

# Behavior of a Scale Factor for Wiener Integrals of an Unbounded Function

Young Sik Kim

Department of Mathematics, College of Natural Sciences, Industry-University Cooperation Foundation, Hanyang University, Seoul, Republic of Korea

Email: yoskim@hanyang.ac.kr

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## Abstract

The purpose of this paper is to investigate the behavior of a scale factor for Wiener integrals about the unbounded function  $F(x) = \exp\left\{a \sum_{j=1}^n \int_0^T \alpha_j dx\right\}$ , where  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  is an orthonormal set of elements in  $L_2[0, T]$  on the Wiener space  $C_0[0, T]$ .

## Keywords

Wiener Space, Wiener Integral, Analytic Wiener Integral, Analytic Feynman Integral, Scale Factor

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## 1. Introduction

In [1], M. D. Brue introduced the functional transform on the Feynman integral (1972). In [2], R. H. Cameron wrote the paper about the translation pathology of a Wiener space (1972). In [3] [4] [5], R. H. Cameron and W. T. Martin proved some theorems on the transformation and the translation and used the expression of the change of scale for Wiener integrals (1944, 1947). In [6] [7], R. H. Cameron and D. A. Storvick proved relationships between Wiener integrals and analytic Feynman integrals to prove the change of scale formula for Wiener integral on the Wiener space in 1987. In [8], M. D. Gaysinsky and M. S. Goldstein proved the Self-Adjointness of a Schrödinger Operator and Wiener Integrals (1992).

In [9], G. W. Johnson and M. L. Lapidus wrote the paper about the Feynman integral and Feynman's Operational Calculus (2000). In [10], G. W. Johnson and D. L. Skoug proved the scale-invariant measurability in Wiener Space (1979).

In [11] and [12], Y. S. Kim proved a **change of scale** formula for Wiener integrals about cylinder functions  $f\left(\left(h_1, x\right)^{\sim}, \dots, \left(h_n, x\right)^{\sim}\right)$  with  $f \in L_p\left(\mathbb{R}^n\right), 1 \leq p \leq \infty$  on the abstract Wiener space: the analytic Wiener integral exists for  $f \in L_p\left(\mathbb{R}^n\right), 1 \leq p \leq \infty$ , and the analytic Feynman integral exists for  $f \in L_1\left(\mathbb{R}^n\right)$  (1998) and (2001). But the Feynman integral does not always exist for  $1 < p$ .

In [13], Y. S. Kim investigates a behavior of a scale factor for the Wiener integral of a function  $F(x) = \exp\left\{\int_0^T \theta(t, x(t)) dt\right\}$ , where  $\theta: [0, T] \times \mathbb{R} \rightarrow \mathbb{C}$  is defined by  $\theta(t, u) = \int_{\mathbb{R}} \exp\{iuv\} d\sigma_t(v)$  which is a Fourier-Stieltjes transform of a complex Borel measure  $\sigma_t \in \mathbf{M}(\mathbb{R})$  and  $\mathbf{M}(\mathbb{R})$  is a set of complex Borel measures defined on  $\mathbb{R}$ .

In this paper, we investigate the behavior of a scale factor  $\rho > 0$  for the Wiener integral  $\int_{C_0[0, T]} F(\rho x) dm(x)$  which is defined on the Wiener space  $C_0[0, T]$  about the unbounded function  $F(x) = \exp\left\{a \sum_{j=1}^n \int_0^T \alpha_j dx\right\}$  with  $a > 0$ , where  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  is an orthonormal set of elements in  $L_2[0, T]$  on the Wiener space  $C_0[0, T]$ .

## 2. Definitions and Preliminaries

Let  $C_0[0, T]$  denote the space of real-valued continuous functions  $x$  on  $[0, T]$  such that  $x(0) = 0$ . Let  $\mathcal{M}$  denote the class of all Wiener measurable subsets of  $C_0[0, T]$  and let  $m$  denote a Wiener measure and  $(C_0[0, T], \mathcal{M}, m)$  be a Wiener measure space and we denote the Wiener integral of a function  $F: C_0[0, T] \rightarrow \mathbb{C}$  by  $\int_{C_0[0, T]} F(x) dm(x)$ .

A subset  $E$  of  $C_0[0, T]$  is said to be scale-invariant measurable if  $\rho E \in \mathcal{M}$  for each  $\rho > 0$ , and a scale-invariant measurable set  $N$  is said to be scale-invariant null if  $m(\rho N) = 0$  for each  $\rho > 0$ . A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals  $F$  and  $G$  are equal s-a.e., we write  $F \approx G$ . A function  $F$  defined on the scale invariant measurable set  $E$  is a scale invariant measurable function if  $F(\alpha x)$  is a Wiener measurable function for all  $\alpha > 0$ .

