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Analysis of an Inventory System for Items with Stochastic Demand and Time Dependent Three-Parameter Weibull Deterioration Function

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Abstract

In recent times, mathematical models have been developed to describe various scenarios obtainable in the management of inventories. These models usually have as objective the minimizing of inventory costs. In this research work we propose a mathematical model of an inventory system with time-dependent three-parameter Weibull deterioration and a stochastic type demand in the form of a negative exponential distribution. Explicit expressions for the optimal values of the decision variables are obtained. Numerical examples are provided to illustrate the theoretical development.

Keywords

Inventory Model, Deteriorating Items, Weibull Distribution, Stochastic Demand, MathCAD14

1. Introduction

Inventory holding refers to producing ahead of demand and sales realizations [1]. The total investment in inventories is enormous and accounts for nearly half of the total logistics cost [2]. In view of this high cost, the management of inventory offers high potential for improvement and results in a relatively rich literature on theoretic inventory models. In inventory planning and control, the performance measures adopted should encourage the positive aspects of holding inventory such as providing flexibility, providing resources for production, providing responsive customer service. We observe that inventory arises in many different situations. It is unlikely that the same inventory planning and control considerations will apply equally to all categories of inventory [3].

Some type of products may undergo change in value in storage. They may become partially or entirely unfit for consumption in the course of time. This change or deterioration can be defined as any process that prevents an item from being used for its intended original purpose. Following its utility, the deteriorating item can be characterized into either an item whose functionality or physical fitness deteriorates over time (e.g. fresh food or medicine) or an item whose functionality does not degrade, but where demand deteriorates over time as customers' perceived utility decreases (e.g. fashion clothes, high technology products or newspapers). Both categories pertain to the same problem but require different actions seeing that items that lose their functional characteristics and quality often cannot, or should not be kept in inventory. However, items that lose perceived utility can be kept in inventory and may be sold on a secondary market.

The main objective of inventory management for deteriorating items is to obtain optimal returns during the useful lifetime of the product [3]. This leads to three main issues: determining reasonable and appropriate methods for issuing inventory, replenishing inventory and allocating inventory. The choice of inventory valuation methods adopted in issuing inventory (*i.e*. the order in which the items are to be issued), such as methods based on time sequence including FIFO (first-in, first-out) and LIFO (last-in, first-out), depends on both the intrinsic characteristics of the inventory (e.g. lifetime, quantity, variety, issuing frequency etc.) and the influence on the company (e.g. inventory balance, cost of goods sold etc.) [4].

1.1. Mathematical Formulation

A rich literature on modelling of deteriorating inventory shows how the deterioration of products has been captured in the research problem up till now. To integrate deterioration into mathematical models, the model type (deterministic or stochastic) and the considered time horizon (infinite or finite) lead to specific methods [4].

1.2. Deterioration Process Modeling Approaches

Many researchers have analyzed inventory control of deteriorating items from different perspectives. Broadly speaking, the existing literature in this field can be divided into the following three classes from the perspective of the modeling approach. These classes are schematically illustrated in **Figure 1**.

$I(t)$: *On-hand inventory as a function of time t.*

$\theta(t)$: *Deterioration function of time t.*

$D(t)$: *Demand function of time t.*

$P(t)$: *Production rate as a function of time t.*

$H(t)$: *Holding cost of one unit in-stock for t units of time.*

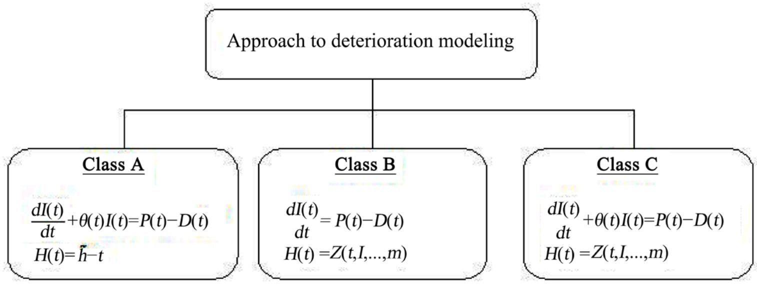

Figure 1. Categorization of deterioration modeling schemes.

$\tilde{h}$: *A positive constant.*

$Z(t,I,\cdots,m)$: *A non-linear increasing positive function of finite number of parameters such as stocking time, t, on-hand inventory, I, etc.*

1.2.1. Class A: Non-Linear Inventory Function

Most researches on deteriorating inventory consider that inventory decays with time, in different patterns. Thus, the on-hand inventory function can be determined by the differential equation:

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t}+\theta(t)I(t)=P(t)-D(t) \tag{1}$$

here $I(t)$ is the inventory level at time *t*, $P(t)$ and $D(t)$ indicate the deterioration rate functions, the production rate and the demand rate as a function of time *t* respectively

In this type of research it is considered that the holding cost per unit item per unit time (holding cost rate) is constant. In other words, the holding cost is linear in terms of parameters like stocking time, *t*, and the on-hand inventory level, *I*, that can be stated as $\tilde{h}tI$, $\tilde{h}>0$ where is constant.

This kind of modeling approach is more appropriate for decaying items and was used in the earliest researches on deteriorating products. Ghare and Schrader [5] seem to have been the first to have developed an exponentially deteriorating inventory model by defining a constant decaying rate.

