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The Inertial Manifold for Class Kirchhoff-Type Equations with Strongly Damped Terms and Source Terms

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Abstract

In this paper, we study the inertial manifolds for a class of the Kirchhoff-type equations with strongly damped terms and source terms. The inertial manifold is a finite dimensional invariant smooth manifold that contains the global attractor, attracting the solution orbits by the exponential rate. Under appropriate assumptions, we firstly exert the Hadamard's graph transformation method to structure a graph norm of a Lipschitz continuous function, and then we prove the existence of the inertial manifold by showing that the spectral gap condition is true.

Keywords

Inertial Manifold, Hadamard's Graph Transformation Method, Lipschitz Continuous, Spectral Gap Condition

1. Introduction

In this paper, we concerned the equation:

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2 + \|\nabla^m v\|^2\right)\Delta u - \beta\Delta u_t + g_1(u,v) = f_1(x), \\ v_{tt} + M\left(\|\nabla u\|^2 + \|\nabla^m v\|^2\right)(-\Delta)^m v + \beta(-\Delta)^m v_t + g_2(u,v) = f_2(x), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0, \quad \left.\dfrac{\partial_i v}{\partial \mu^i}\right|_{\partial\Omega} = 0\ (i = 1,2,\cdots,m-1), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$, $\beta > 1$ is a constant and $f_i(x)(i=1,2)$ is a given source term. Moreover, $M\left(\|\nabla u\|^2+\|\nabla^m v\|^2\right)$ is a scalar function. Then the assumptions on M and $g_i(u,v)$ will be specified later.

Nowadays, the study on the complexity of the space-time of high dimensional and infinite dimensional dynamical systems has gradually become the focus of nonlinear scientific research. In recent years, the inertial manifold has been found in the researches of the long time behavior of the solution and the attractor structure. The inertial manifold is a tool to describe the interaction between the low frequency components and the high frequency components [1]. When the flow has an inertial manifold, its high frequency description depends on the low frequency, and it contains attractors and exponentially attracts solution of the track, which realizes that the infinite dimensional dynamical system is reduced to a finite dimensional dynamical systems of the finite dimensional invariable Lipschitz manifold. Therefore, the inertial manifold is a powerful tool to study the long-time behavior of nonlinear dissipative systems and expose the real or seemingly chaotic structure of nonlinear dynamics.

In addition, the study of inertial manifold is of great significance. The central idea of the methods that people use to solve practical problems such as Galerkin method, Cellular automaton and Coupled map, are to discuss the infinite dimensional problem into a finite dimensional problem. So, the inertial manifold is of great significance to the development of nonlinear science.

In 1988, the concept of inertial manifold was first proposed in the study of infinite dimensional dynamical system by R. Temam, C. Foias and Sell G.R. [2]. They considered the equation as following:

$$u_t+Au+B(u,u)+C(u)-f=0. \tag{1.2}$$

where Au is a linear unbounded self-adjoint operator on H with domain $D(A)$ dense in H.

In 2010, Guoguang Lin and Jingzhu Wu [3] studied the existence of the inertial manifold of Boussinesq equation:

$$\begin{cases} u_{tt}-\alpha\Delta u_t-\Delta u+u^{2k+1}=F(x,y), \\ u(x,y,0)=u_0(x,y), \\ u(x,y,t)=u(x+\pi,y,t)=u(x,y+\pi,t)=0, \quad (x,y)\in\Omega, \end{cases} \tag{1.3}$$

where $\Omega=(0,\pi)^2\in R^2, t>0, \alpha>2$.

In 2016, Ling Chen, Wei Wang and Guoguang Lin [4] established the exponential attractors and inertial manifolds of the higher-order Kirchhoff-type equation:

$$u_{tt}+(-\Delta)^m u_t+\phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m u+g(u)=f(x). \tag{1.4}$$

There are many researches on inertial manifolds for nonlinear wave equations (see [5] [6]). Concerning the inertial manifold, many difficulties are solved. So we take advantage of Hadamard's graph transformation method in this paper.

