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Generated Sets of the Complete Semigroup Binary Relations Defined by Semilattices of the Class $\Sigma_8(X, n+k+1)$

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Abstract

In this article, we study generated sets of the complete semigroups of binary relations defined by X-semilattices unions of the class $\Sigma_8(X, n+k+1)$, and find uniquely irreducible generating set for the given semigroups.

Keywords

Semigroup, Semilattice, Binary Relation

1. Introduction

Let X be an arbitrary nonempty set, D is an X-semilattice of unions which is closed with respect to the set-theoretic union of elements from D, f be an arbitrary mapping of the set X in the set D. To each mapping f we put into correspondence a binary relation α_f on the set X that satisfies the condition $\alpha_f = \bigcup_{x\in X}(\{x\}\times f(x))$. The set of all such α_f ($f: X \to D$) is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by an X-semilattice of unions D.

We denote by $\varnothing$ an empty binary relation or an empty subset of the set X. The condition $(x, y)\in\alpha$ will be written in the form $x\alpha y$. Further, let $x, y\in X$, $Y\subseteq X$, $\alpha\in B_X(D)$, $\breve{D} = \bigcup_{Y\in D} Y$ and $T\in D$. We denote by the symbols $y\alpha$, $Y\alpha$, $V(D,\alpha)$, X^* and $V(X^*,\alpha)$ the following sets:

$$y\alpha = \{x\in X \mid y\alpha x\},\ Y\alpha = \bigcup_{y\in Y} y\alpha,\ V(D,\alpha) = \{Y\alpha \mid Y\in D\},$$

$$X^* = \{Y \mid \varnothing \ne Y \subseteq X\},\ V(X^*,\alpha) = \{Y\alpha \mid \varnothing \ne Y \subseteq X\},$$

$$D_T = \{Z \in D \mid T \subseteq Z\},\ Y_T^{\alpha} = \{y \in X \mid y\alpha = T\}.$$

It is well known the following statements:

Theorem 1.1. *Let* $D = \{\breve{D}, Z_1, Z_2, \cdots, Z_{m-1}\}$ *be some finite X-semilattice of unions and* $C(D) = \{P_0, P_1, P_2, \cdots, P_{m-1}\}$ *be the family of sets of pairwise nonintersecting subsets of the set X (the set* $\varnothing$ *can be repeated several times). If* φ *is a mapping of the semilattice D on the family of sets* $C(D)$ *which satisfies the conditions*

$$\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2 & \cdots & Z_{m-1} \\ P_0 & P_1 & P_2 & \cdots & P_{m-1} \end{pmatrix}$$

and $\hat{D}_Z = D \setminus D_Z$, *then the following equalities are valid*:

$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-1},\ Z_i = P_0 \cup \bigcup_{T \in \hat{D}_{Z_i}} \varphi(T). \quad (1.1)$$

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice D are represented in the form (1.1), then among the parameters P_i $(0 < i \le m-1)$ there exist such parameters that cannot be empty sets for D. Such sets P_i are called bases sources, where sets P_j $(0 \le j \le m-1)$, which can be empty sets too are called completeness sources.

It is proved that under the mapping φ the number of covering elements of the pre-image of a

bases source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see [1] [2] chapter 11).

Definition 1.1. *We say that an element* α *of the semigroup* $B_X(D)$ *is external if* $\alpha \ne \delta \circ \beta$ *for all* $\delta, \beta \in B_X(D) \setminus \{\alpha\}$ (see [1] [2] Definition 1.15.1).

It is well known, that if B is all external elements of the semigroup $B_X(D)$ and B' is any generated set for the $B_X(D)$, then $B \subseteq B'$ (see [1] [2] Lemma 1.15.1).

Definition 1.2. *The representation* $\alpha = \bigcup_{T \in D} (Y_T^{\alpha} \times T)$ *of binary relation* α *is called quasinormal, if* $\bigcup_{T \in D} Y_T^{\alpha} = X$ *and* $Y_T^{\alpha} \cap Y_{T'}^{\alpha} = \varnothing$ *for any* $T, T' \in D$, $T \ne T'$ (see [1] [2] chapter 1.11).

Definition 1.3. *Let* $\alpha, \beta \subseteq X \times X$. *Their product* $\delta = \alpha \circ \beta$ *is defined as follows:* $x\delta y$ $(x, y \in X)$ *if there exists an element* $z \in X$ *such that* $x\alpha z\beta y$ (see [1], chapter 1.3).

2. Result

Let $\Sigma_8(X, n+k+1)$ $(3 \le k \le n)$ be a class of all X-semilattices of unions whose every element is isomorphic to an X-semilattice of unions $D = \{Z_1, Z_2, \cdots, Z_{n+k}, \breve{D}\}$, which satisfies the condition:

$Z_{n+i} \subset Z_i \subset \breve{D}$, $(i = 1, 2, \cdots, k)$; $Z_j \subset \breve{D}$, $(j = 1, 2, \cdots, n+k)$;
$Z_p \setminus Z_q \ne \varnothing$ and $Z_q \setminus Z_p \ne \varnothing$ $(1 \le p \ne q \le n+k)$.
(see Figure 1).

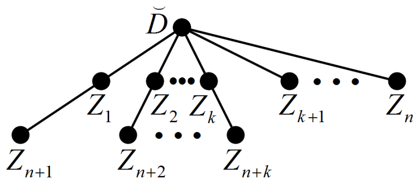

Figure 1. Diagram of the semilattice D.

It is easy to see that $\tilde{D}=\{Z_1,Z_2,\cdots,Z_{n+k}\}$ is irreducible generating set of the semilattice *D*.

