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On q-Analogues of Laplace Type Integral Transforms of q^2-Bessel Functions

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Abstract

The present paper deals with the evaluation of the q-Analogues of Laplece transforms of a product of basic analogues of q^2-special functions. We apply these transforms to three families of q-Bessel functions. Several special cases have been deducted.

Keywords

q-Extensions of Bessel Functions, q-Analogues of Laplace Type Integrals Transforms, q-Analogues of Gamma Function, q-Shift Factorials

1. Introduction

In the second half of twentieth century, there was a significant increase of activity in the area of the q-calculus mainly due to its application in mathematics, statistics and physics. In literature, several aspects of q-calculus were given to enlighten the strong inter disciplinary as well as mathematical character of this subject. Specifically, there have been many q-analogues and q-series representations of various kinds of special functions. In the case of q-Bessel function, there are two related q-Bessel functions introduced by Jackson [1] and denoted by Ismail [2] as

$$J_{\mu}^{(1)}(z;q)=\left(\frac{z}{2}\right)^{\mu}\sum_{n=0}^{\infty}\frac{\left(\frac{-z^2}{4}\right)^n}{(q,q)_{\mu+n}(q;q)_n},|z|<2 \quad (1)$$

$$J_{\mu}^{(2)}(z;q)=\left(\frac{z}{2}\right)^{\mu}\sum_{n=0}^{\infty}\frac{q^{n(n+\mu)}\left(\frac{-z^2}{4}\right)^n}{(q,q)_{\mu+n}(q;q)_n},z\in\mathbb{C} \quad (2)$$

The third related q-Bessel function $J_{\mu}^{(3)}(z;q)$ was introduced in a full case as [3]

$$J_{\mu}^{(3)}(z;q)=z^{\mu}\sum_{n=0}^{\infty}\frac{(-1)^{n}q^{\frac{n(n-1)}{2}}(qz^{2})^{n}}{(q,q)_{\mu+n}(q;q)_{n}},z\in\mathbb{C} \quad (3)$$

A certain type of Laplace transforms, which is called L_2-transform, was introduced by Yürekli and Sadek [4]. Then these transforms were studied in more details by Yürekli [5], [6]. Purohit and Kalla applied the q-Laplace transforms to a product of basic analogues of the Bessel function [7].

On the same manner, integral transforms have different q-analogues in the theory of q-calculus. The q-analogue of the Laplace type integral of the first kind is defined by [8] as

$${}_{q}L_{2}(f(\xi);y)=\frac{1}{1-q^{2}}\int_{0}^{y^{-1}}\xi E_{q^{2}}(q^{2}y^{2}\xi^{2})f(\xi)\mathrm{d}\xi \quad (4)$$

and expressed in terms of series representation as

$${}_{q}L_{2}(f(\xi);y)=\frac{(q^{2};q^{2})_{\infty}}{[2]_{q}y^{2}}\sum_{i=0}^{\infty}\frac{q^{2i}}{(q^{2};q^{2})_{i}}f(q^{i}y^{-1}). \quad (5)$$

On the other hand, the q-analogue of the Laplace type integral of the second kind is defined by [8] as

$${}_{q}\ell_{2}(f(\xi);y)=\frac{1}{1-q^{2}}\int_{0}^{\infty}\xi e_{q^{2}}(-y^{2}\xi^{2})f(\xi)\mathrm{d}_{q}\xi \quad (6)$$

whose q-series representation expressed as

$${}_{q}\ell_{2}(f(\xi);y)=\frac{1}{[2]_{2}(-y^{2};q^{2})_{\infty}}\sum_{i\in\mathbb{Z}}q^{2i}f(q^{i})(-y^{2};q^{2})_{i}. \quad (7)$$

In this paper we build upon analysis of [8]. Following [9], we discuss the q-Laplace type integral transforms (4) and (7) on the q-Bessel functions $J_{\mu}^{(1)}(z;q)$, $J_{\mu}^{(2)}(z;q)$ and $J_{\mu}^{(3)}(z;q)$, respectively. In Section 2, we recall some notions and definitions from the q-calculus. In Section 3, we give the main results to evaluate the q-analogue of Laplace transformation of q^2-Basel function. In Section 4, we discuss some special cases.