Throughout this paper, let  $\mathbb{R}^n$  denote the  $n$ -dimensional Euclidean space and let  $\mathbb{C}, \mathbb{C}_+$ , and  $\mathbb{C}_+^{\sim}$  denote the set of complex numbers, the set of complex numbers with positive real part, and the set of non-zero complex numbers with nonnegative real part, respectively.

**Definition 2.1.** Let  $F$  be a complex-valued measurable function on  $C_0[0, T]$  such that the integral

$$J(F; \lambda) = \int_{C_0[0, T]} F\left(\lambda^{-\frac{1}{2}} x\right) dm(x) \quad (2.1)$$

exists for all real  $\lambda > 0$ . If there exists a function  $J^*(F; z)$  analytic on  $\mathbb{C}_+$  such that  $J^*(F; \lambda) = J(F; \lambda)$  for all real  $\lambda > 0$ , then we define  $J^*(F; z)$  to be the *analytic Wiener integral* of  $F$  over  $C_0[0, T]$  with parameter  $z$ , and for

each  $z \in \mathbf{C}_+$ , we write

$$I^{aw}(F; z) = J^*(F; z) \equiv \int_{C_0[0, T]}^{awz} F(x) dm(x). \tag{2.2}$$

Let  $q$  be a non-zero real number and let  $F$  be a function defined on  $C_0[0, T]$  whose analytic Wiener integral exists for each  $z$  in  $\mathbf{C}_+$ . If the following limit exists, then we call it the *analytic Feynman integral* of  $F$  over  $C_0[0, T]$  with parameter  $q$ , and we write

$$I^{af}(F; q) = \lim_{z \rightarrow -iq} I^{aw}(F; z) \equiv \int_{C_0[0, T]}^{awq} F(x) dm(x), \tag{2.3}$$

where  $z$  approaches  $-iq$  through  $\mathbf{C}_+$  and  $i^2 = -1$ .

Let  $\{e_k\}_{k=1}^n$  be a complete orthonormal set and  $e_k \in C[0, T] \cap B[0, T]$  for  $k = 1, 2, \dots, n$  and  $\alpha \in L_2[0, T]$  and  $x \in C_0[0, T]$ . We define a Paley-Wiener-Zygmund integral (P.W.Z) of  $x$  with respect to  $\alpha$  by

$$\int_0^T \alpha(t) dx(t) \equiv \lim_{n \rightarrow \infty} \int_0^T \sum_{k=1}^n (\alpha, e_k) e_k(t) dx(t).$$

**Theorem 2.2 (Wiener Integration Formula).** *Let  $C_0[0, T]$  be a Wiener space. Then*

$$\begin{aligned} & \int_{C_0[0, T]} f\left(\int_0^T \alpha_1 dx, \int_0^T \alpha_2 dx, \dots, \int_0^T \alpha_n dx\right) dm(x) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbf{R}^n} f(\bar{u}) \exp\left\{-\frac{1}{2} \sum_{j=1}^n u_j^2\right\} d\bar{u} \end{aligned} \tag{2.4}$$

where  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  is an orthonormal set of elements in  $L_2[0, T]$  and  $f: \mathbf{R}^n \rightarrow \mathbf{C}$  is a Lebesgue measurable function and  $\bar{u} = (u_1, u_2, \dots, u_n)$  and  $d\bar{u} = du_1 du_2 \dots du_n$  and  $\int_0^T \alpha_j dx$  is a Paley-Wiener-Zygmund integral for  $1 \leq j \leq n$ .

**Remark.** We will use several times the following well-known integration formula:

$$\int_{\mathbf{R}} \exp\{-au^2 + ibu\} du = \sqrt{\frac{\pi}{a}} \exp\left\{-\frac{b^2}{4a}\right\} \tag{2.5}$$

where  $a$  is a complex number with  $Rea > 0$ ,  $b$  is a real number, and  $i^2 = -1$ .

### 3. Main Results

Define a function  $F: C_0[0, T] \rightarrow \mathbf{C}$  on the Wiener space by

$$F(x) = \exp\left\{a \sum_{j=1}^n \int_0^T \alpha_j dx\right\} \tag{3.1}$$

where  $a > 0$  is a finite real number and  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  is an orthonormal set of elements in  $L_2[0, T]$ .