Let $C(D)=\{P_0,P_1,P_2,\cdots,P_{n+k}\}$ be a family of sets, where $P_0,P_1,P_2,\cdots,P_{n+k}$ are pairwise disjoint subsets of the set *X* and $\varphi=\begin{pmatrix}\breve{D} & Z_1 & Z_2 & \cdots & Z_{n+k}\\ P_0 & P_1 & P_2 & \cdots & P_{n+k}\end{pmatrix}$ is a mapping of the semilattice *D* onto the family of sets $C(D)$. Then the formal equalities of the semilattice *D* have a form:

$$\breve{D}=\bigcup_{i=0}^{n+k}P_i;\quad Z_j=\bigcup_{\substack{i=0,\\ i\neq j}}^{n+k}P_i,\ j=1,2,\cdots,n;Z_{n+q}=\bigcup_{\substack{i=0,\\ i\neq q,n+q}}^{n+k}P_i,\quad q=1,2,\cdots,k.\qquad(2.0)$$

Here the elements $P_i\,(i=1,2,\cdots,n+k)$ are bases sources, the element P_0 are sources of completeness of the semilattice *D*. Therefore $|X|\geq n+k$ (by symbol $|X|$ we denoted the power of a set *X*), since $|P_i|\geq 1\,(i=1,2,\cdots,n+k)$ (see [1] [2] chapter 11).

In this paper we are learning irreducible generating sets of the semigroup $B_X(D)$ defined by semilattices of the class $\Sigma_8(X,n+k+1)$.

Note, that it is well known, when $k=2$, then generated sets of the complete semigroup of binary relations defined by semilattices of the class $\Sigma_8(X,2+2+1)=\Sigma_8(X,5)$.

In this paper we suppose, that $3\leq k\leq n$.

Remark, that in this case (*i.e.* $k\geq 3$), from the formal equalities of a semilattice *D* follows, that the intersections of any two elements of a semilattice *D* is not empty.

Lemma 2.0 If $D\in\Sigma_8(X,n+k+1)$, then the following statements are true:

a) $\bigcap_{i=1}^{n+k}Z_i=P_0;$

b) $Z_{j+1}\setminus Z_j=P_j,\ j=1,2,\cdots,n-1;$

c) $Z_q\setminus Z_{n+q}=P_{n+q},\ q=1,\cdots,k.$

Proof. From the formal equalities of the semilattise *D* immediately follows the following statements:

$$\bigcap_{i=1}^{n+k}Z_i=P_0,\quad Z_{j+1}\setminus Z_j=\left(\bigcup_{\substack{i=0,\\ i\neq j+1}}^{n+k}P_i\right)\setminus\left(\bigcup_{\substack{i=0,\\ i\neq j}}^{n+k}P_i\right)=P_j,\ j=1,2,\cdots,n-1;$$

$$Z_q\setminus Z_{n+q}=\left(\bigcup_{\substack{i=0,\\ i\neq q}}^{n+k}P_i\right)\setminus\left(\bigcup_{\substack{i=0,\\ i\neq q,n+q}}^{n+k}P_i\right)=P_{n+q},\quad q=1,\cdots,k.$$

The statements a), b) and c) of the lemma 2.0 are proved.

Lemma 2.0 is proved.

We denoted the following sets by symbols D_1, D_2 and D_3:

$$D_1=\{Z_1,Z_2,\cdots,Z_k\},\ D_2=\{Z_{k+1},Z_{k+2},\cdots,Z_n\},\ D_3=\{Z_{n+1},Z_{n+2},\cdots,Z_{n+k}\}.$$

Lemma 2.1. *Let* $D\in\Sigma_{8.0}(X,n+k+1)$ *and* $\alpha\in B_X(D)$. *Then the following statements are true:*

1) *Let* $T,T'\in D_2\cup D_3, T\neq T'$. *If* $T,T'\in V(D,\alpha)$, *then* α *is external element of the semigroup* $B_X(D)$;

2) *Let* $T\in D_1$, $T'\in D_2\cup D_3$. *If* $T'\not\subset T$ *and* $T,T'\in V(D,\alpha)$, *then* α *is external element of the semigroup* $B_X(D)$.

3) *Let* $T,T'\in D_1$ *and* $T\neq T'$. *If* $T,T'\in V(D,\alpha)$ *and* $k\geq 3$, *then* α *is external element of the semigroup* $B_X(D)$;

Proof. Let $Z_0=\breve{D}$ and $\alpha=\delta\circ\beta$ for some $\delta,\beta\in B_X(D)\setminus\{\alpha\}$. If quasinormal representation of binary relation δ has a form

$$\delta=\bigcup_{T\in V(D,\delta)}\left(Y_T^{\delta}\times T\right),$$

then

$$\alpha=\delta\circ\beta=\bigcup_{T\in V(D,\delta)}\left(Y_T^{\delta}\times T\beta\right). \tag{2.1}$$

From the formal equalities (2.0) of the semilattice D we obtain that:

$$Z_0\beta=\bigcup_{i=0}^{n+k}P_i\beta;\ Z_j\beta=\bigcup_{\substack{i=0,\\ i\neq j}}^{n+k}P_i\beta,\ j=1,2,\cdots,n;$$
$$Z_{n+q}\beta=\bigcup_{\substack{i=0,\\ i\neq q,n+q}}^{n+k}P_i\beta,\ q=1,2,\cdots,k. \tag{2.2}$$

where $P_i\beta\neq\varnothing$ for any $P_i\neq\varnothing$ $(i=0,1,2,\cdots,n+k)$ and $\beta\in B_X(D)$ by definition of a semilattice D from the class $\Sigma_{8.0}(X,n+k+1)$.

Now, let $Z_m\beta=T$ and $Z_j\beta=T'$ for some $T\neq T'$, $T,T'\in D_2\cup D_3$, then from the equalities (2.3) follows that $T=P_0\beta=T'$ since T and T' are minimal elements of the semilattice D and $P_0\neq\varnothing$ by preposition. The equality $T=T'$ contradicts the inequality $T\neq T'$.

The statement a) of the Lemma 2.1 is proved.