2. Definitions and Preliminaries

In this section, we recall some usual notions and notations used in the q-theory. It is assumed in this paper wherever it appears that $0<q<1$. For a complex number a, the q-analogue of a is introduced as $[a]_{q}=\frac{1-q^{a}}{1-q}$. Also, by fixing $a\in\mathbb{C}$, the q-shifted factorials are defined as

$$(a;q)_{0}=1;(a,q)_{n}=\prod_{k=0}^{n-1}(1-aq^{k}),n=1,2,\cdots;(a;q)_{\infty}=\lim_{n\to\infty}(a;q)_{n}. \quad (8)$$

This indeed lead to the conclusion

$$\left([n]_q\right)! = \frac{(q;q)_n}{(1-q)^n}, n \in \mathbb{N} \text{ and } (a;q)_x = \frac{(a;q)_\infty}{(aq^x;q)_\infty}. \quad (9)$$

The q-analogue of the exponential function of first and second type are respectively given in [10] by

$$e_q(x) = \sum_0^\infty \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}, |x| < 1. \quad (10)$$

and

$$E_q(x) = \sum_0^\infty \frac{(-1)^n q^{n\frac{n-1}{2}} x^n}{(q;q)_n}, x \in \mathbb{C}. \quad (11)$$

Indeed it has been shown that

$$e_q(x) = \frac{1}{(x;q)_\infty}, |x| < 1 \text{ and } E_q(x) = (x,q)_\infty, x \in \mathbb{C} \quad (12)$$

The finite q-Jackson and improper integrals are respectively defined by [11]

$$\int_0^x f(t)\,\mathrm{d}_q t = x(1-q)\sum_{k=0}^\infty q^k f(xq^k) \quad (13)$$

and

$$\int_0^{\infty/A} f(t)\,\mathrm{d}_q t = (1-q)\sum_{k\in\mathbb{Z}} \frac{q^k}{A} f\left(\frac{q^k}{A}\right). \quad (14)$$

The q-analogues of the gamma function of first and second type are respectively defined in [9] as

$$\Gamma_q(\alpha) = \int_0^{1/(1-q)} x^{\alpha-1} E_q(q(1-q)x)\,\mathrm{d}_q x, (\alpha > 0) \quad (15)$$

and

$$_q\Gamma(\alpha) = K(A;\alpha)\int_0^{\infty/A(1-q)} x^{\alpha-1} e_q(-(1-q)x)\,\mathrm{d}_q x \quad (16)$$

where, $\alpha_1 > 0$, where $K(A;\alpha)$ is the function given by

$$K(A;\alpha) = A^{\alpha-1}\frac{(-q/\alpha;q)_\infty(-\alpha;q)_\infty}{(-q^t/\alpha;q)_\infty(-\alpha q^{1-t};q)_\infty}. \quad (17)$$

Some useful results, for $x \neq 0,-1,-2,\cdots$, we use here are given by

$$\Gamma_q(\alpha) = \frac{(q;q)_\infty}{(1-q)^{\alpha-1}}\sum_{k=0}^\infty \frac{q^{k\alpha}}{(q;q)_k} = \frac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-x}, \quad (18)$$

and

$$_q\Gamma(\alpha) = \frac{K(A;\alpha)}{(1-q)^{\alpha-1}\left(-\frac{1}{A};q\right)_\infty}\sum_{k\in\mathbb{Z}}\left(\frac{q^k}{A}\right)\left(-\frac{1}{A};q\right)_k. \quad (19)$$

3. Main Theorems

Theorem 1. *Let* $J_{2\mu_1}^{(1)}\left(2\sqrt{a_1 t};q^2\right),\cdots,J_{2\mu_n}^{(1)}\left(2\sqrt{a_n t};q^2\right)$ *be a set of first kind of*

q^2-Bessel functions, $f(t)=t^{\Delta-1}\prod_{j=1}^{n}J_{2\mu_j}^{(1)}\left(2\sqrt{a_j t};q^2\right)$, *where* Δ, a_j *and* μ_j *for* $j=1,2,\cdots,n$ *are constants, then the q-analogue of Lablace transformation* ${}_qL_2$ *of* $f(t)$ *is given as:*