1.2.2. Class B: Non-Linear Holding Cost

The deterioration process directly affects the on-hand inventory function and thereby inventory holding cost modeling. In this category, the on-hand inventory function form is similar to its form of non-deteriorating products and can be obtained by the differential equation:

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t}=P(t)-D(t) \tag{2}$$

Here, instead of considering the deterioration rate function, $\theta(t)$ in the on-hand inventory function, the holding cost, *H*, is considered as a non-linear increasing positive function of parameters like stocking time, *t* or on-hand inventory *I*.

Considering a non-linear time-dependent holding cost is more suitable for deteriorating items-especially perishable ones—when the value and quality of the unsold items decrease with time, as in the case of green vegetables. For products such as electronic components, radioactive substances, volatile liquids etc., where more sophisticated tools are required for their security and safety in stock, a non-linear stock-dependent holding cost can be appropriate.

1.2.3. Class C: Non-Linear Inventory Function and Non-Linear Holding Cost

This modeling approach is more complicated than the other two. Here, both the deterioration rate function, $\theta(t)$, a feature of Class A, and the non-linear holding cost, a feature of Class B, are considered to model the inventory system of deteriorating products. In [6] the authors discussed Goh's model, considering a constant $\theta(t)$ in addition to non-linear holding cost in two time-dependent and stock-dependent cases.

1.3. The Demand Characteristics

The customer arrival rate per time period may be deterministic or stochastic, each individual demand may be deterministic or stochastic and each individual demand may also be discrete or continuous [7] [8]. Demand plays a key role in the modeling of deteriorating inventory. Aiming towards satisfying customer demand, companies employ demand forecasts as a prediction of customer behaviour. The following variations of demand labeled from the point of view of real life situations have been recognized and studied by a number of researchers such as Khanra *et al*. [9]. It is assumed that demand is known with certainty in a deterministic demand process. Stochastic demand process on the other hand basically incorporates randomness and unpredictability.

A deterministic demand distribution can be categorized into:

1) Uniform demand, *i.e*. demand is a constant, fixed number of items.

2) Time-varying demand.

3) Stock-dependent demand.

4) Price-dependent demand.

A combination of the above is also possible.

In the case of stochastic demand models, a further distinction is made between a specific type of probability distribution and an arbitrary probability distribution. Although modeling in a deterministic setting is more straightforward, a stronger focus on stochastic modeling of deteriorating inventory is suggested in order to better represent inventory control in practice since customer demand is variable in time and uncertain in terms of quantification.

1.4. Stochastic Demand Function

From a real life point of view, a stochastic demand distribution is more reasonable, because demand and supply is not always known but can be controlled by using probability distribution function. Although less than 20% of the developed models in the literature (after 2001) can be classified as stochastic demand models, Bakker *et al.* [10]. However, before 2001 researchers mostly concentrated on developing basic models under certain conditions, such as inventory models with stock dependent items. Based on Goyal and Giri [11], stochastic demand functions in the existing literature can be seen in two ways:

- Taking into consideration a specific type of probability distribution function (PDF) such as Ravichandram [12] and Weiss [13] who developed inventory models for deteriorating products assuming Poisson demand function.
- Considering an arbitrary probability distribution function (PDF) for end customer's demand such as Aggoun *et al.* [14] and Lian *et al.* [15]. According to Bakker *et al.*, since 2001 only about 4% of developed researches on deteriorating inventories provide models with an arbitrary probability distribution for demand.

1.5. Proposed Deterioration Model

The Weibull distribution $W(t)=\alpha\beta(t-\gamma)^{\beta-1}\exp\left(-\alpha(t-\gamma)^{\beta}\right), t>0$, having exponential and Rayleigh as submodels, is often used for modeling lifetime data. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shape. Rinne [16] suggested that a three-parameter generalization of the Weibull distribution deals with general situations in modeling survival process with various shapes in the hazard function. Chakrabarty *et al.* [17] provided rationale for considering three-parameter Weibull deterioration rate. They discovered that many products that start deteriorating appreciably only after a certain period (e.g. after they are produced) and for which the rate of deterioration increases over time have a deterioration rate best described by a Weibull distribution (**Figure 2**).

1.6. Negative Exponential Distribution

The low flow of traffic can be modeled using the negative exponential distribution. The probability density of the negative exponential distribution is given as

$$f(t)=\lambda e^{-\lambda t},\ t\geq 0 \tag{3}$$

where λ is a parameter that determines the shape of the distribution. **Figure 3** displays the exponential distribution for some values of λ.

We observe that the probability that the random variable t is greater than or equal to zero is given by;

$$p(t\geq 0)=\int_0^{\infty} f(t)\mathrm{d}t=\int_0^{\infty}\lambda e^{-\lambda t}\mathrm{d}t=1$$

The probability that the random variable t is greater than a specific value h is

$$p(t\geq h)=1-p(t<h)=1-\int_0^{h}\lambda e^{-\lambda t}\mathrm{d}t=e^{-\lambda h}$$

Unlike many other distributions, one of the key advantages of the negative exponential distribution is the existence of a closed form solution for the probability density function.

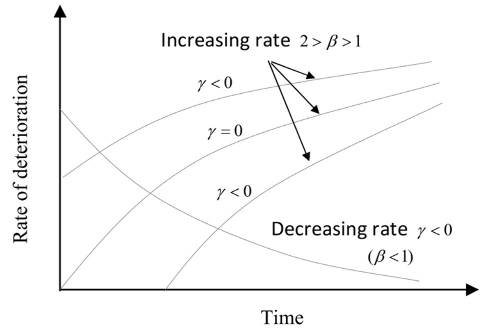

Figure 2. Rate of deterioration-time relationship for a three-parameter Weibull distribution.