Now, let $Z_m\beta=T$ and $Z_j\beta=T'$, for some $T\in D_1$, $T'\in D_2\cup D_3$ and $T'\not\subset T$, then from the equalities 2.3 follows, that

$T'=Z_j\beta=Z_0\beta=\bigcup_{i=0}^{n+k}P_i\beta$, if $j=0$, or $T'=Z_j\beta=\bigcup_{\substack{i=0,\\ i\neq j}}^{n+k}P_i\beta$, $1\leq j\leq n$, or

$$T'=Z_{n+q}\beta=\bigcup_{\substack{i=0,\\ i\neq q,n+q}}^{n+k}P_i\beta$$

where $j=n+q$. For the $Z_j\beta=T'$ we consider the following cases:

1) If $T'=Z_0\beta=\bigcup_{i=0}^{n+k}P_i\beta$, then we have

$$P_0\beta=P_1\beta=\cdots=P_{n+k}\beta=T',$$

since T' is a minimal element of a semilattice D. On the other hand,

$$T = Z_m\beta = \begin{cases} \bigcup\limits_{\substack{i=0,\\ i\neq m}}^{n+k} P_i\beta = \bigcup\limits_{\substack{i=0,\\ i\neq m}}^{n+k} T' = T', \text{ if } 1 \le m \le n; \\ \bigcup\limits_{\substack{i=0,\\ i\neq q,n+q}}^{n+k} P_i\beta = \bigcup\limits_{\substack{i=0,\\ i\neq q,n+q}}^{n+k} T' = T', \text{ if } m = n+q. \end{cases}$$

But the equality $T = T'$ contradicts the inequality $T \neq T'$. Thus we have, that $j \neq 0$.

2) Let $1 \le j \le n$, *i.e.* $T' = Z_j\beta = \bigcup\limits_{\substack{i=0,\\ i\neq j}}^{n+k} P_i\beta$, then we have, that

$$P_0\beta = P_1\beta = \cdots = P_{j-1}\beta = P_{j+1}\beta = \cdots = P_{n+k}\beta = T',$$

since T' is a minimal element of a semilattice *D*. On the other hand:

$$T = Z_m\beta = \begin{cases} \left(\bigcup\limits_{i=0}^{n+k} P_i\beta\right) = \left(\bigcup\limits_{\substack{i=0,\\ i\neq j}}^{n+k} P_i\beta\right) \cup P_j\beta = T' \cup P_j\beta, \text{ if } m = 0; \\ \left(\bigcup\limits_{\substack{i=0,\\ i\neq m}}^{n+k} P_i\beta\right) = \left(\bigcup\limits_{\substack{i=0,\\ i\neq m,j}}^{n+k} P_i\beta\right) \cup P_j\beta = T' \cup P_j\beta, \text{ if } 1 \le m \le n,\ m \neq j; \\ \left(\bigcup\limits_{\substack{i=0,\\ i\neq j,n+j}}^{n+k} P_i\beta\right) = T', \text{ if } m = n+j; \\ \left(\bigcup\limits_{\substack{i=0,\\ i\neq q,n+q}}^{n+k} P_i\beta\right) = \left(\bigcup\limits_{\substack{i=0,\\ i\neq q,n+q,j}}^{n+k} P_i\beta\right) \cup P_j\beta = T' \cup P_j\beta, \text{ if } m = n+q,\ q \neq j. \end{cases}$$

The equality $T = T'$ contradicts the inequality $T \neq T'$. Also, the equality $T = T' \cup P_j\beta$ $\left(P_j\beta \in D\right)$ contradicts the inequality $T \neq T' \cup Z$ for any $Z \in D$ and $T' \not\subset T$ ($T' \not\subset T$, by preposition) by definition of a semilattice *D*.

3) If $j = n+q$ $\left(1 \le q \le k\right)$, *i.e.* $T' = Z_{n+q}\beta = \bigcup\limits_{\substack{i=0,\\ i\neq q,n+q}}^{n+k} P_i\beta$, then we have, that

$$P_0\beta = P_1\beta = \cdots = P_{q-1}\beta = P_{q+1}\beta = \cdots = P_{n+q-1}\beta = P_{n+q+1}\beta = \cdots = P_{n+k}\beta = T',$$

since T' is a minimal element of a semilattice *D*. On the other hand:

$$T = Z_m\beta = \begin{cases} \left(\bigcup\limits_{i=0,}^{n+k} P_i\beta\right) = \left(\bigcup\limits_{\substack{i=0,\\ i\neq q,n+q}}^{n+k} P_i\beta\right) \cup P_q\beta \cup P_{n+q}\beta = T' \cup P_q\beta \cup P_{n+q}\beta, \text{ if } m = 0; \\ \left(\bigcup\limits_{\substack{i=0,\\ i\neq q}}^{n+k} P_i\beta\right) = \left(\bigcup\limits_{\substack{i=0,\\ i\neq q,n+q}}^{n+k} P_i\beta\right) \cup P_{n+q}\beta = T' \cup P_{n+q}\beta, \text{ if } 1 \le m = q \le n; \\ \left(\bigcup\limits_{\substack{i=0,\\ i\neq m}}^{n+k} P_i\beta\right) = \left(\bigcup\limits_{\substack{i=0,\\ i\neq m,q,n+q}}^{n+k} P_i\beta\right) \cup P_q\beta \cup P_{n+q}\beta = T' \cup P_q\beta \cup P_{n+q}\beta, \text{ if } 1 \le m \neq q \le n; \\ \left(\bigcup\limits_{\substack{i=0,\\ i\neq m,}}^{n+k} P_i\beta\right) = \left(\bigcup\limits_{\substack{i=0,\\ i\neq q',n+q'}}^{n+k} P_i\beta\right) = \left(\bigcup\limits_{\substack{i=0,\\ i\neq,q',n+q',q,n+q}}^{n+k} P_i\beta\right) \cup P_q\beta \cup P_{n+q} = T \cup P_q\beta \cup P_{n+q}, \\ \text{if } n+1 \le m = n+q' \le n+k,\ q \neq q' \text{ since } j \neq m. \end{cases}$$

The equality $T = T'$ contradicts the inequality $T \neq T'$. Also, the equality $T = T' \cup P_q\beta \cup P_{n+q}\beta$, or $T = T' \cup P_{n+q}\beta$ $\left(P_q\beta, P_{n+q}\beta \in D\right)$ contradicts the inequality $T \neq T' \cup Z$ for any $Z \in D$ and $T' \not\subset T$ by definition of a semilattice *D*.