$$\begin{aligned}&{}_qL_2\left(f(t);s\right)\\&=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\left(\frac{-a_j}{s}\right)^{m_j}B_{m_j}\left(q^2\right)\Gamma_{q^2}\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)\end{aligned}\tag{20}$$

and the q-analogue of Laplace transformation ${}_ql_2$ of $f(t)$ is given as:

$$\begin{aligned}&{}_ql_2\left(f(t);s\right)\\&=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_j}{s}\right)^{m_j}B_{m_j}\left(q^2\right)_{q^2}\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)}.\end{aligned}\tag{21}$$

where $Re(s)>0$, $Re(\Delta)>0$ and

$$A_\Delta=\frac{\left(1-q^2\right)^{\Delta/2}}{[2]s^{\Delta+1}\left(q^2;q^2\right)_\infty},B_{m_j}\left(q^2\right)=\frac{\left(q^{2\mu_j+m_j+2};q^2\right)_\infty\left(1-q^2\right)\frac{m_j+\mu_j-1}{2}}{\left(q^2;q^2\right)_{m_j}}$$

Proof. Now,

$${}_qL_2\left(f(t);s\right)=\frac{\left(q^2;q^2\right)_\infty}{[2]s^2}\sum_{k=0}^{\infty}\frac{q^{2k}f\left(q^ks^{-1}\right)}{\left(q^2;q^2\right)_k}$$

since

$$J_{2\mu_j}^{(1)}\left(2\sqrt{a_jt};q^2\right)=\left(\frac{2\sqrt{a_jt}}{2}\right)^{2\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-\left(2\sqrt{a_jt}\right)^2}{4}\right)^{m_j}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}}$$

so

$$\begin{aligned}&{}_qL_a\left(f(t);s\right)\\&=\frac{\left(q^2;q^2\right)_\infty}{[2]s^2}\sum_{k=0}^{\infty}\frac{q^{2k}}{\left(q^2;q^2\right)_k}\left(q^ks^{-1}\right)^{\Delta-1}\prod_{j=1}^{n}\left(\sqrt{a_jq^ks^{-1}}\right)^{2\mu_j}\\&\cdot\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}\left(a_jq^ks^{-1}\right)^{m_j}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}}\\&=\frac{\left(q^2;q^2\right)_\infty}{[2]s^{\Delta+1}}\sum_{k=0}^{\infty}\frac{q^{k(\Delta+1)}}{\left(q^2;q^2\right)_k}\prod_{j=1}^{n}\left(\frac{a_jq^k}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}\left(\frac{a_jq^k}{s}\right)^{m_j}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}}\\&=\frac{\left(q^2;q^2\right)_\infty}{[2]s^{\Delta+1}}\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}\left(\frac{a_j}{s}\right)^{m_j}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}}\sum_{k=0}^{\infty}\frac{q^{k\left(\Delta+1+m_j+\mu_j\right)}}{\left(q^2;q^2\right)_k}\end{aligned}\tag{22}$$

Since

$$\Gamma_{q^2}(\alpha)=\frac{\left(q^2;q^2\right)_\infty}{\left(1-q^2\right)^{\alpha-1}}\sum_{k=0}^{\infty}\frac{q^{2k\alpha}}{\left(q^2;q^2\right)_k}$$

putting $\alpha=\frac{1+\Delta+m_j+\mu_j}{2}$, so (22) becomes:

$$_qL_s\left(f(t);s\right)=\frac{1}{[2]s^{\Delta+1}}\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}\left(\frac{a_j}{s}\right)^{m_j}\left(1-q^2\right)^{\frac{1+\Delta+m_j+\mu_j}{2}}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}}.$$

$$\Gamma_{q^2}\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)\tag{23}$$

Since

$$\left(q^2;q^2\right)_{2\mu_j+m_j}=\frac{\left(q^2;q^2\right)_\infty}{\left(q^2q^{2\mu_j+m_j};q^2\right)_\infty}$$

so (23) becomes:

$$_qL_2\left(f(t);s\right)=\frac{\left(1-q^2\right)^{\Delta/2}}{[2]s^{\Delta+1}\left(q^2;q^2\right)_\infty}\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}$$

$$\cdot\sum_{m_j=0}^{\infty}\frac{\left(\frac{a_j}{s}\right)(-1)^{m_j}\left(q^{2\mu_j+m_j+2};q^2\right)_\infty\left(1-q^2\right)^{\frac{m_j+\mu_j-1}{2}}}{\left(q^2;q^2\right)_{m_j}}.$$

$$\Gamma_{q^2}\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)$$

$$=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\left(\frac{a_j}{s}\right)(-1)^{m_j}B_{m_j}\left(q^2\right)\Gamma_{q^2}\frac{\left(m_j+\mu_j+\Delta+1\right)}{2}$$