The paper is arranged as follows. In Section 2, some assumptions, notations and lemmas are stated. In Section 3, the existence of the inertial manifold is established.

2. Preliminaries

For convenience, we first introduce the following notations:

$$X = H_0^2(\Omega)\times H_0^{2m}(\Omega)\times H_0^1(\Omega)\times H_0^m(\Omega),\ X_0 = H_0^1(\Omega)\times H_0^m(\Omega),$$

$c_i\ (i=1,2,\cdots)$ denotes different positive constants, $(\cdot,\cdot)$ and $\|\cdot\|$ are the inner product and norm of $L^2(\Omega)$, $\|\cdot\|_{-m}$ is the norm of $H^{-m}(\Omega)$.

Next, we give some assumptions and definition needed in the proof of our results.

$$(A_1)\ g_i(u,v)\in C^1(\Omega),\quad (i=1,2).$$

$$(A_2)\ \varepsilon \le m_0 \le M(s) \le m_1 \le \frac{(\beta-1)\mu_k}{4}.$$

Definition 2.1. [7] Let $A: X\to X$ be an operator and assume that $F\in C_b(X,X)$ satisfies the Lipschitz condition

$$\|F(U)-F(V)\|_X \le l_F\|U-V\|_X,\quad U,V\in X. \tag{2.1}$$

The operator A is called satisfy the spectral gap condition relative to F, if the point spectrum of the operator A can be divided into two parts σ_1 and σ_2, of which σ_1 is finite, and such that, if

$$\Lambda_1 = \sup\{\operatorname{Re}\lambda \mid \lambda\in\sigma_1\},\ \Lambda_2 = \inf\{\operatorname{Re}\lambda \mid \lambda\in\sigma_2\}, \tag{2.2}$$

and

$$X_i = span\{w_j \mid \lambda_j\in\sigma_i\},\quad i=1,2. \tag{2.3}$$

Then

$$\Lambda_2-\Lambda_1 > 4l_F, \tag{2.4}$$

and the orthogonal decomposition $X = X_1\oplus X_2$ holds with continuous orthogonal projections $P_1: X\to X_1, P_2: X\to X_2$.

Lemma 2.1. [8] Let the eigenvalues $\mu_j^{\pm}, j\ge 1$ be arranged in nondecreasing order. For all $\forall m\in N$, there exists $N\ge m$ such that μ_N^- and μ_{N+1}^- are consecutive.

3. The Inertial Manifold

Equation (1.1) is equivalent to the following one order evolution equation:

$$U_t + AU = F(U), \tag{3.1}$$

where $U=(u,v,p,q)^{\mathrm{T}}\in X, p=u_t, q=v_t$,

$$A=\begin{pmatrix} 0 & 0 & -I & 0\\ 0 & 0 & 0 & -I\\ -M(s)\Delta & 0 & -\beta\Delta & 0\\ 0 & M(s)(-\Delta)^m & 0 & \beta(-\Delta)^m \end{pmatrix}, F(U)=\begin{pmatrix} 0\\ 0\\ f_1(x)-g_1(u,v)\\ f_2(x)-g_2(u,v)\end{pmatrix} \tag{3.2}$$

$$D(A)=\{(u,v)\in H_0^2(\Omega)\times H_0^{2m}(\Omega)\}\times H_0^1(\Omega)\times H_0^m(\Omega).$$