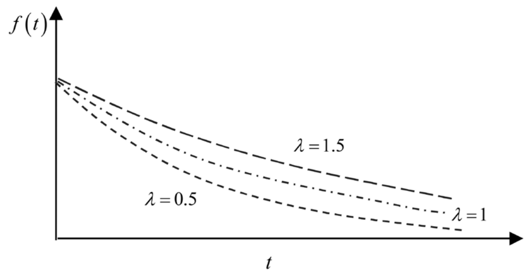

Figure 3. Graphical profile of the negative exponential distribution for various values of λ.

1.7. Notations and Assumptions of the Model

We adopt the following notations and assumptions in the derivation of our model.

Notations:

c_1: inventory holding cost per unit per unit time.

c_2: shortage cost per unit per unit time.

c_3: ordering cost per order.

c_4: unit purchasing cost.

$D(t)$: demand rate at any time, $t \ge 0$.

T: cycle time.

I_0: initial inventory size.

$\theta(t)=\alpha\beta(t-\gamma)^{\beta-1}$: instantaneous rate function for a three-parameter Weibull distribution; where α is the scale parameter, β is the shape parameter and γ is the location parameter. Also, $0<\alpha\ll 1$.

t_1: time during which there is no shortage.

κ: a constant value between 0 and 1.

T^*: optimal value of T.

I_0^*: optimal value of I_0.

t_1^*: optimal value of t_1.

κ^*: optimal value of κ.

Assumptions

1) The inventory system under consideration deals with single item.

2) The planning horizon is infinite.

3) The demand rate is stochastic and given by the negative exponential distribution as a function of time *t*, *i.e.* $D(t)=\lambda e^{-\lambda t}$, where $\lambda>0$, is the parameter of the distribution.

4) Shortages in the inventory are allowed and completely backlogged.

5) The supply is instantaneous and the lead time is zero.

6) Deteriorated unit is not repaired or replaced during a given cycle.

7) The holding cost, ordering cost, shortage cost and unit cost remain constant over time.

8) There are no quantity discounts.

9) The distribution of the time to deterioration of the items follows the three-parameter Weibull distribution, *i.e.*
$W(t)=\alpha\beta(t-\gamma)^{\beta-1}\exp\left(-\alpha(t-\gamma)^{\beta}\right), t>0$. The instantaneous rate function is $\theta(t)=\alpha\beta(t-\gamma)^{\beta-1}$.

2. Mathematical Formulation of the Model

At the beginning of the cycle, the inventory level $I(t)$ reaches its maximum $I(0)=I_0$ units of item at time $t=0$. During the interval $[0,t_1]$, the inventory level depletes due to the combine effects of demand and deterioration. At $t=t_1$, the inventory level is zero and all the demand hereafter (*i.e.* $T-t_1$) is completely backlogged. The total number of backordered items is replaced by the next replenishment. A graphical representation of this inventory system is depicted in Figure 4. Since the depletion of the units is due to demand and deterioration, the rate of change of the inventory level at any time t is governed by the differential equations:

$$\frac{dI(t)}{dt}+\theta(t)I(t)=P(t)-D(t),\quad 0\le t<t_1 \tag{4}$$

with boundary conditions $I(0)=I_0$ and $I(t_1)=0$. Furthermore the production rate $P(t)$ is zero in this case, thus in the interval $0\le t<t_1$, the initial value problem to be solved is;

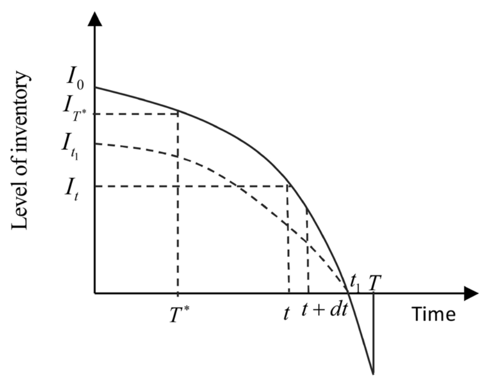

Figure 4. An Economic Order Quantity (EOQ) model with shortages and deterioration.

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t}+\theta(t)I(t)=-D(t),\quad I(0)=I_0,\ I(t_1)=0 \tag{5}$$

In the interval $t_1 \le t \le T$, the initial value problem becomes;

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t}=-D(t),\quad I(t_1)=0 \tag{6}$$

Employing the previously stated assumptions, we have:

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t}+\alpha\beta(t-\gamma)^{\beta-1}I(t)=-\lambda\mathrm{e}^{-\lambda t},\ 0\le t<t_1 \tag{7}$$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t}=-\lambda\mathrm{e}^{-\lambda t},\quad t_1\le t\le T \tag{8}$$

2.1. Solution of the Model

Equation (7) is a first order differential equation and its integrating factor is:

$$\exp\left[\alpha\beta\int(t-\gamma)^{\beta-1}\,\mathrm{d}t\right]=\mathrm{e}^{\alpha(t-\gamma)^{\beta}} \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[I(t)\mathrm{e}^{\alpha(t-\gamma)^{\beta}}\right]=-\lambda\mathrm{e}^{-\lambda t}\mathrm{e}^{\alpha(t-\gamma)^{\beta}}$$

$$\therefore\quad \left[I(t)\mathrm{e}^{\alpha(t-\gamma)^{\beta}}\right]_t^{t_1}=-\lambda\int_t^{t_1}\mathrm{e}^{-\lambda t+\alpha(t-\gamma)^{\beta}}\,\mathrm{d}t$$