The statement 2) of the Lemma 2.1 is proved.

Let $T, T' \in D_1$ and $T \neq T'$. If $k \geq 3$ and $Z_j\beta = T'$, $Z_m\beta = T$, then from the formal equalities (2.0) of a semilattice D there exists such an element, that $P_q \subseteq Z_j$ and $P_q \subseteq Z_m$, where $0 \leq q \leq m + k$. So, from the equalities (2.3) follows that $P_q\beta \subseteq Z_j\beta = T'$ and $P_q\beta \subseteq Z_m\beta = T$. Of from this and from the equalities (2.3) we obtain that there exists such an element $Z \in D$, for which the equalities $T' = Z \cup Z'$ and $T = Z \cup Z''$, where $Z', Z'' \in D$. But such elements by definition of a semilattice D do not exist.

The statement c) of the Lemma 2.1 is proved.

Lemma 2.1 is proved.

Lemma 2.2. *Let* $D \in \Sigma_8\left(X, n+k+1\right)$ *and* $\alpha \in B_X\left(D\right)$. *Then the following statements are true:*

1) *Let* $V\left(D,\alpha\right) \cap D_1 \neq \varnothing, V\left(D,\alpha\right) \cap D_2 = \varnothing, V\left(D,\alpha\right) \cap D_3 = \varnothing$. *If* $\left|V\left(D,\alpha\right) \cap D_1\right| \geq 2$, *then* α *is external element of the semigroup* $B_X\left(D\right)$;

2) *Let* $V\left(D,\alpha\right) \cap D_1 = \varnothing, V\left(D,\alpha\right) \cap D_2 \neq \varnothing, V\left(D,\alpha\right) \cap D_3 = \varnothing$. *If* $\left|V\left(D,\alpha\right) \cap D_2\right| \geq 2$, *then* α *is external element of the semigroup* $B_X\left(D\right)$;

3) *Let* $V\left(D,\alpha\right) \cap D_1 = \varnothing, V\left(D,\alpha\right) \cap D_2 = \varnothing, V\left(D,\alpha\right) \cap D_3 \neq \varnothing$. *If* $\left|V\left(D,\alpha\right) \cap D_3\right| \geq 2$, *then* α *is external element of the semigroup* $B_X\left(D\right)$;

4) *Let* $V\left(D,\alpha\right) \cap D_1 = \varnothing, V\left(D,\alpha\right) \cap D_2 \neq \varnothing, V\left(D,\alpha\right) \cap D_3 \neq \varnothing$, *then* α *is external element of the semigroup* $B_X\left(D\right)$;

5) *Let* $V\left(D,\alpha\right) \cap D_1 \neq \varnothing, V\left(D,\alpha\right) \cap D_2 = \varnothing, V\left(D,\alpha\right) \cap D_3 \neq \varnothing$. *If* $\left|V\left(D,\alpha\right) \cap D_1\right| \geq 2$, $\left|V\left(D,\alpha\right) \cap D_3\right| = 1$, *or* $\left|V\left(D,\alpha\right) \cap D_1\right| = 1$, $\left|V\left(D,\alpha\right) \cap D_3\right| \geq 2$ *then* α *is external element of the semigroup* $B_X\left(D\right)$;

6) *Let* $V\left(D,\alpha\right) \cap D_1 \neq \varnothing, V\left(D,\alpha\right) \cap D_2 \neq \varnothing, V\left(D,\alpha\right) \cap D_3 = \varnothing$, *then* α *is external element of the semigroup* $B_X\left(D\right)$;

7) *Let* $V\left(D,\alpha\right) \cap D_1 \neq \varnothing, V\left(D,\alpha\right) \cap D_2 \neq \varnothing, V\left(D,\alpha\right) \cap D_3 \neq \varnothing$, *then* α *is external element of the semigroup* $B_X\left(D\right)$.

Proof. Let α be any element of the semigroup $B_X\left(D\right)$. It is easy that $V\left(D,\alpha\right) \in D$. We consider the following cases:

Let $V\left(D,\alpha\right) \cap D_1 = \varnothing, V\left(D,\alpha\right) \cap D_2 = \varnothing, V\left(D,\alpha\right) \cap D_3 = \varnothing$, then $V\left(D,\alpha\right) \in \left\{\breve{D}\right\}$ since $V\left(D,\alpha\right)$ is subsemilattice of the semilattice *D*.

1) Let $V\left(D,\alpha\right) \cap D_1 \neq \varnothing, V\left(D,\alpha\right) \cap D_2 = \varnothing, V\left(D,\alpha\right) \cap D_3 = \varnothing$.

If $\left|V\left(D,\alpha\right) \cap D_1\right| = 1$, then $V\left(D,\alpha\right) \in \left\{Z_j\right\}$, or $V\left(D,\alpha\right) \in \left\{Z_j, \breve{D}\right\}$, where $j = 1, 2, \cdots, k$, since $V\left(D,\alpha\right)$ is subsemilattice of the semilattice *D*.

If $\left|V\left(D,\alpha\right) \cap D_1\right| \geq 2$, then by statement c) of the Lemma 2.1 follows that α is external element of the semigroup $B_X\left(D\right)$.

2) Let $V\left(D,\alpha\right) \cap D_1 = \varnothing, V\left(D,\alpha\right) \cap D_2 \neq \varnothing, V\left(D,\alpha\right) \cap D_3 = \varnothing$.