We consider the usual graph norm in *X*, as follows

$$(U,V)_X=\left(M(s)\nabla u,\nabla\overline{u_1}\right)+\left(M(s)\nabla^m v,\nabla^m\overline{v_1}\right)+\left(\overline{p_1},p\right)+\left(\overline{q_1},q\right), \quad (3.3)$$

where $U=(u,v,p,q)^{\mathrm{T}}, V=(u_1,v_1,p_1,q_1)^{\mathrm{T}}\in X, \overline{u_1}, \overline{v_1}, \overline{p_1}, \overline{q_1}$ respectively represent the conjugation of u_1,v_1,p_1,q_1. Evidently, the operator A is monotone, for $U\in D(A)$, we obtain

$$\begin{aligned}(AU,U)_X&=\left(-M(s)\nabla p,\nabla\overline{u}\right)+\left(-M(s)\nabla^m q,\nabla^m\overline{v}\right)\\&\quad+\left(\overline{p},-M(s)\Delta u-\beta\Delta p\right)+\left(\overline{q},M(s)(-\Delta)^m v+\beta(-\Delta)^m q\right)\\&=\left(-M(s)\nabla p,\nabla\overline{u}\right)+\left(\nabla\overline{p},M(s)\nabla u\right)+\beta\|\nabla p\|^2\\&\quad+\left(-M(s)\nabla^m q,\nabla^m\overline{v}\right)+\left(\nabla^m\overline{q},M(s)\nabla^m v\right)+\beta\|\nabla^m q\|^2\\&=\beta\|\nabla p\|^2+\beta\|\nabla^m q\|^2\geq 0.\end{aligned} \quad (3.4)$$

So, $(AU,U)_X$ is a nonnegative and real number.

In order to determine the eigenvalues of *A*, we consider the eigenvalues equation:

$$AU=\lambda U,\quad U=(u,v,p,q)^{\mathrm{T}}\in X. \quad (3.5)$$

That is

$$\begin{cases}-p=\lambda u, & (3.6)\\ -q=\lambda v, & (3.7)\\ -M(s)\Delta u-\beta\Delta p=\lambda p, & (3.8)\\ M(s)(-\Delta)^m v+\beta(-\Delta)^m q=\lambda q. & (3.9)\end{cases}$$

Substitute (3.6), (3.7) into (3.8), (3.9), we obtain

$$\begin{cases}\lambda^2 u-M(s)\Delta u+\beta\lambda\Delta u=0, & (3.9)\\ \lambda^2 v+M(s)(-\Delta)^m v-\beta\lambda(-\Delta)^m v=0. & (3.10)\end{cases}$$

Replacing u,v with u_k,v_k, taking u,v inner product with the Equations (3.9), (3.10), and adding them together, we have

$$\lambda_k^k\left(\|u_k\|^2+\|v_k\|^2\right)+M(s)\left(\|\nabla u_k\|^2+\|\nabla^m v_k\|^2\right)-\lambda_k\beta\left(\|\nabla u_k\|^2+\|\nabla^m v_k\|^2\right)=0. \quad (3.11)$$

(3.11) is regard as a quadratic equation with one unknown about λ_k, so we get

$$\lambda_k^{\pm}=\frac{\beta\mu_k\pm\sqrt{\beta^2\mu_k^2-4M(s)\mu_k}}{2}, \quad (3.12)$$

for $\forall k\geq 1$, we have

$$\|u_k\|^2+\|v_k\|^2=1,\ \|\nabla u_k\|^2+\|\nabla^m v_k\|^2=\mu_k,\ \|\nabla^{-1}u_k\|^2+\|\nabla^{-m}v_k\|^2=\frac{1}{\mu_k}. \quad (3.13)$$

and μ_k is non-derogatory. If $\beta\geq\frac{4M(s)}{\mu_k}+1$, because of $\beta>1$, then

$\beta^2 \ge \frac{4M(s)}{\mu_k}$, we can get the eigenvalues of A are all positive and real numbers.

The corresponding **eigenfunction** is as follows

$$U_k^{\pm} = \left(u_k, v_k, -\lambda_k^{\pm} u_k, -\lambda_k^{\pm} v_k\right). \tag{3.14}$$

Lemma 3.1. $g_i : X_0 \to X_0, (i = 1, 2)$ is uniformly bounded and globally Lipschitz continuous.