Taking first order approximation of the integrand, we have

$$\mathrm{e}^{-\lambda t+\alpha(t-\gamma)^{\beta}}\approx 1+\left\{-\lambda t+\alpha(t-\gamma)^{\beta}\right\}=1-\lambda t+\alpha(t-\gamma)^{\beta}$$

$$\begin{aligned}\Rightarrow\ & I(t_1)\mathrm{e}^{\alpha(t_1-\gamma)^{\beta}}-I(t)\mathrm{e}^{\alpha(t-\gamma)^{\beta}}\\ &=-\lambda\int_t^{t_1}\left\{1-\lambda t+\alpha(t-\gamma)^{\beta}\right\}\mathrm{d}t\\ &=\frac{2\alpha\lambda\left[(t-\gamma)^{\beta+1}-(t_1-\gamma)^{\beta+1}\right]+\lambda(t-t_1)(2\beta-\lambda t-\lambda t_1+2)-\beta\lambda^2\left(t^2-t_1^2\right)}{2(\beta+1)}\end{aligned}$$

Applying the boundary condition $I(t_1)=0$, we get

$$\begin{aligned}& I(t)\mathrm{e}^{\alpha(t-\gamma)^{\beta}}\\ &=\frac{2\alpha\lambda\left[(t-\gamma)^{\beta+1}-(t_1-\gamma)^{\beta+1}\right]+\lambda(t-t_1)(2\beta-\lambda t-\lambda t_1+2)-\beta\lambda^2\left(t^2-t_1^2\right)}{2(\beta+1)}\end{aligned}$$

$$\Rightarrow I(t)=\frac{2\alpha\lambda\left[(t-\gamma)^{\beta+1}-(t_1-\gamma)^{\beta+1}\right]+\lambda(t-t_1)(2\beta-\lambda t-\lambda t_1+2)-\beta\lambda^2\left(t^2-t_1^2\right)}{2(\beta+1)}\mathrm{e}^{-\alpha(t-\gamma)^{\beta}} \tag{10}$$

Hence

$$I(0)=I_0=\frac{2\alpha\lambda\left[(-\gamma)^{\beta+1}-(t_1-\gamma)^{\beta+1}\right]-\lambda t_1(2\beta-\lambda t_1+2)+\beta\lambda^2t_1^2}{2(\beta+1)}\mathrm{e}^{-\alpha(-\gamma)^{\beta}} \tag{11}$$

From Equation (8), in the interval $t_1\le t\le T$ we obtain the solution

$$\left[I(t)\right]_{t_1}^{t} = -\lambda\int_{t_1}^{t} \mathrm{e}^{-\lambda t}\mathrm{d}t = -\lambda\left[-\frac{1}{\lambda}\mathrm{e}^{-\lambda t}\right]_{t_1}^{t} = \mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda t_1}$$

$$\therefore \; I(t) = \mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda t_1} \tag{12}$$

Hence, the inventory level at any time $t \in [0,T]$ is given by

$$I(t) = \begin{cases} \dfrac{2\alpha\lambda\left[(t-\gamma)^{\beta+1} - (t_1-\gamma)^{\beta+1}\right] + \lambda(t-t_1)(2\beta - \lambda t - \lambda t_1 + 2) - \beta\lambda^2\left(t^2 - t_1^2\right)}{2(\beta+1)}\mathrm{e}^{-\alpha(t-\gamma)^{\beta}} & 0 \le t < t_1 \\ \mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda t_1} & t_1 \le t \le T \end{cases} \tag{13}$$

The total cost per unit time, $\phi(T,t_1)$, of the inventory system consist of the deterioration cost (DC), the shortage cost (SC), the holding cost (HC) and the ordering cost (OC). Put differently, the total cost per unit time is:

$$\phi(T,t_1) = \frac{1}{T}(DC + SC + HC + OC) \tag{14}$$

We derive the components of the total relevant cost as follows:

The total quantity of deteriorated items in the time interval $[0,t_1]$ is given by

$$\begin{aligned} D &= \text{Initial inventory} - \text{Total demand within } [0,t_1] \\ &= I_0 - \int_0^{t_1} \lambda\mathrm{e}^{-\lambda t}\mathrm{d}t = I_0 - \left(1 - \mathrm{e}^{-\lambda t_1}\right) \end{aligned} \tag{15}$$

Thus, the deterioration cost per unit time is

$$DC = c_1\left(I_0 - 1 + \mathrm{e}^{-\lambda t_1}\right) \tag{16}$$

The average shortage cost within $[t_1,T]$ is

$$SC = c_2\int_{t_1}^{T} \lambda\mathrm{e}^{-\lambda t}(T-t)\mathrm{d}t = \frac{c_2}{\lambda}\left[(\lambda T - \lambda t_1 - 1)\mathrm{e}^{-\lambda t_1} + \mathrm{e}^{-\lambda T}\right] \tag{17}$$

The average inventory holding cost accumulated over the period $[0,t_1]$ is:

$$HC = c_3\int_0^{t_1} I(t)\mathrm{d}t \tag{18}$$

The total inventory cost per unit time is:

$$\phi(T,t_1) = \frac{1}{T}\left\{c_1\left(I_0 - 1 + \mathrm{e}^{-\lambda t_1}\right) + \frac{c_2}{\lambda}\left[(\lambda T - \lambda t_1 - 1)\mathrm{e}^{-\lambda t_1} + \mathrm{e}^{-\lambda T}\right] + c_3\int_0^{t_1} I(t)\mathrm{d}t + c_4\right\} \tag{19}$$

Here c_1, c_2, c_3 are constants as well as c_4 the ordering cost, assumed constant.

