x)\end{aligned}$$

for all $x\in X$. We take $\Phi=\Psi\in V_1$ in the first position, then for all $x\in X$, we have

$$\left\|f\left(\frac{1}{2^{n}}x\right)-\frac{1}{4^{n}}f(x)\right\|^{p}\le\Psi^{p}\left(\frac{1}{2^{n}}x\right)+\frac{1}{4^{n\beta p}}\Phi^{p}(x)$$

and the condition (ii) holds.

Conversely we suppose that (i) and (ii) hold. It follows from condition (ii) that for all $x\in X$, we have

$$\left\|4^{n}f\left(\frac{1}{2^{n}}x\right)-f(x)\right\|^{p}\le4^{n\beta p}\Psi^{p}\left(\frac{1}{2^{n}}x\right)+\Phi^{p}(x).\qquad(2)$$

Then $\left\{4^{n}f\left(\frac{1}{2^{n}}x\right)\right\}$ is a Cauchy sequence. Indeed, by using $\frac{1}{2^{m}}x$ replace x, we get

$$\left\|4^n f\left(\frac{1}{2^{n+m}}x\right)-f\left(\frac{1}{2^m}x\right)\right\|^p \le 4^{n\beta p}\Psi^p\left(\frac{1}{2^{n+m}}x\right)+\Phi^p\left(\frac{1}{2^m}x\right),$$

and by multipling $4^{m\beta p}$, for all $x\in X$, we have

$$\left\|4^{n+m} f\left(\frac{1}{2^{n+m}}x\right)-4^m f\left(\frac{1}{2^m}x\right)\right\|^p \le 4^{(n+m)\beta p}\Psi^p\left(\frac{1}{2^{n+m}}x\right)+4^m\Phi^p\left(\frac{1}{2^m}x\right).$$

Hence, for all $x\in X$,

$$\left\|4^{n+m} f\left(\frac{1}{2^{n+m}}x\right)-4^m f\left(\frac{1}{2^m}x\right)\right\|^p \to 0$$

as $m,n\to\infty$. Since Y is a (β,p)-Banach space, the limit $Q(x):=\lim_{n\to\infty}4^n f\left(\frac{1}{2^n}x\right)$ exists. Let $n\to\infty$ in relation (2), we get condition (1).

Now we show that Q satisfies the required conditions. From the hypothesis, for all $x,y\in X$,

$$\lim_{n\to\infty}4^{n\beta p}\left\|f\left(\frac{1}{2^n}x+\frac{1}{2^n}y\right)+f\left(\frac{1}{2^n}x-\frac{1}{2^n}y\right)-2f\left(\frac{1}{2^n}x\right)-2f\left(\frac{1}{2^n}y\right)\right\|^p=0.$$

Hence for all $x,y\in X$,

$$\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\|=0.$$

Therefore

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$$

and Q is a quadratic map. Now we show the uniqueness of Q. We suppose that Q satisfies

$$\|f(x)-Q(x)\|\le\Phi(x)$$

for all $x\in X$ and there exists a Q' satisfying

$$\|f(x)-Q'(x)\|\le\Phi(x).$$

Since Q and Q' are quadratic mappings, we have

$$\left\|f\left(\frac{1}{2^n}x\right)-Q\left(\frac{1}{2^n}x\right)\right\|=\left\|f\left(\frac{1}{2^n}x\right)-\frac{1}{4^n}Q(x)\right\|\le\Phi\left(\frac{1}{2^n}x\right)$$

for all $x\in X$. Hence for all $x,y\in X$,

$$\begin{aligned}\|Q(x)-Q'(x)\|^p &\le \left\|Q(x)-4^n f\left(\frac{1}{2^n}x\right)\right\|^p+\left\|4^n f\left(\frac{1}{2^n}x\right)-Q'(x)\right\|^p\\ &\le 2\cdot 4^{n\beta p}\Phi^p\left(\frac{1}{2^n}x\right).\end{aligned}$$

Since $\Phi\in V_1$, for all $x\in X$, we have

$$\|Q(x)-Q'(x)\|^p\le 2\lim_{n\to\infty}4^{n\beta p}\Phi^p\left(\frac{1}{2^n}x\right)=0.$$