Proof. $\forall (u,v),(u_1,v_1) \in X_0$, by (A_1), we have

$$\begin{aligned}
&\left\|g_i(u,v) - g_i(u_1,v_1)\right\|_{X_0} \\
&\le \left\|g_{iu}(\xi,v)(u-u_1)\right\|_{H_0^1(\Omega)} + \left\|g_{iv}(u,\eta)(v-v_1)\right\|_{H_0^m(\Omega)} \\
&\le c_{i_1}\left\|u-u_1\right\|_{H_0^1(\Omega)} + c_{i_2}\left\|v-v_1\right\|_{H_0^m(\Omega)} \\
&\le c_{i_3}\left(\left\|u-u_1\right\|_{H_0^1(\Omega)} + \left\|v-v_1\right\|_{H_0^m(\Omega)}\right),
\end{aligned} \tag{3.15}$$

where

$$c_{i_3} = \max\left\{c_{i_1}, c_{i_2}\right\}, \xi = u + (1-\theta_1)u_1, \eta = v + (1-\theta_2)v_1.$$

Let $l_i = c_{i_3}$, then l_i is Lipschitz coefficient of $g_i(u,v)$.

Theorem 3.1. l_i is Lipschitz constant of $g_i(u,v)$, when $\beta \ge \frac{4M(s)}{\mu_k} + 1$, set $N_1 \in N$, such that $N \ge N_1$, we obtain

$$\left(\mu_{N+1} - \mu_N\right)\left(\frac{1}{2} - \frac{1}{2}\sqrt{(\beta-1)\mu_1 - 4m_1}\right) \ge \frac{4l}{\sqrt{(\beta-1)\mu_1 - 4m_1}} + 1, \tag{3.16}$$

where $l = \max\{l_1, l_2\}$. By (A_2) and Lemma 3.1, the operator A satisfies the spectral gap condition of (2.4).

Proof. when $\beta \ge \frac{4M(s)}{\mu_k} + 1$, the eigenvalues of A are all positive and real numbers, meanwhile $\left\{\lambda_k^-\right\}_{k\ge1}$ and $\left\{\lambda_k^+\right\}_{k\ge1}$ are increasing order.

Next, we divided the whole process of proof into four steps.

Step 1 By Lemma 2.1, since $\left\{\lambda_k^{\pm}\right\}$ is nondecreasing order, so there exists *N*, such that λ_N^- and λ_{N+1}^- are continuous. Then the eigenvalues of A are separate as

$$\sigma_1 = \left\{\lambda_j^-, \lambda_k^+ \middle| \max\left\{\lambda_j^-, \lambda_k^+\right\} \le \lambda_N^-\right\}, \sigma_2 = \left\{\lambda_j^-, \lambda_k^{\pm} \middle| \lambda_j^- \le \lambda_N^- \le \min\left\{\lambda_j^+, \lambda_k^{\pm}\right\}\right\}. \tag{3.17}$$

Step 2 The corresponding X is decomposed into

$$X_1 = span\left\{U_j^-, U_k^+ \middle| \lambda_j^-, \lambda_k^+ \in \sigma_1\right\}, X_2 = span\left\{U_j^-, U_k^{\pm} \middle| \lambda_j^-, \lambda_k^{\pm} \in \sigma_2\right\}. \tag{3.18}$$

We aim at madding two orthogonal subspaces of X and verifying the spectral gap condition (2.4) is true when $\Lambda_1 = \lambda_N^-, \Lambda_2 = \lambda_{N+1}^-$. Therefore, we further decompose $X_2 = X_c \oplus X_R$, where

$$X_c = span\left\{U_j^- \middle| \lambda_j^- \le \lambda_N^- < \lambda_j^+\right\}, X_R = span\left\{U_k^{\pm} \middle| \lambda_N^- < \lambda_k^{\pm}\right\}. \tag{3.19}$$

Set $X_N = X_1 \oplus X_c$, in order to verify the X_1 and X_2 are orthogonal, we need to introduce two functions $\Phi : X_N \to R$, $\Psi : X_R \to R$.