We assume $t_1 = \kappa T$; $0 < \kappa < 1$. This assumption appears reasonable since the length of the shortage interval is a fraction of the cycle time. Substituting $t_1 = \kappa T$ in Equation (19), we get:

$$\phi(T,\kappa) = \frac{1}{T}\left\{c_1\left(I_0^{\kappa} - 1 + \mathrm{e}^{-\lambda\kappa T}\right) + \frac{c_2}{\lambda}\left[(\lambda T - \lambda\kappa T - 1)\mathrm{e}^{-\lambda\kappa T} + \mathrm{e}^{-\lambda T}\right] + c_3\int_0^{\kappa T} I(t)\mathrm{d}t + c_4\right\} \tag{20}$$

$$I_0^{\kappa} = \frac{2\alpha\lambda\left[(-\gamma)^{\beta+1} - (\kappa T - \gamma)^{\beta+1}\right] - \lambda\kappa T(2\beta - \lambda\kappa T + 2) + \beta\lambda^2\kappa^2 T^2}{2(\beta+1)}\mathrm{e}^{-\alpha(-\gamma)^{\beta}} \tag{21}$$

We now proceed to determine the optimal values of T and κ. The total average cost per unit time $\phi(T,\kappa)$ is now a function of two variables T and κ, its partial derivatives with respect to T and κ are computed and the result equated to zero. We have

$$\frac{\partial}{\partial T}\phi(T,\kappa)=\frac{1}{T}\left\{c_1\left(\frac{\partial I_0^\kappa}{\partial T}+\frac{\partial \mathrm{e}^{-\lambda\kappa T}}{\partial T}\right)+\frac{c_2}{\lambda}\left[\frac{\partial}{\partial T}(\lambda T-\lambda\kappa T-1)\mathrm{e}^{-\lambda\kappa T}+\frac{\partial}{\partial T}\mathrm{e}^{-\lambda T}\right]\right.$$
$$\left.+c_3\frac{\partial}{\partial T}\int_0^{\kappa T}I(t)\mathrm{d}t+c_4\right\}$$

$$\frac{\partial I_0^\kappa}{\partial T}=\frac{\mathrm{e}^{-\alpha(-\gamma)^\beta}}{2(\beta+1)}\left[\kappa\lambda(2\beta-\kappa\lambda T+2)-\kappa^2\lambda^2T-2\kappa^2\lambda^2\beta T+2\alpha\kappa\lambda(\beta+1)(\kappa T-\gamma)^\beta\right] \quad (22)$$

$$\frac{\partial}{\partial T}(\lambda T-\lambda\kappa T-1)\mathrm{e}^{-\lambda\kappa T}=\lambda\mathrm{e}^{-\lambda\kappa T}(1-\kappa)-\lambda\kappa\mathrm{e}^{-\lambda\kappa T}(\lambda\kappa T-\lambda T+1) \quad (23)$$

The Lebnitz rule for differentiating the integral $I(\alpha)=\int_{a(\alpha)}^{b(\alpha)}f(x,\alpha)\mathrm{d}x$ is given by

$$\frac{\mathrm{d}I(\alpha)}{\mathrm{d}\alpha}=f(b,\alpha)\frac{\mathrm{d}b}{\mathrm{d}\alpha}-f(b,\alpha)\frac{\mathrm{d}a}{\mathrm{d}\alpha}+\int_a^b\frac{\partial f(x,\alpha)}{\partial\alpha}\mathrm{d}x$$

Applying this rule to $\frac{\partial}{\partial T}\int_0^{\kappa T}I(t,T)\mathrm{d}t$, we get

$$\frac{\partial}{\partial T}\int_0^{\kappa T}I(t,T)\mathrm{d}t=\int_0^{\kappa T}\frac{\partial}{\partial T}I(t,T)\mathrm{d}t+\kappa I(\kappa,T) \quad (24)$$

Hence

$$\frac{\partial}{\partial T}\phi(T,\kappa)=\frac{1}{T}\left\{\frac{c_1\mathrm{e}^{-\alpha(-\gamma)^\beta}}{2(\beta+1)}\left[\kappa\lambda(2\beta-\kappa\lambda T+2)-\kappa^2\lambda^2T-2\kappa^2\lambda^2\beta T\right.\right.$$
$$\left.+2\alpha\kappa\lambda(\beta+1)(\kappa T-\gamma)^\beta\right]-\lambda\kappa\mathrm{e}^{-\lambda\kappa T}$$
$$+\frac{c_2}{\lambda}\left[\lambda\mathrm{e}^{-\lambda\kappa T}(1-\kappa)-\lambda\kappa\mathrm{e}^{-\lambda\kappa T}(\lambda\kappa T-\lambda T+1)-\lambda\mathrm{e}^{-\lambda T}\right] \quad (25)$$
$$\left.+c_3\int_0^{\kappa T}\frac{\partial}{\partial T}I(t,T)\mathrm{d}t+\kappa I(\kappa,T)\right\}$$
$$=0$$