Similarly we have

$$_ql_2\left(f(t);s\right)=\frac{1}{[2]}\frac{1}{\left(-s^2;q^2\right)_\infty}\sum_{k=0}^{\infty}q^{2k}\left(-s^2;q^2\right)_k\left(q^k\right)^{\Delta-1}\prod_{j=1}^{n}J_{2\mu_j}^{(1)}\left(2\sqrt{a_jq^k};q^2\right)$$

$$=\frac{1}{[2]}\frac{1}{\left(-s^2;q^2\right)_\infty}\sum_{k=0}^{\infty}q^{2k}\left(-s^2;q^2\right)_k\left(q^k\right)^{\Delta-1}\prod_{j=1}^{n}\left(a_jq^k\right)^{\mu_j}.$$

$$\sum_{m_j=0}^{\infty}\frac{\left(-a_jq^k\right)^{m_j}}{\left(q^2;q^2\right)_{m_j}+2\mu_j}$$

$$=\prod_{j=1}^{n}\frac{\left(a_j\right)^{\mu_j}}{[2]}\sum_{m_j=0}^{\infty}\frac{\left(-a_j\right)^{m_j}}{\left(q^2;q^2\right)_{m_j+2\mu_j}\left(q^2;q^2\right)_{m_j}}\sum_{k=0}^{\infty}\frac{\left(-s^2;q^2\right)_kq^{k\left(m_j+\mu_j+\Delta+1\right)}}{\left(-s^2;q^2\right)_\infty}$$

Now using

$$
{}_{q^2}\Gamma(\alpha)=\frac{K(A;\alpha)}{\left(1-q^2\right)^{\alpha-1}\left(-\frac{1}{A};q^2\right)_\infty}\sum_{k\in Z}\left(\frac{q^K}{A}\right)^\alpha\left(-\frac{1}{A};q^2\right)_K
$$

with $A=\frac{1}{s^2}$, $\alpha=\frac{m_j+\mu_j+\Delta+1}{2}$ we get

$$
\begin{aligned}
&{}_ql_2\left(f(t);s\right)\\
&=\prod_{j=1}^{m}\frac{\left(a_j\right)^{\mu_j}}{[2]s^{\mu_j+\Delta+1}}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_j}{s}\right)^{m_j}\left(1-q^2\right)^{\frac{m_j+\mu_j+\Delta+1}{2}}q^2\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)\left(q^2;q^2\right)_{m_j+2\mu_j}\left(q^2;q^2\right)_{m_j}}\\
&=\frac{\left(1-q^2\right)^{\frac{\Delta}{2}}}{[2]s^{\Delta+1}\left(q^2;q^2\right)_\infty}\prod_{j=1}^{m}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_j}{s}\right)^{m_j}\left(1-q^2\right)^{\frac{m_j+\mu_j-1}{2}}\left(q^{m_j+2\mu_j+2};q^2\right)_\infty}{K\left(\frac{1}{s^2},\frac{m_j+\mu_j+\Delta+1}{2}\right)\left(q^2;q^2\right)_{m_j}}\cdot\\
&\quad {}_{q^2}\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)\\
&=A_\Delta\prod_{j=1}^{m}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_j}{s}\right)^{m_j}}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)}B_{m_j}\left(q^2\right){}_{q^2}\Gamma\left(m_j+\mu_j+\Delta+1\right)
\end{aligned}
$$