$$\begin{aligned}\Phi(U,V) &= \beta\left(\nabla u,\nabla \overline{u_1}\right) - 4M(s)\left(u,\overline{u_1}\right) + \left(\nabla^{-1}\overline{p_1},\nabla u\right) + \left(\nabla^{-1}\overline{p},\nabla u_1\right) \\ &+4\left(\nabla^{-1}p,\nabla^{-1}\overline{p_1}\right) + \beta\left(\nabla^m v,\nabla^m \overline{v_1}\right) - 4M(s)\left(v,\overline{v_1}\right) \\ &+\left(\nabla^{-m}\overline{q_1},\nabla^m v\right) + \left(\nabla^{-m}\overline{q},\nabla^m v_1\right) + 4\left(\nabla^{-m}q,\nabla^{-m}\overline{q_1}\right),\end{aligned} \tag{3.20}$$

$$\begin{aligned}\Psi(U,V) &= 2\beta\left(\nabla u,\nabla \overline{u_1}\right) - \beta\left(\nabla^{-1}\overline{p_1},\nabla u\right) - \beta\left(\nabla^{-1}\overline{p},\nabla u_1\right) \\ &+ \beta\left(\nabla^{-1}p,\nabla^{-1}\overline{p_1}\right) + 2\beta\left(\nabla^m v,\nabla^m \overline{v_1}\right) - \beta\left(\nabla^{-m}\overline{q_1},\nabla^m v\right) \\ &- \beta\left(\nabla^{-m}\overline{q},\nabla^m v_1\right) + \beta\left(\nabla^{-m}q,\nabla^{-m}\overline{q_1}\right),\end{aligned} \tag{3.21}$$

where $U,V \in X$ are defined before.

Let $U = (u,v,p,q) \in X_N$, by (A_2), then

$$\begin{aligned}\Phi(U,U) &= \beta(\nabla u,\nabla \overline{u}) - 4M(s)(u,\overline{u}) + \left(\nabla^{-1}\overline{p},\nabla u\right) + \left(\nabla^{-1}\overline{p},\nabla u\right) \\ &+4\left(\nabla^{-1}p,\nabla^{-1}\overline{p}\right) + \beta\left(\nabla^m v,\nabla^m \overline{v}\right) - 4M(s)(v,\overline{v}) \\ &+\left(\nabla^{-m}\overline{q},\nabla^m v\right) + \left(\nabla^{-m}\overline{q},\nabla^m v\right) + 4\left(\nabla^{-m}q,\nabla^{-m}\overline{q}\right) \\ &\geq (\beta-1)\left(\|\nabla u\|^2 + \|\nabla^m v\|^2\right) - 4M(s)\left(\|u\|^2 + \|v\|^2\right) \\ &\geq \left[(\beta-1)\mu_1 - 4m_1\right]\left(\|u\|^2 + \|v\|^2\right).\end{aligned} \tag{3.22}$$

Since for $\forall k, m_1 \leq \frac{(\beta-1)\mu_k}{4}$, therefore $\Phi(U,U) \geq 0$, for $\forall U \in X_N$, then Φ is positive definite.

Similarly, for $U \in X_R$, we have

$$\begin{aligned}\Psi(U,U) &= 2\beta(\nabla u,\nabla \overline{u}) - \beta\left(\nabla^{-1}\overline{p},\nabla u\right) - \beta\left(\nabla^{-1}\overline{p},\nabla u\right) \\ &+ \beta\left(\nabla^{-1}p,\nabla^{-1}\overline{p}\right) + 2\beta\left(\nabla^m v,\nabla^m \overline{v}\right) - \beta\left(\nabla^{-m}\overline{q},\nabla^m v\right) \\ &- \beta\left(\nabla^{-m}\overline{q},\nabla^m v\right) + \beta\left(\nabla^{-m}q,\nabla^{-m}\overline{q}\right) \\ &\geq \beta\left(\|\nabla u\|^2 + \|\nabla^m v\|^2\right) \geq \beta\mu_1\left(\|u\|^2 + \|v\|^2\right).\end{aligned} \tag{3.23}$$

So, for $\forall U \in X_R, \Psi(U,U) \geq 0$, the Ψ is also positive definite.