**Lemma 3.1.** *For a finite real number  $a > 0$ , the unbounded cylinder function  $F(x)$  in (3.1) is a Wiener integrable function.*

*Proof.* By the Wiener integration Formula (2.4), we have that for a finite real

number  $a > 0$ ,

$$\begin{aligned} & \int_{C_0[0,T]} F(x) dm(x) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \cdot \int_{R^n} \exp\left\{a \sum_{j=1}^n u_j - \frac{1}{2} \sum_{j=1}^n u_j^2\right\} d\bar{u} \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \cdot (2\pi)^{\frac{n}{2}} \cdot \exp\left\{+\frac{n}{2} a^2\right\} \\ &= \exp\left\{+\frac{n}{2} a^2\right\} < \infty \end{aligned} \tag{3.2}$$

**Remark.** If we let  $F(x) = f\left(\int_0^T \alpha_1 dx, \int_0^T \alpha_2 dx, \dots, \int_0^T \alpha_n dx\right)$  and  $f: R^n \rightarrow C$ , then  $f(\bar{u}) = \exp\left\{a \sum_{j=1}^n u_j\right\}$  is unbounded for a finite real number  $a > 0$ .

**Lemma 3.2.** Let  $F: C_0[0,T] \rightarrow C$  be defined by (3.1). For a finite real  $\rho > 0$  and a finite real  $a > 0$ ,

$$\int_{C_0[0,T]} F(\rho x) dm(x) = \exp\left\{+\frac{n}{2} a^2 \rho^2\right\} \tag{3.3}$$

*Proof.* By the Wiener integration Formula (2.4), we have that

$$\begin{aligned} & \int_{C_0[0,T]} F(\rho x) dm(x) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \cdot \int_{R^n} \exp\left\{a \rho \sum_{j=1}^n u_j - \frac{1}{2} \sum_{j=1}^n u_j^2\right\} d\bar{u} \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \cdot (2\pi)^{\frac{n}{2}} \cdot \exp\left\{+\frac{n}{2} a^2 \rho^2\right\} \\ &= \exp\left\{+\frac{n}{2} a^2 \rho^2\right\} < \infty \end{aligned} \tag{3.4}$$

**Lemma 3.3.** Let  $F: C_0[0,T] \rightarrow C$  be defined by (3.1). For a finite real  $\rho > 0$  and a finite real  $a > 0$ ,

$$\int_{C_0[0,T]} F(\rho x) dm(x) = \left[\int_{C_0[0,T]} F(x) dm(x)\right]^{\rho^2} \tag{3.5}$$

*Proof.* By the above Lemma, we have that

$$\begin{aligned} & \int_{C_0[0,T]} F(\rho x) dm(x) \\ &= \exp\left\{+\frac{n}{2} a^2 \rho^2\right\} \\ &= \left[\exp\left\{+\frac{n}{2} a^2\right\}\right]^{\rho^2} \\ &= \left[\int_{C_0[0,T]} F(x) dm(x)\right]^{\rho^2} \end{aligned} \tag{3.6}$$

Now we define a concept of the **scale factor for the Wiener integral** which was first defined in [13]:

**Definition 3.4.** We define the scale factor for the Wiener integral by the

real number  $\rho > 0$  of the absolute value of the Wiener integral:

$$G(\rho) = \left| \int_{C_0[0,T]} F(\rho x) dm(x) \right| \tag{3.7}$$

where  $G : R \rightarrow R$  is a real valued function defined on  $R$ .

**Property.** <Behavior of a scale factor for the Wiener integral.>

We investigate the interesting behavior of the scale factor for the Wiener integral by analyzing the Wiener integral as followings: For real  $\rho > 0$  and for a finite real number  $a > 0$ ,

$$\int_{C_0[0,T]} F(\rho x) dm(x) = \exp \left\{ + \frac{n}{2} a^2 \rho^2 \right\}. \tag{3.8}$$

**Example.** For the scale factor  $\rho = \left\{ \dots, \frac{1}{100}, \frac{1}{10}, 1, 10, 10^2, \dots \right\}$ , we can investigate the very interesting behavior of the Wiener integral:

$$\begin{aligned} \text{(a)} \quad & \int_{C_0[0,T]} F\left(\frac{1}{100}x\right) dm(x) = \left[ \int_{C_0[0,T]} F(x) dm(x) \right]^{\frac{1}{10000}} \\ \text{(b)} \quad & \int_{C_0[0,T]} F\left(\frac{1}{10}x\right) dm(x) = \left[ \int_{C_0[0,T]} F(x) dm(x) \right]^{\frac{1}{100}} \\ \text{(c)} \quad & \left[ \int_{C_0[0,T]} F\left(\frac{1}{100}x\right) dm(x) \right]^{100} = \left[ \int_{C_0[0,T]} F\left(\frac{1}{10}x\right) dm(x) \right] \\ \text{(d)} \quad & \int_{C_0[0,T]} F(\rho x) dm(x) = \left[ \int_{C_0[0,T]} F(x) dm(x) \right]^{\rho^2} \end{aligned} \tag{3.9}$$

**Remark.** <Interpretation of a scale factor for Wiener integrals of an unbounded cylinder function.>

1) Whenever the scale factor  $\rho > 1$  is increasing, the Wiener integral increases very rapidly. Whenever the scale factor  $0 < \rho < 1$  is decreasing, the Wiener integral decreases very rapidly.