Theorem 2. *Let* $J^{(2)}_{2\mu_1}\left(2\sqrt{a,t};q^2\right),\cdots,J^{(2)}_{2\mu_n}\left(2\sqrt{at};q^2\right)$ *be a set of second order* q^2*-Bessel function,* $f(t)=t^{\Delta-1}\prod_{j=1}^{n}J^{(2)}_{2\mu_j}\left(2\sqrt{a_jt};q^2\right)$ *where* Δ,a_j *and* μ_j *for* $j=1,2,\cdots,n$ *are constants then* ${}_qL_2$*-transform of* $f(t)$ *is given as:*

$$
\begin{aligned}
{}_qL_2\left(f(t),s\right)=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}(-1)^{m_j}q^{2m_j\left(m_j+2\mu_j\right)}\left(\frac{a_j}{s}\right)^{m_j+\mu_j}\\
\cdot B_{m_j}\left(q^2\right)\Gamma_{q^2}\left(m_j+\mu_j+\Delta+1\right)
\end{aligned}
\tag{24}
$$

and the q-analogue of Laplace transformation ${}_ql_2$ of $f(t)$ is given as:

$$
\begin{aligned}
{}_ql_2\left(f(t);s\right)=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_j}{s}\right)^{m_j}q^{2m_j\left(m_j+2\mu_j\right)}}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)}\\
\cdot B_{m_j}\left(q^2\right){}_{q^2}\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)
\end{aligned}
\tag{25}
$$

Proof. Now,

$$
J^{(2)}_{2\mu_j}\left(2\sqrt{a_jt};q^2\right)=\left(\frac{2\sqrt{a_jt}}{2}\right)^{2\mu_j}\sum_{m_j}^{\infty}\frac{\left(-\frac{\left(2\sqrt{a_jt}\right)^2}{4}\right)^{m_j}q^{2m_j\left(m_j+2a_j\right)}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}}
$$

so

$$
{}_qL_2\left(f(t);s\right)=\frac{\left(q^2;q^2\right)_\infty}{[2]s^2}\sum_{k=0}^{\infty}\frac{q^{2k}}{\left(q^2;q^2\right)_k}\left(q^k s^{-1}\right)^{\Delta-1}\prod_{j=1}^{n}\left(\frac{2\sqrt{a_jq^ks^{-1}}}{2}\right)^{2\mu_j}
\cdot\sum_{m_j=0}^{\infty}\frac{\left(-\frac{\left(2\sqrt{a_jq^ks^{-1}}\right)^2}{4}\right)^{m_j}q^{2m_j(m_j+2\mu_j)}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}} \tag{26}
$$

By the same argument we can write (26) as

$$
{}_qL_2\left(f(t);s\right)=\frac{\left(q^2;q^2\right)_\infty}{[2]s^{\Delta+1}\left(q^2;q^2\right)_\infty}\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}q^{2m_j(m_j+2\mu_j)}}{\left(q^2;q^2\right)_{m_j}}
\cdot\left(\frac{a_j}{s}\right)^{m_j+\mu_j}\left(q^{2\mu_j+m_j+2};q^2\right)_\infty\sum_{k=0}^{\infty}\frac{q^{k(m_j+\mu_j+1+\Delta)}}{\left(q^2;q^2\right)_k}
$$

put $\alpha=\dfrac{m_j+\mu_j+\Delta+1}{2}$ in $\Gamma_{q^2}(\alpha)$, then

So (25) becomes:

$$
{}_qL_2\left(f(t);s\right)=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}(-1)^{m_j}q^{2m_j(m_j+2\mu_j)}\left(\frac{a_j}{s}\right)^{m_j+\mu_j}
\cdot B_{m_j}\left(q^2\right)\Gamma_{q^2}\left(m_j+\mu_j+\Delta+1\right)
$$

Similarly

$$
{}_ql_2\left(f(t);s\right)=\frac{1}{[2]}\frac{1}{\left(-s^2;q^2\right)_\infty}\sum_{k=0}^{\infty}q^{2k}\left(-s^2;q^2\right)_k\left(q^k\right)^{\Delta-1}\prod_{j=1}^{n}\left(a_jq^k\right)^{\mu_j}
\cdot\sum_{m_j=0}^{\infty}\frac{\left(-a_jq^k\right)^{m_j}q^{2m_j(m_j+2\mu_j)}}{\left(q^2;q^2\right)_{m_j+2\mu_j}\left(q^2;q^2\right)_{m_j}}
$$