Next, we need to define a scale product in X

$$\langle\langle U,V\rangle\rangle_X = \Phi(P_N U, P_N V) + \Psi(P_R U, P_R V), \tag{3.24}$$

where P_N and P_R are projection: $X \to X_N, X \to X_R$ respectively, for convenience, we rewrite (3.24) as follows

$$\langle\langle U,V\rangle\rangle_X = \Phi(U,V) + \Psi(U,V). \tag{3.25}$$

We will proof that two subspaces X_1 and X_2 in (3.18) are orthogonal. In fact, we only need to show X_N and X_C are orthogonal, that is

$$\langle\langle U_j^-, U_j^+\rangle\rangle_X = 0 \quad \left(\forall U_j^- \in X_N, U_j^+ \in X_c\right). \tag{3.26}$$

By (3.20), (3.25), we have

$$
\begin{aligned}
\left\langle\left\langle U_j^-,U_j^+\right\rangle\right\rangle_X &= \Phi\left(U_j^-,U_j^+\right)\\
&= \beta\left(\nabla u_j,\nabla\overline{u_j}\right)-4M(s)\left(u_j,\overline{u_j}\right)-\lambda_j^+\left(\nabla^{-1}\overline{u_j},\nabla u_j\right)-\lambda_j^-\left(\nabla^{-1}\overline{u_j},\nabla u_j\right)\\
&\quad +4\lambda_j^-\lambda_j^+\left(\nabla^{-1}u_j,\nabla^{-1}\overline{u_j}\right)+\beta\left(\nabla^m v_j,\nabla^m\overline{v_j}\right)-4M(s)\left(v_j,\overline{v_j}\right)\\
&\quad -\lambda_j^+\left(\nabla^{-m}\overline{v_j},\nabla^m v_j\right)-\lambda_j^-\left(\nabla^{-m}\overline{v_j},\nabla^m v_j\right)+4\lambda_j^-\lambda_j^+\left(\nabla^{-m}v_j,\nabla^{-m}\overline{v_j}\right)\\
&= \beta\left(\left\|\nabla u_j\right\|^2+\left\|\nabla^m v_j\right\|^2\right)-M(s)\left(\left\|u_j\right\|^2+\left\|v_j\right\|^2\right)\\
&\quad -\left(\lambda_j^++\lambda_j^-\right)\left(\left\|u_j\right\|^2+\left\|v_j\right\|^2\right)+4\lambda_j^-\lambda_j^+\left(\left\|u_j\right\|_{-1}^2+\left\|v_j\right\|_{-m}^2\right).
\end{aligned}
\tag{3.27}
$$

By (3.12), we can get $\lambda_j^++\lambda_j^-=\beta\mu_j,\lambda_j^-\lambda_j^+=M(s)\mu_j$, therefore

$$
\left\langle\left\langle U_j^-,U_j^+\right\rangle\right\rangle_X=\Phi\left(U_j^-,U_j^+\right)=0. \tag{3.28}
$$

Step 3 Further, we estimate the Lipschitz constant l_F of $F(U)$ (3.2). According to Lemma 3.1, $g_i:X_3\to X_3$ is Lipschitz continuous with Lipschitz constant l_i. Let $P_i:X\to X_i\,(i=1,2)$ is orthogonal projection. From (3.22), (3.23) and (3.24), we have

$$
\begin{aligned}
\|U\|_X^2 &= \Phi\left(P_1U,P_1U\right)+\Psi\left(P_2U,P_2U\right)\\
&\ge\left[(\beta-1)\mu_1-4m_1\right]\left(\|P_1u\|^2+\|P_1v\|^2\right)+\beta\mu_1\left(\|P_2u\|^2+\|P_2v\|^2\right)\\
&\ge\left[(\beta-1)\mu_1-4m_1\right]\left(\|P_1u\|^2+\|P_2u\|^2+\|P_1v\|^2+\|P_2v\|^2\right)\\
&\ge\left[(\beta-1)\mu_1-4m_1\right]\left(\|u\|^2+\|v\|^2\right).
\end{aligned}
\tag{3.29}
$$