If $\left|V\left(D,\alpha\right) \cap D_2\right| = 1$, then $V\left(D,\alpha\right) \in \left\{Z_j\right\}$, or $V\left(D,\alpha\right) \in \left\{Z_j, \breve{D}\right\}$, where $j = k+1, k+2, \cdots, n$, since $V\left(D,\alpha\right)$ is a subsemilattice of the semilattice *D*.

If $\left|V(D,\alpha)\cap D_2\right|\ge 2$, then by statement a) of the Lemma 2.1 follows that α is external element of the semigroup $B_X(D)$.

3) Let $V(D,\alpha)\cap D_1=\varnothing, V(D,\alpha)\cap D_2=\varnothing, V(D,\alpha)\cap D_3\ne\varnothing$.

If $\left|V(D,\alpha)\cap D_3\right|=1$, then $V(D,\alpha)\in\{Z_j\}$, or $V(D,\alpha)\in\{Z_j,\breve{D}\}$, $j=n+1,n+2,\cdots,n+k$, since $V(D,\alpha)$ is subsemilattice of the semilattice *D*.

If $\left|V(D,\alpha)\cap D_3\right|\ge 2$, then by statement a) of the Lemma 2.1 follows that α is external element of the semigroup $B_X(D)$.

4) Let $V(D,\alpha)\cap D_1=\varnothing, V(D,\alpha)\cap D_2\ne\varnothing, V(D,\alpha)\cap D_3\ne\varnothing$, then by the statement a) of the Lemma 2.1 follows that α is external element of the semigroup $B_X(D)$.

5) Let $V(D,\alpha)\cap D_1\ne\varnothing, V(D,\alpha)\cap D_2=\varnothing, V(D,\alpha)\cap D_3\ne\varnothing$.

If $\left|V(D,\alpha)\cap D_1\right|=1, \left|V(D,\alpha)\cap D_3\right|=1$, then $V(D,\alpha)=\{Z_{n+q},Z_q\}$, or $V(D,\alpha)=\{Z_{n+q},Z_q,\breve{D}\}$, or $V(D,\alpha)=\{Z_{n+q},Z_j,\breve{D}\}$ where $Z_1\le Z_j\le Z_k$ and $q=1,2,\cdots,k$.

If $V(D,\alpha)=\{Z_{n+q},Z_j,\breve{D}\}$ where $j\ne q, q=1,2,\cdots,k$, then by the statement 2) of the Lemma 2.1 follows that α is external element of the semigroup $B_X(D)$;

If $\left|V(D,\alpha)\cap D_1\right|=1, \left|V(D,\alpha)\cap D_3\right|\ge 2$, or $\left|V(D,\alpha)\cap D_1\right|\ge 2, \left|V(D,\alpha)\cap D_3\right|=1$, then from the statement 1) and 3) of the Lemma 2.1 follows that α is external element of the semigroup $B_X(D)$ respectively.

6) Let $V(D,\alpha)\cap D_1\ne\varnothing, V(D,\alpha)\cap D_2\ne\varnothing, V(D,\alpha)\cap D_3=\varnothing$. Then from the statement b) of the Lemma 2.1 follows that α is external element of the semigroup $B_X(D)$.

7) Let $V(D,\alpha)\cap D_1\ne\varnothing, V(D,\alpha)\cap D_2\ne\varnothing, V(D,\alpha)\cap D_3\ne\varnothing$, then by the statement a) of the Lemma 2.1 follows that α is external element of the semigroup $B_X(D)$.

Lemma 2.2 is proved.

Now we learn the following subsemilattices of the semilattice *D*:

$$\mathfrak{A}_1=\left\{\left\{Z_{n+j},Z_j,\breve{D}\right\}\right\}, \text{where } j=1,2,\cdots,k;$$

$$\mathfrak{A}_2=\left\{\left\{Z_j,\breve{D}\right\}\right\}, \text{where } j=1,2,\cdots,n+k;$$

$$\mathfrak{A}_3=\left\{\left\{Z_{n+j},Z_j\right\}\right\}, \text{where } j=1,2,\cdots,k;$$

$$\mathfrak{A}_4=\left\{\left\{Z_j\right\},\left\{\breve{D}\right\}\right\}, \text{where } j=1,2,\cdots,n+k.$$

We denoted the following sets by symbols $\mathfrak{A}_0$ and $B(\mathfrak{A}_0)$:

$$\mathfrak{A}_0=\left\{V(D,\alpha)\subseteq D \mid V(D,\alpha)\notin\mathfrak{A}_1\cup\mathfrak{A}_2\cup\mathfrak{A}_3\cup\mathfrak{A}_4\right\},$$

$$B(\mathfrak{A}_0)=\left\{\alpha\in B_X(D) \mid V(D,\alpha)\in\mathfrak{A}_0\right\}.$$

By definition of a set $B(\mathfrak{A}_0)$ follows that any element of the set is external element of the semigroup $B_X(D)$.

Lemma 2.3. *Let* $D\in\Sigma_8(X,n+k+1)$. *If quasinormal representation of a binary relation* α *has a form*

$$\alpha=\left(Y_{n+j}^{\alpha}\times Z_{n+j}\right)\cup\left(Y_j^{\alpha}\times Z_j\right)\cup\left(Y_0^{\alpha}\times\breve{D}\right),$$

where $Y_{n+j}^{\alpha},Y_j^{\alpha},Y_0^{\alpha}\notin\{\varnothing\}$ *and* $j=1,2,\cdots,k$ *, then* α *is generated by elements of the elements of set* $B(\mathfrak{A}_0)$ *.*