Put $A=\dfrac{1}{s^2},\alpha=\dfrac{m_j+\mu_j+\Delta+1}{2}$ we get

$$
{}_ql_2\left(f(t);s\right)
$$

$$
=\frac{1}{[2]}\prod_{j=1}^{n}\left(a_j\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(-a_j\right)^{m_j}q^{2m_j(m_j+2\mu_j)}\left(1-q^2\right)^{\frac{m_j+\mu_j+\Delta+1}{2}}q^2\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)}{\left(q^2;q^2\right)_{m_j+2\mu_j}K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)s^{m_j+\mu_j+\Delta+1}}
$$

$$
=A_\Delta\prod_{m_j=0}^{\infty}\frac{\left(\frac{-a_j}{s}\right)^{m_j}q^{2m_j(m_j+2\mu_j)}}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)}B_{m_j}\left(q^2\right)_{q^2}\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)
$$

Theorem 3. *Let* $J^{(3)}_{2\mu_j}\left(\sqrt{q^{-1}a_1t};q^2\right),\cdots,J^{(3)}_{2\mu_n}\left(\sqrt{q^{-1}a_nt};q^2\right)$ *be s set of* q^2*-Bessel* *functions,* $f(t)=t^{\Delta-1}\prod_{j=1}^{n}J_{2\mu_j}^{(3)}\left(\sqrt{q^{-1}a_jt};q^2\right)$ *where* Δ, a_j *and* μ_j *for* $j=1,2,\cdots,n$ *are constants. Then we have*

$$ {}_qL_2\left(f(t);s\right)=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{qs}\right)^{\mu_j}\sum_{m_j=0}^{\infty}(-1)^{m_j}q^{m_j(m_j-1)}\left(\frac{a_jq}{s}\right)^{m_j} \cdot B_{m_j}\left(q^2\right)\Gamma_{q^2}\left(\frac{m_j+\mu_j+\Delta+1}{2}\right) \tag{27} $$

and the q-analogue of Laplace transformation ${}_ql_2$ of $f(t)$ is given by:

$$ {}_ql_2\left(f(t);s\right)=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{qs}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_jq}{s}\right)^{m_j}q^{m_j(m_j-1)}}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)} \cdot B_{m_j}\left(q^2\right){}_{q^2}\Gamma\frac{\left(m_j+\mu_j+\Delta+1\right)}{2} \tag{28} $$

Proof. Now

$$ J_{2\mu_j}^{(3)}\left(\sqrt{a_jq^{k-1}s^{-1}};q^2\right)=\left(\sqrt{a_jq^{k-1}s^{-1}}\right)^{2\mu_j}\sum_{m_j=0}^{\infty}(-1)^{m_j}\frac{q^{\frac{2m_j(m_j-1)}{2}}\left(q^2a_jq^{k-1}s^{-1}\right)^{m_j}}{\left(q^2;q^2\right)_{m_j+2\mu_j}\left(q^2;q^2\right)_{m_j}} $$

$$ {}_qL_2\left(f(t);s\right)=\frac{\left(q^2;q^2\right)_\infty}{[2]s^2}\sum_{k=0}^{\infty}\frac{q^{2k}\left(q^ks^{-1}\right)^{\Delta-1}}{\left(q^2;q^2\right)_k}\prod_{j=1}^{n}\left(a_jq^{k-1}s^{-1}\right)^{\mu_j} \cdot\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}q^{m_j(m_j-1)}\left(q^2a_jq^{k-1}s^{-1}\right)^{m_j}}{\left(q^2;q^2\right)_{m_j+2\mu_j}\left(q^2;q^2\right)_{m_j}} $$

put $\alpha=\frac{m_j+\mu_j+\Delta+1}{2}$, we get

$$ {}_qL_2\left(f(t);s\right) =A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{qs}\right)^{\mu_j}\sum_{m_j=0}^{\infty}(-1)^{m_j}q^{m_j(m_j-1)}\left(\frac{a_jq}{s}\right)^{m_j}B_{m_j}\left(q^2\right)\Gamma_{q^2}\left(\frac{m_j+\mu_j+\Delta+1}{2}\right) $$