2) The function  $G(\rho) = \left| \int_{C_0[0,T]} F(\rho x) dm(x) \right|$  for  $F(x)$  in (3.1) is an increasing function of a scale factor  $\rho > 0$ , because the exponential function  $y = e^{x^2}$  is an increasing function of  $x \in R$ .

3) Whenever the scale factor  $\rho > 0$  is increasing and decreasing, the Wiener integral varies very rapidly.

### 4. Conclusions

What we have done in this research is that we investigate the very interesting behavior of the scale factor for the Wiener integral of an **unbounded** function.

From these results, we find an interesting property for the Wiener integral as a function of a scale factor which was first defined in [13].

Note that the function in [13] is bounded and the function of this paper is unbounded!

Finally, we introduce the motivation and the application of the Wiener

integral and the Feynman integral and the relationship between the scale factor and the heat (or diffusion) equation:

**Remark.**

1) The solution of the heat (or diffusion) equation

$$\frac{\partial \psi}{\partial t} = -\frac{i}{h} \left[ -\frac{h^2}{2m} \frac{\partial \psi^2}{\partial \xi^2} + V(\xi) \psi \right], \quad (3.10)$$

is that for a real  $\lambda > 0$ ,

$$\psi_\lambda(t, \xi) = \int_{C_0^t} \exp \left\{ -\frac{i}{h} \int_0^t V \left( \lambda^{-\frac{1}{2}} x(s) + \xi \right) ds \right\} \cdot \psi \left( \lambda^{-\frac{1}{2}} x(s) + \xi \right) dm(s) \quad (3.11)$$

where  $\psi_\lambda(\cdot, \xi) = \phi(\xi)$  and  $\phi \in L_2(R^d)$  and  $\xi \in R^d$  and  $x(\cdot)$  is a  $R^d$ -valued continuous function defined on  $[0, t]$  such that  $x(0) = 0$ .

2)  $H = -\Delta + V$  is the energy operator (or, Hamiltonian) and  $\Delta$  is a Laplacian and  $V: R^d \rightarrow R$  is a potential. This Formula (3.11) is called the Feynman-Kac formula. The application of the Feynman-Kac Formula (in various settings) has been given in the area: diffusion equations, the spectral theory of the Schrödinger operator, quantum mechanics, statistical physics, for more details, see the paper [8] and the book [12].

3) If we let  $\lambda = \rho^{-2}$ , the solution of this heat (or diffusion) equation is

$$\psi_\rho(t, \xi) = \int_{C_0^t} \exp \left\{ -\frac{i}{h} \int_0^t V(\rho x(s) + \xi) ds \right\} \cdot \phi(\rho x(s) + \xi) dm(s) \quad (3.12)$$

4) If we let  $h = \frac{m}{i\lambda} = -im\rho^{-2}$ , then

$$\psi_\rho(t, \xi) = \int_{C_0^t} \exp \left\{ +m\rho^2 \int_0^t V(\rho x(s) + \xi) ds \right\} \cdot \phi(\rho x(s) + \xi) dm(s) \quad (3.13)$$

is a solution of a heat (or diffusion) equation:

$$\frac{\partial \psi}{\partial t} = \frac{1}{m\rho^2} \left[ \left( \frac{m^2 \rho^2}{2m} \right) \frac{\partial \psi^2}{\partial \xi^2} + V(\xi) \psi \right]. \quad (3.14)$$

This equation is of the form:

$$\frac{\partial \psi}{\partial t} = \frac{1}{2} \frac{\partial \psi^2}{\partial \xi^2} + \frac{1}{m\rho^2} V(\xi) \psi. \quad (3.15)$$

5) If we let  $F(x) = \exp \left\{ -\frac{i}{h} \int_0^t V \left( \lambda^{-\frac{1}{2}} x(s) + \xi \right) ds \cdot \phi \left( \lambda^{-\frac{1}{2}} x(s) + \xi \right) \right\}$ , then we

can express the solution of the heat (or diffusion) equation by the formula

$$\psi_\rho(t, \xi) = \int_{C_0^t} F(\rho x) dm(x), \psi_\lambda(t, \xi) = \int_{C_0^t} F \left( \lambda^{-\frac{1}{2}} x \right) dm(x) \quad (3.16)$$

6) By this motivation, we first define the scale factor of the Wiener integral by the real number  $\rho > 0$  in the paper [13].

**Remark.** <Gratitude for the Referee> I am very grateful for the referee to comment in details.

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## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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