Proof. 1). Let quasinormal representation of binary relations δ and β have a form

$$\delta=\left(Y_{n+j}^{\delta}\times Z_{n+j}\right)\cup\left(Y_j^{\delta}\times Z_j\right)\cup\left(Y_q^{\delta}\times Z_q\right)\cup\left(Y_0^{\delta}\times\breve{D}\right),$$

$$\beta=\left(Z_{n+j}\times Z_{n+j}\right)\cup\left(\left(Z_j\setminus Z_{n+j}\right)\times Z_j\right)\cup\left(\left(\breve{D}\setminus Z_j\right)\times Z_q\right)\cup\left(\left(X\setminus\breve{D}\right)\times\breve{D}\right),$$

where $Y_{n+j}^{\alpha},Y_j^{\alpha},Y_q^{\alpha}\notin\{\varnothing\},Z_1\le Z_q\le Z_k,q\ne j,\ j=1,\cdots,k$.

$$Z_{n+j}\cup\left(Z_j\setminus Z_{n+j}\right)\cup\left(\breve{D}\setminus Z_j\right)\cup\left(X\setminus\breve{D}\right)$$
$$=\left(P_0\cup\bigcup_{\substack{i=1,\\ i\ne j,n+j}}^{n+k}P_i\right)\cup P_{n+j}\cup P_j\cup\left(X\setminus\breve{D}\right)=\breve{D}\cup\left(X\setminus\breve{D}\right)=X\ ,$$

since the representation of a binary relation β is quasinormal and by statement 3) of the Lemma 2.1 binary relations δ and β are external elements of the semigroup $B_X(D)$. It is easy to see, that:

$$Z_{n+j}\beta=Z_{n+j},$$

$$Z_j\beta=\left(P_0\cup\bigcup_{\substack{i=1,\\ i\ne j}}^{n+k}P_i\right)\beta=\left(\left(P_0\cup\bigcup_{\substack{i=1,\\ i\ne j,n+j}}^{n+k}P_i\right)\cup P_{n+j}\right)\beta$$
$$=Z_{n+j}\beta\cup P_{n+j}\beta=Z_{n+j}\cup Z_j=Z_j,$$

$$Z_q\beta=\left(P_0\cup\bigcup_{\substack{i=1,\\ i\ne q}}^{n+k}P_i\right)\beta=Z_{n+j}\cup Z_j\cup Z_q=\breve{D},$$

$$\breve{D}\beta=\bigcup_{i=0}^{n+k}P_i\beta=Z_{n+j}\cup Z_j\cup Z_q=\breve{D}$$

since $Z_q\cap Z_{n+j}\ne\varnothing,Z_q\cap\left(Z_j\setminus Z_{n+j}\right)=P_{n+j}\ne\varnothing,Z_q\cap\left(D\setminus Z_j\right)=P_j\ne\varnothing$ (see equality (2.0))

$$\alpha=\delta\circ\beta=\left(Y_{n+j}^{\delta}\times Z_{n+j}\beta\right)\cup\left(Y_j^{\delta}\times Z_j\beta\right)\cup\left(Y_q^{\delta}\times Z_q\beta\right)\cup\left(Y_0^{\delta}\times\breve{D}\beta\right)$$
$$=\left(Y_{n+j}^{\delta}\times Z_{n+j}\right)\cup\left(Y_j^{\delta}\times Z_j\right)\cup\left(Y_q^{\delta}\times\breve{D}\right)\cup\left(Y_0^{\delta}\times\breve{D}\right)$$
$$=\left(Y_{n+j}^{\delta}\times Z_{n+j}\right)\cup\left(Y_j^{\delta}\times Z_j\right)\cup\left(\left(Y_q^{\delta}\cup Y_0^{\delta}\right)\times\breve{D}\right)=\alpha,$$

if $Y_{n+j}^{\delta}=Y_{n+j}^{\alpha}$, $Y_j^{\delta}=Y_j^{\alpha}$ and $Y_q^{\delta}\cup Y_0^{\delta}=Y_0^{\alpha}$. Last equalities are possible since $\left|Y_q^{\delta}\cup Y_0^{\delta}\right|\ge 1$ ($\left|Y_0^{\delta}\right|\ge 0$, by preposition).

Lemma 2.3 is proved.

Lemma 2.4. *Let* $D\in\Sigma_8(X,n+k+1)$*. If quasinormal representation of a binary relation* α *has a form* $\alpha=\left(Y_j^{\alpha}\times Z_j\right)\cup\left(Y_0^{\alpha}\times\breve{D}\right)$*, where* $Y_j^{\alpha},Y_0^{\alpha}\notin\{\varnothing\}$, $j=1,2,\cdots,n+k$ *, then binary relation* α *is generated by elements of the elements of set* $B(\mathfrak{A}_0)$*.*

Proof. Let quasinormal representation of the binary relations δ and β have a form:

$$\delta = \left(Y_j^\delta \times Z_j\right) \cup \left(Y_q^\delta \times Z_q\right) \cup \left(Y_0^\delta \times \breve{D}\right),$$

$$\beta = \left(Z_j \times Z_j\right) \cup \left(\left(\breve{D} \setminus Z_j\right) \times Z_q\right) \cup \left(\left(X \setminus \breve{D}\right) \times \breve{D}\right),$$

where $Y_j^\delta, Y_q^\delta \notin \{\varnothing\}$ and $Z_1 \le Z_j \ne Z_q \le Z_{n+k}$. Then from the statements a), b) and c) of the Lemma 2.1 follows, that δ and β are generated by elements of the set $B(\mathfrak{A}_0)$ and

$$Z_j\beta = Z_j,$$

$Z_q\beta = \breve{D}$, since $Z_q \cap \left(D \setminus Z_j\right) = P_j \ne \varnothing$,

$$\breve{D}\beta = \breve{D};$$

$$\begin{aligned}\delta \circ \beta &= \left(Y_j^\delta \times Z_j\beta\right) \cup \left(Y_q^\delta \times Z_q\beta\right) \cup \left(Y_0^\delta \times \breve{D}\beta\right)\\ &= \left(Y_j^\delta \times Z_j\right) \cup \left(Y_q^\delta \times \breve{D}\right) \cup \left(Y_0^\delta \times \breve{D}\right)\\ &= \left(Y_j^\delta \times Z_j\right) \cup \left(\left(Y_q^\delta \cup Y_0^\delta\right) \times \breve{D}\right) = \alpha,\end{aligned}$$

if $Y_j^\delta = Y_j^\alpha$, $Y_q^\delta \cup Y_0^\delta = Y_0^\alpha$ and $q = 1, 2, \cdots, n+k$. Last equalities are possible since $\left|Y_q^\delta \cup Y_0^\delta\right| \ge 1$ ($\left|Y_0^\delta\right| \ge 0$ by preposition).