Similarly

$$ {}_ql_2\left(f(t);s\right) =\frac{1}{[2]}\frac{1}{\left(-s^2;q^2\right)_\infty}\prod_{j=1}^{n}\left(q^{k-1}\right)^{\mu_j}\left(a_j\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}q^{m_j(m_j-1)}\left(q^{km_j}\right)\left(qa_j\right)^{m_j}}{\left(q^2;q^2\right)_{m_j+\mu_2}\left(q^2;q^2\right)_{m_j}} \cdot\sum_{k=0}^{\infty}q^{k(\Delta+1)}\left(-s^2;q^2\right)_k . $$

Put $\alpha=\frac{m_j+\mu_j+\Delta+1}{2}$, $A=\frac{1}{s^2}$ we get

$$
{}_{q}l_2\left(f\left(t\right);s\right)=\frac{\left(1-q^2\right)^{\frac{\Delta}{2}}}{[2]s^{\Delta+1}\left(q^2;q^2\right)_\infty}\prod_{j=1}^{n}\left(\frac{a_j}{qs}\right)^{\mu_j}
\cdot\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_jq}{s}\right)^{m_j}q^{m_j\left(m_j-1\right)}B_{m_j}\left(q^2\right){}_{q^2}\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)}.
$$

4. Special Cases

1) Let $n=1$, $\mu_1=\mu$, $a_1=a$ in above theorems, respectively we have:

$$
{}_{q}L_2\left(t^{\Delta-1}J_{2\mu}^{(1)}\left(2\sqrt{at};q^2\right);s\right)
=A_\Delta\left(\frac{a}{s}\right)^{\mu}\sum_{m=0}^{\infty}\left(\frac{-a}{s}\right)^{m}B_m\left(q^2\right)\Gamma_{q^2}\left(\frac{m+\mu+\Delta+1}{2}\right) \tag{29}
$$

$$
{}_{q}l_2\left(t^{\Delta-1}J_{2\mu}^{(1)}\left(2\sqrt{at};q^2\right);s\right)
=A_\Delta\left(\frac{a}{s}\right)^{\mu}\sum_{m=0}^{\infty}\frac{\left(\frac{-a}{s}\right)^{m}}{K\left(\frac{1}{s^2};\frac{m+\mu+\Delta+1}{2}\right)}B_m\left(q^2\right){}_{q^2}\Gamma\left(\frac{m+\mu+\Delta+1}{2}\right) \tag{30}
$$

$$
{}_{q}L_2\left(t^{\Delta-1}J_{2\mu}^{(2)}\left(2\sqrt{at};q^2\right);s\right)
=A_\Delta\left(\frac{a}{s}\right)^{\mu}\sum_{m=0}^{\infty}(-1)^m q^{2m(m+2\mu)}B_m\left(q^2\right)\Gamma_{q^2}\left(\frac{m+\mu+\Delta+1}{2}\right) \tag{31}
$$

$$
{}_{q}l_2\left(t^{\Delta-1}J_{2\mu}^{(2)}\left(2\sqrt{at};q^2\right);s\right)
=A_\Delta\left(\frac{a}{s}\right)^{\mu}\sum_{m=0}^{\infty}\frac{\left(\frac{-a}{s}\right)^{m}q^{2m(m+2\mu)}}{K\left(\frac{1}{s^2};\frac{m+\mu+\Delta+1}{2}\right)}B_m\left(q^2\right){}_{q^2}\Gamma\left(\frac{m+\mu+\Delta+1}{2}\right) \tag{32}
$$

$$
{}_{q}L_2\left(t^{\Delta-1}J_{2\mu}^{(3)}\left(2\sqrt{aq^{-1}t};q^2\right);s\right)
=A_\Delta\left(\frac{a}{qs}\right)^{\mu}\sum_{m=0}^{\infty}(-1)^m q^{m(m-1)}\left(\frac{aq}{s}\right)^{m}B_m\left(q^2\right)\Gamma_{q^2}\left(\frac{m+\mu+\Delta+1}{2}\right) \tag{33}
$$

$$
{}_{q}l_2\left(t^{\Delta-1}J_{2\mu}^{(3)}\left(2\sqrt{aq^{-1}t};q^2\right);s\right)
=A_\Delta\left(\frac{a}{s}\right)^{\mu}\sum_{m=0}^{\infty}\frac{\left(\frac{aq}{s}\right)^{m}q^{m(m-1)}}{K\left(\frac{1}{s^2};\frac{m+\mu+\Delta+1}{2}\right)}B_m\left(q^2\right){}_{q^2}\Gamma\left(\frac{m+\mu+\Delta+1}{2}\right) \tag{34}
$$