Hence for all $x \in X$, $Q(x) = Q'(x)$. This completes the proof. □

Corollary 2.3. Let $\varphi : X \times X \to [0, \infty)$ be a mapping satisfying

$$\Phi_1^p(x, y) = \sum_{n=0}^{\infty} 4^{n\beta p} \varphi^p\left(\frac{1}{2^{n+1}}x, \frac{1}{2^{n+1}}y\right) < \infty$$

and

$$\lim_{n\to\infty} 4^{n\beta p} \Phi^p\left(\frac{1}{2^n}x\right) = 0$$

for all $x, y \in X$ where $\Phi(x) = \Phi_1(x, x)$. Suppose $f : X \to Y$ a function with $f(0) = 0$ and satisfying

$$\left\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \right\|^p \le \varphi^p(x, y) \tag{3}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\left\| f(x) - Q(x) \right\| \le \Phi(x), x \in X$$

which is defined

$$Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{1}{2^n}x\right)$$

for all $x \in X$.

Proof. Replace x and y by $\frac{1}{2}x$ in (3), we have

$$\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\|^p \le \varphi^p\left(\frac{x}{2}, \frac{x}{2}\right).$$

Dividing by $4^{\beta p}$, we have

$$\left\| \frac{1}{4}f(x) - f\left(\frac{x}{2}\right) \right\|^p \le \frac{1}{4^{\beta p}} \varphi^p\left(\frac{x}{2}, \frac{x}{2}\right). \tag{4}$$

Replacing x by $\frac{1}{2}x$ in (4), we get

$$\left\| \frac{1}{4}f\left(\frac{x}{2}\right) - f\left(\frac{x}{4}\right) \right\|^p \le \frac{1}{4^{\beta p}} \varphi^p\left(\frac{x}{4}, \frac{x}{4}\right). \tag{5}$$

Then we have

$$\begin{aligned} \left\| \frac{1}{4^2}f(x) - f\left(\frac{1}{2^2}x\right) \right\|^p &= \left\| \frac{1}{4^2}f(x) - \frac{1}{4}f\left(\frac{x}{2}\right) \right\|^p + \left\| \frac{1}{4}f\left(\frac{x}{2}\right) - f\left(\frac{1}{2^2}x\right) \right\|^p \\ &\le \frac{1}{4^{2\beta p}} \varphi^p\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{1}{4^{\beta p}} \varphi^p\left(\frac{x}{4}, \frac{x}{4}\right) \\ &= \frac{1}{4^{2\beta p}} \left[\varphi^p\left(\frac{x}{2}, \frac{x}{2}\right) + 4^{\beta p} \varphi^p\left(\frac{x}{4}, \frac{x}{4}\right) \right] \\ &\le \frac{1}{4^{2\beta p}} \Phi^p(x) \end{aligned}$$

for all $x \in X$. We claim that

$$\left\|\frac{1}{4^m} f(x)-f\left(\frac{1}{2^m} x\right)\right\|^p \leq \frac{1}{4^{m\beta p}} \Phi^p(x). \tag{6}$$

holds for all $m \geq 1$ and $x \in X$. When $m=1$, this is obviously by (4). Suppose (6) holds when $m=k$, *i.e.* for all $x \in X$,

$$\left\|\frac{1}{4^k} f(x)-f\left(\frac{1}{2^k} x\right)\right\|^p \leq \frac{1}{4^{k\beta p}} \Phi^p(x).$$

Then for $m=k+1$, we have

$$\begin{aligned}
&\left\|\frac{1}{4^{k+1}} f(x)-f\left(\frac{1}{2^{k+1}} x\right)\right\|^p \\
&\leq\left\|\frac{1}{4^{k+1}} f(x)-\frac{1}{4^k} f\left(\frac{x}{2}\right)\right\|^p+\left\|\frac{1}{4^k} f\left(\frac{x}{2}\right)-f\left(\frac{1}{2^{k+1}} x\right)\right\|^p \\
&\leq \frac{1}{4^{(k+1)\beta p}}\left[\varphi^p\left(\frac{x}{2}, \frac{x}{2}\right)+4^{\beta p} \Phi^p\left(\frac{x}{2}\right)\right] \\
&\leq \frac{1}{4^{(k+1)\beta p}} \Phi^p(x)
\end{aligned}$$