Given $U=(u,v,p,q)^{\mathrm T},V=(u_1,v_1,p_1,q_1)^{\mathrm T}\in X$, we have

$$
\begin{aligned}
\|F(U)-F(V)\|_X &= \left\|g_1(u,v)-g_1(u_1,v_1)\right\|_{X_3}+\left\|g_2(u,v)-g_2(u_1,v_1)\right\|_{X_3}\\
&\le l_1\left(\|u-u_1\|+\|v-v_1\|\right)+l_2\left(\|u-u_1\|+\|v-v_1\|\right)\\
&\le l\left(\|u-u_1\|+\|v-v_1\|\right)\\
&\le\frac{l}{\sqrt{(\beta-1)-4m_1}}\|U-V\|_X,
\end{aligned}
\tag{3.30}
$$

where $l=\max\{l_1,l_2\}$.

So, we obtain

$$
l_F\le\frac{l}{\sqrt{(\beta-1)-4m_1}}. \tag{3.31}
$$

Step 4 Now, we will show the spectral gap condition (2.4) holds.

Since $\Lambda_1=\lambda_N^-,\Lambda_2=\lambda_{N+1}^-$, then

$$
\Lambda_2-\Lambda_1=\lambda_{N+1}^--\lambda_N^-=\frac12\left(\mu_{N+1}-\mu_N\right)+\frac12\sqrt{Q(N)}-\sqrt{Q(N+1)}, \tag{3.32}
$$

where $Q(N)=\beta^2\mu_N^2-4M(s)\mu_N$.

Let $N_1>0$, for $\forall N\ge N_1$, then

$Q_1(N)=1-\sqrt{\dfrac{\beta^2}{(\beta-1)\mu_1-4m_1}-\dfrac{4m_1}{\mu_N\left[(\beta-1)\mu_1-4m_1\right]}}$, we can obtain

$$\begin{aligned}&\sqrt{Q(N)}-\sqrt{Q(N+1)}+\sqrt{(\beta-1)-4m_1}\left(\mu_{N+1}-\mu_N\right)\\&=\sqrt{(\beta-1)\mu_1-4m_1}\left(\mu_{N+1}Q_1(N+1)-\mu_N Q_1(N)\right)\end{aligned} \quad (3.33)$$

From (A_2), we can easily obtain

$$\lim_{N\to\infty}\left(\sqrt{Q(N)}-\sqrt{Q(N+1)}+\sqrt{(\beta-1)\mu_1-4m_1}\left(\mu_{N+1}-\mu_N\right)\right)=0\,. \quad (3.34)$$

Then, according to (3.16), (3.31), (3.32) and (3.34), we have

$$\begin{aligned}\Lambda_2-\Lambda_1&>\left(\mu_{N+1}-\mu_N\right)\left(\frac{1}{2}-\frac{1}{2}\sqrt{(\beta-1)\mu_1-4m_1}\right)-1\\&\geq\frac{4l}{\sqrt{(\beta-1)\mu_1-4m_1}}\geq 4l_F\,.\end{aligned} \quad (3.35)$$

Therefore, Theorem 3.1 is true.

Theorem 3.2. Under the condition of Theorem 3.1, the problem (1.1)-(1.5) exist an inertial manifold μ in *X*,

$$\mu=graph(m):=\left\{\zeta+m(\zeta):\zeta\in X_1\right\}, \quad (3.36)$$

where X_1, X_2 defined in (3.18) and $m: X_1\to X_2$ is a Lipschitz continuous function.

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