Similarly;

$$\frac{\partial}{\partial\kappa}\phi(T,\kappa)=\frac{1}{T}\left\{c_1\left(\frac{\partial I_0^\kappa}{\partial\kappa}+\frac{\partial \mathrm{e}^{-\lambda\kappa T}}{\partial\kappa}\right)+\frac{c_2}{\lambda}\left[\frac{\partial}{\partial\kappa}(\lambda T-\lambda\kappa T-1)\mathrm{e}^{-\lambda\kappa T}+\frac{\partial}{\partial\kappa}\mathrm{e}^{-\lambda T}\right]\right.$$
$$\left.+c_3\frac{\partial}{\partial\kappa}\int_0^{\kappa T}I(t)\mathrm{d}t\right\}$$

$$\frac{\partial I_0^\kappa}{\partial\kappa}=\frac{\mathrm{e}^{-\alpha(-\gamma)^\beta}}{2(\beta+1)}\left[\lambda T(2\beta-\kappa\lambda T+2)-\kappa\lambda^2T^2-2\kappa\beta\lambda^2T^2+2\alpha\lambda T(\beta+1)(\kappa T-\gamma)^\beta\right] \quad (26)$$

$$\frac{\partial}{\partial\kappa}(\lambda T-\lambda\kappa T-1)\mathrm{e}^{-\lambda\kappa T}=-\lambda T\mathrm{e}^{-\lambda\kappa T}-\lambda T(\lambda\kappa T-\lambda T+1)\mathrm{e}^{-\lambda\kappa T} \quad (27)$$

and

$$\frac{\partial}{\partial \kappa}\int_0^{\kappa T} I(t,T)\,\mathrm{d}t = \int_0^{\kappa T} \frac{\partial}{\partial \kappa} I(t,T)\,\mathrm{d}t + TI(\kappa,T) \tag{29}$$

Hence

$$\begin{aligned}\frac{\partial}{\partial \kappa}\phi(T,\kappa) = \frac{1}{T}\Bigg\{ & \frac{c_1 \mathrm{e}^{-\alpha(-\gamma)^\beta}}{2(\beta+1)}\Big[\lambda T(2\beta - \kappa\lambda T + 2) - \kappa\lambda^2 T^2 - 2\kappa\beta\lambda^2 T^2 \\ & + 2\alpha\lambda T(\beta+1)(\kappa T - \gamma)^\beta\Big] - \lambda\kappa \mathrm{e}^{-\lambda\kappa T} \\ & + \frac{c_2}{\lambda}\Big[-\lambda T \mathrm{e}^{-\lambda\kappa T} - \lambda T(\lambda\kappa T - \lambda T + 1)\mathrm{e}^{-\lambda\kappa T}\Big] \\ & + c_3 \int_0^{\kappa T} \frac{\partial}{\partial \kappa} I(t,T)\,\mathrm{d}t + TI(\kappa,T)\Bigg\} \\ = 0 \end{aligned} \tag{30}$$

where

$$I(t,T) = \frac{2\alpha\lambda\left[(t-\gamma)^{\beta+1} - (\kappa T - \gamma)^{\beta+1}\right] + \lambda(t-\kappa T)(2\beta - \lambda t - \lambda\kappa T + 2) - \beta\lambda^2\left(t^2 - \kappa^2 T^2\right)}{2(\beta+1)}\mathrm{e}^{-\alpha(t-\gamma)^\beta}$$

and

$$\begin{cases}\dfrac{\partial}{\partial T} I(t,T) = -\dfrac{\kappa\lambda^2(t-\kappa T) + \kappa\lambda(2\beta - \lambda t - \kappa\lambda T + 2) - 2\beta\kappa^2\lambda^2 T + 2\alpha\kappa\beta(\beta+1)(\kappa T - \gamma)^\beta}{2(\beta+1)}\mathrm{e}^{-\alpha(t-\gamma)^\beta} \\ \dfrac{\partial}{\partial \kappa} I(t,T) = -\dfrac{T\lambda^2(t-\kappa T) + T\lambda(2\beta - \lambda t - \kappa\lambda T + 2) - 2\beta\kappa\lambda^2 T^2 + 2T\alpha\beta(\beta+1)(\kappa T - \gamma)^\beta}{2(\beta+1)}\mathrm{e}^{-\alpha(t-\gamma)^\beta}\end{cases}$$

2.2. Remark

Equations (25) and (30) provide the necessary condition for T^* and κ^* to be minimum points of $\phi(T,\kappa)$.

The sufficient condition for these values to minimize $\phi(T,\kappa)$ is that the Hesssian matrix H must be positive definite. Here

$$H = \nabla\phi^2(T,\kappa) = \begin{pmatrix} \dfrac{\partial^2\phi}{\partial T^2} & \dfrac{\partial^2\phi}{\partial T\partial\kappa} \\ \dfrac{\partial^2\phi}{\partial T\partial\kappa} & \dfrac{\partial^2\phi}{\partial\kappa^2}\end{pmatrix}$$

Thus the sufficient condition for optimality is $\dfrac{\partial^2\phi}{\partial T^2} > 0, \dfrac{\partial^2\phi}{\partial\kappa^2} > 0$ and $\dfrac{\partial^2\phi}{\partial T^2}\dfrac{\partial^2\phi}{\partial\kappa^2} - \left(\dfrac{\partial^2\phi}{\partial T\partial\kappa}\right)^2 > 0$.