Lemma 2.4 is proved.

Lemma 2.5. *Let* $D \in \Sigma_8(X, n+k+1)$. *If quasinormal representation of a binary relation* α *has a form* $\alpha = \left(Y_{n+j}^\alpha \times Z_{n+j}\right) \cup \left(Y_j^\alpha \times Z_j\right)$, *where* $Y_{n+j}^\alpha, Y_j^\alpha \notin \{\varnothing\}$, $j = 1, 2, \cdots, k$, *then binary relation* α *is generated by elements of the elements of set* $B(\mathfrak{A}_0)$.

Proof. Let quasinormal representation of a binary relations δ, β have a form

$$\delta = \left(Y_{n+j}^\delta \times Z_{n+j}\right) \cup \left(Y_q^\delta \times Z_q\right) \cup \left(Y_0^\delta \times \breve{D}\right),$$

$$\beta = \left(Z_{n+j} \times Z_{n+j}\right) \cup \left(\left(\breve{D} \setminus Z_{n+j}\right) \times Z_j\right) \cup \left(\left(X \setminus \breve{D}\right) \times \breve{D}\right),$$

where $Y_{n+j}^\delta, Y_q^\delta \notin \{\varnothing\}$, $j \ne q$ and $j = 1, 2, \cdots, k$. Then from the Lemma 2.2 follows that β is generated by elements of the set $B(\mathfrak{A}_0)$, $\delta \in B(\mathfrak{A}_0)$ and

$$Z_{n+j}\beta = Z_{n+j},$$

$Z_q\beta = Z_{n+j} \cup Z_j = Z_j$, since $Z_q \cap Z_{n+j} \ne \varnothing$, $Z_q \cap \left(\breve{D} \setminus Z_{n+j}\right) = P_{n+j} \ne \varnothing,\ j \ne q$ (see equality (2.0))

$\breve{D}\beta = Z_j$ since $\breve{D} \cap \left(X \setminus \breve{D}\right) = \varnothing$,

$$\begin{aligned}\delta \circ \beta &= \left(Y_{n+j}^\delta \times Z_{n+j}\beta\right) \cup \left(Y_q^\delta \times Z_q\beta\right) \cup \left(Y_0^\delta \times \breve{D}\right)\\ &= \left(Y_{n+j}^\delta \times Z_{n+j}\right) \cup \left(Y_q^\delta \times Z_j\right) \cup \left(Y_0^\delta \times Z_j\right)\\ &= \left(Y_{n+j}^\delta \times Z_{n+j}\right) \cup \left(\left(Y_q^\delta \cup Y_0^\delta\right) \times Z_j\right) = \alpha,\end{aligned}$$

if $Y_{n+j}^\delta = Y_{n+j}^\alpha$ and $Y_q^\delta \cup Y_0^\delta = Y_j^\alpha$. Last equalities are possible since $\left|Y_q^\delta \cup Y_0^\delta\right| \ge 1$ ($\left|Y_0^\delta\right| \ge 0$ by preposition).

Lemma 2.5 is proved.

Lemma 2.6. *Let* $D \in \Sigma_8(X, n+k+1)$. *Then the following statements are true:*

1) *If* quasinormal representation of a binary relation α has a form $\alpha = X \times Z_j$

$(j=1,2,\cdots,k)$, then binary relation α is generated by elements of the set $B(\mathfrak{A}_0)$.

2) *If* quasinormal representation of a binary relation α has a form $\alpha = X \times \breve{D}$, then binary relation α is generated by elements of the set $B(\mathfrak{A}_0)$.

Proof. 1) Let $T \in D\setminus(D_2 \cup D_3)$. If quasinormal representation of a binary relations δ, β have a form

$$\delta = \left(Y_j^\delta \times Z_j\right) \cup \left(Y_0^\delta \times \breve{D}\right),$$
$$\beta = \left(Z_{n+j} \times Z_{n+j}\right) \cup \left(\left(Z_j \setminus Z_{n+j}\right) \times Z_j\right) \cup \left(\left(X \setminus \breve{D}\right) \times \breve{D}\right),$$

where $Y_j^\delta, Y_0^\delta \in \{\varnothing\}$, $j=1,2,\cdots,k$

$$Z_{n+j} \cup \left(Z_j \setminus Z_{n+j}\right) \cup \left(X \setminus \breve{D}\right)$$
$$= \left(\bigcup_{\substack{i=0,\\ i\neq j,n+j}}^{n+k} P_i\right) \cup \left(P_j \cup P_{n+j}\right) \cup \left(X \setminus \breve{D}\right) = \breve{D} \cup \left(X \setminus \breve{D}\right) = X$$

(see equalities (2.0) and (2.1)), then from the Lemma 2.4 follows that δ is generated by elements of the set $B(\mathfrak{A}_0)$ and from the Lemma 2.3 element β is generated by elements of the set $B(\mathfrak{A}_0)$ and

$$Z_j\beta = Z_{n+j} \cup Z_j = Z_j,$$
$$\breve{D}\beta = Z_j, \text{ since } \breve{D} \cap \left(X \setminus \breve{D}\right) = \varnothing,$$
$$\delta \circ \beta = \left(Y_j^\delta \times Z_j\beta\right) \cup \left(Y_0^\delta \times \breve{D}\beta\right) = \left(Y_j^\delta \times Z_j\right) \cup \left(Y_0^\delta \times Z_j\right) = X \times Z_j = \alpha,$$

since representation of a binary relation δ is quasinormal.