2) Put $\Delta-1=\mu$ in part (29) above, then

$$
{}_{q}L_2\left(t^{\mu}J_{2\mu}^{(1)}\left(2\sqrt{at};q^2\right);s\right)=\frac{\left(1-q^2\right)^{\frac{\mu+1}{2}}}{[2]s^{\mu+2}\left(q^2;q^2\right)_\infty}\left(\frac{a}{s}\right)^{\mu}
$$

$$\sum_{m=0}^{\infty}\left(\frac{a}{s}\right)^{m}\frac{\left(q^{2\mu+m+2};q^{2}\right)_{\infty}\left(1-q^{2}\right)^{\frac{m+\mu-1}{2}}}{\left(q^{2};q^{2}\right)_{m}}\Gamma_{q^{2}}\left(\frac{m+2\mu+2}{2}\right)$$

$$=\frac{\left(\frac{a}{s}\right)^{\mu}}{[2]s^{\mu+2}}\sum_{m=0}^{\infty}\frac{\left(\frac{-a}{s}\right)^{m}}{\left(q^{2};q^{2}\right)_{m}}=\frac{(a)^{\mu}}{[2]s^{2\mu+2}}e_{q^{2}}\left(\frac{-a}{s}\right).$$

3) Put $\mu=0$ we get

$${}_{q}L_{2}\left(J_{0}^{(1)}\left(2\sqrt{at};q^{2}\right);s\right)=\frac{1}{[2]s^{2}}e_{q^{2}}\left(\frac{-a}{s}\right).$$

which is the same result cited by [7].

4) Put $\Delta-1$ in (33), then

$${}_{q}L_{2}\left(t^{\mu}J_{2\mu}^{(3)}\left(2\sqrt{q^{-1}at}\right);s\right)=\frac{\left(1-q^{2}\right)^{\frac{\mu+1}{2}}}{[2]s^{\mu+2}\left(q^{2};q^{2}\right)_{\infty}}\left(\frac{a}{qs}\right)^{\mu}.$$

$$\sum_{m=0}^{\infty}(-1)^{m}\frac{q^{m(m-1)}\left(\frac{aq}{s}\right)^{m}\left(q^{2\mu+m+2};q^{2}\right)\left(1-q^{2}\right)^{\frac{m+\mu-1}{2}}\Gamma_{q^{2}}\left(\frac{m+2\mu+2}{2}\right)}{\left(q^{2};q^{2}\right)_{m}}$$

$$=\frac{\left(\frac{a}{q}\right)^{\mu}}{[2]s^{2\mu+2}}\sum_{m=0}^{\infty}(-1)^{m}\frac{\left(\frac{aq}{s}\right)^{m}q^{2m\frac{m-1}{2}}}{\left(q^{2};q^{2}\right)_{m}}=\frac{\left(\frac{a}{q}\right)^{\mu}}{[2]s^{2\mu+2}}E_{q^{2}}\left(\frac{aq}{s}\right).$$

5) Let $\mu=0$ and $a=0$ in (34), then

$${}_{q}L_{2}\left(t^{\Delta-1};s\right)=\frac{\left(1-q^{2}\right)^{\frac{\Delta}{2}}}{[2]s^{\Delta+1}}\frac{1}{K\left(\frac{1}{s^{2}};\frac{\Delta+1}{2}\right)}\left(1-q^{2}\right)^{-\frac{1}{2}}\Gamma_{q^{2}}\left(\frac{\Delta+1}{2}\right)$$

replacing $\Delta-1$ by α, we get

$${}_{q}L_{2}\left(t^{\alpha};s\right)=\frac{\left(1-q^{2}\right)^{\frac{\alpha}{2}}}{[2]s^{\alpha+2}}\frac{1}{K\left(\frac{1}{s^{2}};1+\frac{\alpha}{2}\right)}\Gamma_{q^{2}}\left(1+\frac{\alpha}{2}\right)$$

which is the same result in [8].

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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