Since $I(t) = \mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda t_1}$ for $t_1 \le t \le T$, the total back-order quantity for the cycle is $I^* = I_0^* + \mathrm{e}^{-\lambda T^*} - \mathrm{e}^{-\lambda t_1^*}$.

2.3. Optimal Inventory Policy for the Model

In this section, we provide the optimal inventory policy for the proposed model.

The procedure for reaching this optimum policy is also given. The optimal inventory policy for the proposed model is:

Order I^* units for every T^* time units. Use $e^{-\lambda T^*} - e^{-\lambda t_1^*}$ units to offset the backordered quantity and begin a new cycle with I_0^κ units. The total inventory cost per unit time associated with the proposed model is:

$$\phi(T,\kappa) = \frac{1}{T}\left\{c_1\left(I_0^\kappa - 1 + e^{-\lambda\kappa T}\right) + \frac{c_2}{\lambda}\left[\left(\lambda T - \lambda\kappa T - 1\right)e^{-\lambda\kappa T} + e^{-\lambda T}\right] + c_3\int_0^{\kappa T} I(t)\,dt + c_4\right\}$$

2.4. Solution Algorithm

We give the following steps for computing the optimal ordering quantity, optimal cycle time and the optimal total cost for the model:

Step 1: Solve Equations (25) and (30) simultaneously to get the optimal values T^* and κ^* for T and κ respectively.

Step 2: If at T^* and κ^* the sufficiency condition is satisfied, then go to step 3 else stop and declare the solution infeasible.

Step 3: Substitute T^* and κ^* into $t_1 = \kappa T$ to obtain t_1^*.

Step 4: Determine the optimal EOQ I_0^* by substituting the values of T^* and κ^* into Equation (11).

Step 5: Substitute the values of I_0^*, T^* and κ^* into Equation (20) to get the optimal total average cost $\phi(T,\kappa)$.

2.5. Numerical Analysis and Results

In this section we employ MathCAD 14 [18] to obtain numerical solution to the highly nonlinear system of Equations (25) and (30). This will provide us with the optimal solutions for the average cost function for some specified data. We consider the following inventory data adapted from Ghosh and Chaudhuri (2004):

$c_1 = 2.40$, $c_2 = 5$, $c_3 = 100.00$, $c_4 = 20.00$, $\alpha = 0.001$, $\beta = 8$, $\gamma = 0.1$, $\lambda = 0.1$, $\kappa = 0.85$, $T = 2$.

The format for the MathCAD 14 solve block follows;

- *Initial values for the unknown variables* (κ, T).
- *Given.*
- *Equation* 1.
- *Equation* 2.
- *Find* (κ, T).

2.6. Mathcad Solve Block Solution

$$c_1 := 2.40 \quad c_2 := 5 \quad c_3 := 100 \quad \alpha := 0.01 \quad \beta := 8 \quad \lambda := 1.5 \quad \gamma := 0.4$$

$\kappa := 0.85$ $T := 2$ Initial values of the variables

Given

$$\frac{c_1\cdot\exp\left[-\alpha\cdot(-\gamma)^\beta\right]}{2\cdot(\beta+1)}\cdot\left[\kappa\cdot\lambda\cdot(2\cdot\beta-\kappa\cdot\lambda\cdot T+2)-\kappa^2\cdot\lambda^2\cdot T-2\cdot\kappa^2\cdot\lambda^2\cdot\beta\cdot T+2\cdot\alpha\cdot\lambda\cdot\kappa\cdot(\beta+1)\cdot(\kappa\cdot T-\gamma)^\beta\right]$$

$$-\lambda\cdot\kappa\cdot\exp(-\lambda\cdot\kappa\cdot T)+\frac{c_2}{\lambda}\cdot\left[\lambda\cdot\exp(-\lambda\cdot\kappa\cdot T)\cdot(1-\kappa)-\lambda\cdot\kappa\cdot\exp(-\lambda\cdot\kappa\cdot T)\cdot(\lambda\cdot\kappa\cdot T-\lambda\cdot T+1)-\lambda\cdot\exp(-\lambda\cdot T)\right]$$

$$+\frac{(-1)\cdot c_3}{2\cdot(\beta+1)}\cdot\int_0^{\kappa\cdot T}\lambda\cdot\exp\left[-\lambda\cdot t-\alpha\cdot(t-\gamma)^\beta\right]$$

$$\cdot\left[\kappa\cdot\lambda^2(t-\kappa\cdot T)+\kappa\cdot\lambda(2\cdot\beta-\lambda\cdot t+\kappa\cdot\lambda\cdot t+2)-2\cdot\beta\cdot\kappa^2\cdot\lambda^2\cdot T+2\alpha\cdot\kappa\cdot\beta\cdot(\beta+1)\cdot(\kappa\cdot T-\gamma)^\beta\right]\mathrm{d}t$$

$$+\kappa\cdot\left[\frac{2\cdot\alpha\cdot\lambda\cdot\left[(\kappa-\gamma)^{\beta+1}-(\kappa\cdot T-\gamma)^{\beta+1}\right]+\lambda\cdot(\kappa-\kappa\cdot T)(2\cdot\beta-\lambda\cdot\kappa-\lambda\cdot\kappa\cdot T+2)-\beta\cdot\lambda^2\cdot\left(\kappa^2-\kappa^2\cdot T^2\right)}{2\cdot(\beta+1)}\right]=0$$