for all $x \in X$. By induction, (6) is true for all $m \geq 1$ and $x \in X$. Replacing (x, y) by $\left(\frac{1}{2^n} x, \frac{1}{2^n} y\right)$ in (3) and multiplying both side by $4^{n\beta p}$, we have

$$\begin{aligned}
&4^{n\beta p}\left\|f\left(\frac{1}{2^n} x+\frac{1}{2^n} y\right)+f\left(\frac{1}{2^n} x-\frac{1}{2^n} y\right)-2 f\left(\frac{1}{2^n} x\right)-2 f\left(\frac{1}{2^n} y\right)\right\|^p \\
&\leq 4^{n\beta p} \varphi^p\left(\frac{1}{2^n} x, \frac{1}{2^n} y\right).
\end{aligned}$$

Since

$$\Phi_1^p(x, y)=\sum_{n=0}^{\infty} 4^{n\beta p} \varphi^p\left(\frac{1}{2^{n+1}} x, \frac{1}{2^{n+1}} y\right)<\infty,$$

we have

$$\lim_{n \to \infty} 4^{n\beta p} \varphi^p\left(\frac{1}{2^{n+1}} x, \frac{1}{2^{n+1}} y\right)=0$$

for all $x, y \in X$. Hence for all $x, y \in X$,

$$\lim_{n \to \infty} 4^{n\beta p}\left\|f\left(\frac{1}{2^n} x+\frac{1}{2^n} y\right)+f\left(\frac{1}{2^n} x-\frac{1}{2^n} y\right)-2 f\left(\frac{1}{2^n} x\right)-2 f\left(\frac{1}{2^n} y\right)\right\|^p=0.$$

It follows from Theorem 2.2 (with $\Psi=0$ there) that there exists a unique quadratic function Q such that

$$\|f(x)-Q(x)\| \leq \Phi(x)$$

for all $x \in X$. □

Theorem 2.4. Let $V_2=\left\{\Phi: X \rightarrow \mathbb{R}_{+}: \lim_{n \to \infty} \frac{1}{4^{n\beta p}} \Phi^p\left(2^n x\right)=0, \forall x \in X\right\}$. Suppose $\Phi \in V_2$. Then f is Φ-approximable by a quadratic map if and only if the following two condition

(i) $\lim_{n\to\infty}\frac{1}{4^{n\beta p}}\left\|f\left(2^n x+2^n y\right)+f\left(2^n x-2^n y\right)-2f\left(2^n x\right)-2f\left(2^n y\right)\right\|^p=0$;

(ii) There exists a $\Psi\in V_2$ such that

$$\left\|f\left(2^n x\right)-4^n f\left(x\right)\right\|^p\le\Psi^p\left(2^n x\right)+4^{n\beta p}\Phi^p\left(x\right)$$

hold for all $x,y\in X$. In this case, the quadratic Φ-approximation of *f* is unique and is given by

$$Q\left(x\right)=\lim_{n\to\infty}\frac{1}{4^n}f\left(2^n x\right),\ x\in X.$$

Proof. The proof is similar to that of Theorem 2.2 and we omit it. □

Corollary 2.5. Let $\varphi:X\times X\to\left[0,\infty\right)$ be a mapping such that

$$\Phi_1^p\left(x,y\right)=\sum_{n=0}^{\infty}4^{-(n+1)\beta p}\varphi^p\left(2^n x,2^n y\right)<\infty$$

for all $x,y\in X$. Let $\Phi\left(x\right)=\Phi_1\left(x,x\right)$. Suppose $\lim_{n\to\infty}\frac{1}{4^{n\beta p}}\Phi^p\left(2^n x\right)=0$ all $x\in X$. Let $f:X\to Y$ a function with $f\left(0\right)=0$ and satisfying

$$\left\|f\left(x+y\right)+f\left(x-y\right)-2f\left(x\right)-2f\left(y\right)\right\|^p\le\varphi^p\left(x,y\right)$$

for all $x,y\in X$. Then there exists a unique quadratic function $Q:X\to Y$ such that

$$\left\|f\left(x\right)-Q\left(x\right)\right\|\le\Phi\left(x\right)$$

for all $x\in X$.

Proof. The proof is similar to that of Corollary 2.3 and we omit it. □

Funding

This article is partially supported by NSFC (11871303 and 11671133) and NSF of Shandong Province (ZR2019MA039).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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