$$\frac{c_1\cdot\exp\left[-\alpha\cdot(-\gamma)^\beta\right]}{2\cdot(\beta+1)}\cdot\left[\lambda\cdot T\cdot(2\cdot\beta-\kappa\cdot\lambda\cdot T+2)-\kappa\cdot\lambda^2\cdot T^2-2\cdot\kappa\cdot\beta\cdot\lambda^2\cdot T^2+2\cdot\alpha\cdot\lambda\cdot T\cdot(\beta+1)\cdot(\kappa T-\gamma)^\beta\right]$$

$$-\lambda\cdot\kappa\cdot\exp(-\lambda\cdot T)+\frac{c_2}{\lambda}\cdot\left[-\lambda\cdot T\cdot\exp(-\lambda\cdot\kappa\cdot T)-\lambda\cdot T\cdot\exp(-\lambda\cdot\kappa\cdot T)\cdot(\lambda\cdot\kappa\cdot T-\lambda\cdot T+1)\right]$$

$$+\frac{(-1)\cdot c_3}{2\cdot(\beta+1)}\cdot\int_0^{\kappa\cdot T}\lambda\cdot\exp\left[-\lambda\cdot t-\alpha\cdot(t-\gamma)^\beta\right]$$

$$\cdot\left[T\cdot\lambda^2\cdot(t-\kappa\cdot T)+T\cdot\lambda\cdot(2\cdot\beta-\lambda\cdot t-\kappa\cdot\lambda\cdot T+2)-2\cdot\beta\cdot\kappa\cdot\lambda^2\cdot T^2+2\cdot T\cdot\alpha\cdot\beta\cdot(\beta+1)\cdot(\kappa\cdot T-\gamma)^\beta\right]\mathrm{d}t$$

$$+T\cdot\left[\frac{2\cdot\alpha\cdot\lambda\cdot\left[(\kappa-\gamma)^{\beta+1}-(\kappa\cdot T-\gamma)^{\beta+1}\right]+\lambda\cdot(\kappa-\kappa\cdot T)(2\cdot\beta-\lambda\cdot\kappa-\lambda\cdot\kappa\cdot T+2)-\beta\cdot\lambda^2\cdot\left(\kappa^2-\kappa^2\cdot T^2\right)}{2\cdot(\beta+1)}\right]=0$$

$$\textbf{\textit{Find}}(\kappa,T)=\begin{pmatrix}0.9460303\\2.0306513\end{pmatrix}$$

2.7. Remark

- From the solve block solution we obtain the optimal T^* and κ^* as $T^*=2.0306513$, $\kappa^*=0.9460303$.
- It is not difficult to show, using MathCAD, that for these optimal values the sufficient conditions for minimizing $\phi(T,\kappa)$ are satisfied.
- We proceed to use these values to compute the optimal t_1^* and I_0^* to be

$$t_1^*=\kappa^*T^*=1.921,$$

$$I_0^*=\frac{2\alpha\lambda\left[(-\gamma)^{\beta+1}-\left(t_1^*-\gamma\right)^{\beta+1}\right]-\lambda t_1^*\left(2\beta-\lambda t_1^*+2\right)+\beta\lambda^2t_1^{*2}}{2(\beta+1)}\mathrm{e}^{-\alpha(-\gamma)^\beta}=1.197$$

- Finally, we have;

$$\begin{aligned}\phi(T,\kappa)&=\frac{1}{T}\left\{c_1\left(I_0^\kappa-1+\mathrm{e}^{-\lambda\kappa T}\right)+\frac{c_2}{\lambda}\left[(\lambda T-\lambda\kappa T-1)\mathrm{e}^{-\lambda\kappa T}+\mathrm{e}^{-\lambda T}\right]\right.\\&\qquad\left.+c_3\int_0^{\kappa T}I(t)\,\mathrm{d}t+c_4\right\}\\&=11.334\end{aligned}$$

In summary, for the *mathematical model of an inventory system with time* *dependent three-parameter Weibull deterioration and a stochastic type demand in the form of a negative exponential distribution, we obtained the following results.*

The optimum cycle time $T^* = 2.031$ *days.*

The optimum value $\kappa^* = 0.94603$.

The optimum stock-period $t_1^* = 1.921$ *days.*

The economic order quantity $I_0^* = 1.197$ *units.*

The optimum total average cost $\phi(T,\kappa)^* = \$11.334$ *per day.*

The optimum number of order, $N^* = 1/1.197 = 0.8354$ *order per day.*

2.8. Conclusions

In this work we developed an inventory model for a three-parameter Weibull deteriorating items with stochastic demand in the form of a negative exponential distribution. We derived the optimal inventory policy for the proposed model and also established the necessary and sufficient conditions for the optimal policy. In the solution of the differential equation obtained, because of the cumbersome nature of the associated integral, we were forced to make a first order approximation for the integrand involving an exponential function. This in turn enabled us to obtain a closed form solution for our model. We provided a numerical example illustrating our solution procedure. Though our solution is only approximate, we were still able to obtain very reasonable results which compared favourably with that of Ghosh and Chaudhuri [6] ($T^* = 2.145$ days, $\kappa^* = 0.8832$) for the deterministic demand case.

It is important to state that the numerical procedure for this problem relied heavily on the power of MathCAD14, which was used to solve a highly nonlinear system of equations in two unknowns, and involving a definite integral. The advantage of this numerical software is that the equations are composed as they appear in the text and need not be recast in a special format for computation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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