The statement a) of the lemma 2.6 is proved.

2) Let quasinormal representation of a binary relation δ have a form

$$\delta = \left(Z_{n+j} \times Z_q\right) \cup \left(\left(X \setminus Z_{n+j}\right) \times \breve{D}\right),$$

where $j \neq q$, then from the Lemma 2.4 follows that δ is generated by elements of the set $B(\mathfrak{A}_0)$ and

$$Z_q\delta = \left(\bigcup_{\substack{i=0,\\ i\neq q}}^{n+k} P_i\right)\delta = \left(\bigcup_{\substack{i=0,\\ i\neq q}}^{n+k} P_i\delta\right) = Z_q \cup \breve{D} = \breve{D}, \breve{D}\delta = \breve{D}, \text{ since}$$

$j \neq q, Z_q\delta \cap Z_{n+1} \neq \varnothing$ and $Z_q\delta \cap \left(X \setminus Z_{n+1}\right) \neq \varnothing$;

$$\delta \circ \delta = \left(Z_{n+j} \times Z_q\delta\right) \cup \left(\left(X \setminus Z_{n+j}\right) \times \breve{D}\delta\right)$$
$$= \left(Z_{n+j} \times \breve{D}\right) \cup \left(\left(X \setminus Z_{n+j}\right) \times \breve{D}\right) = X \times \breve{D} = \alpha,$$

since representation of a binary relation δ is quasinormal.

The statement b) of the lemma 2.6 is proved.

Lemma 2.6 is proved.

Lemma 2.7. *Let* $D \in \Sigma_8(X, n+k+1)$. *Then the following statements are true:*

a) *If* $\left|X \setminus \breve{D}\right| \geq 1$ *and* $T \in D_2 \cup D_3$, *then binary relation* $\alpha = X \times T$ *is generated by elements of the elements of set* $B(\mathfrak{A}_0)$;

b) *If* $X = \breve{D}$ *and* $T \in D_2 \cup D_3$, *then binary relation* $\alpha = X \times T$ *is external element for the semigroup* $B_X(D)$.

Proof. 1) If quasinormal representation of a binary relation δ has a form

$$\delta = \left(Y_0^\delta \times \breve{D}\right) \cup \bigcup_{j=k+1}^{n+k} \left(Y_j^\delta \times Z_j\right),$$

where $Y_j^\delta \neq \varnothing$ for all $j = k+1, k+2, \cdots, n+k$, then $\delta \in B(\mathfrak{A}_0) \setminus \{\alpha\}$. Let quasinormal representation of a binary relations β have a form $\beta = \left(\breve{D} \times T\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times f(t')\right)$, where f is any mapping of the set $X \setminus \breve{D}$ in the set $(D_2 \cup D_3) \setminus \{T\}$. It is easy to see, that $\beta \neq \alpha$ and two elements of the set $D_2 \cup D_3$ belong to the semilattice $V(D, \beta)$, i.e. $\delta \in B(\mathfrak{A}_0) \setminus \{\alpha\}$. In this case we have that $Z_j \beta = T$ for all $j = k+1, k+2, \cdots, n+k$.

$$\begin{aligned}\delta \circ \beta = \delta &= \left(Y_0^\delta \times \breve{D}\beta\right) \cup \bigcup_{j=k+1}^{n+k} \left(Y_j^\delta \times Z_j \beta\right) \\ &= \left(Y_0^\delta \times T\right) \cup \bigcup_{j=k+1}^{n+k} \left(Y_j^\delta \times T\right) \\ &= \left(\left(Y_0^\delta \cup \bigcup_{j=k+1}^{n+k} Y_j^\delta\right) \times T\right) = X \times T = \alpha,\end{aligned}$$

since the representation of a binary relation δ is quasinormal. Thus, the element α is generated by elements of the set $B(\mathfrak{A}_0)$.

The statement a) of the lemma 2.7 is proved.

2) Let $X = \breve{D}$, $\alpha = X \times T$, for some $T \in D_2 \cup D_3$ and $\alpha = \delta \circ \beta$ for some $\delta, \beta \in B_X(D) \setminus \{\alpha\}$. Then we obtain that $Z_j \beta = T$ since T is a minimal element of the semilattice D.

Now, let subquasinormal representations $\overline{\beta}$ of a binary relation β have a form

$$\overline{\beta} = \left(\left(\bigcup_{i=0}^{n+k} P_i\right) \times T\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \overline{\beta}_2(t')\right),$$

where $\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & \cdots & P_{n+k} \\ T & T & T & \cdots & T \end{pmatrix}$ is normal mapping. But complement mapping $\overline{\beta}_2$ is empty, since $X \setminus \breve{D} = \varnothing$, i.e. in the given case, subquasinormal representation $\overline{\beta}$ of a binary relation β is defined uniquely. So, we have that $\beta = \overline{\beta} = X \times T = \alpha$ (see property 2) in the case 1.1), which contradict the condition, that $\beta \notin B_X(D) \setminus \{\alpha\}$.

Therefore, if $X = \breve{D}$ and $\alpha = X \times T$, for some $T \in D_2 \cup D_3$, then α is external element of the semigroup $B_X(D)$.

The statement 2) of the Lemma 2.7 is proved.

Lemma 2.7 is proved.

Theorem 2.1. *Let* $D \in \Sigma_8(X, n+k+1)$, $k \geq 3$, *and*

$$D_1 = \{Z_1, Z_2, \cdots, Z_k\},\ D_2 = \{Z_{k+1}, Z_{k+2}, \cdots, Z_n\},\ D_3 = \{Z_{n+1}, Z_{n+2}, \cdots, Z_{n+k}\};$$

$$\mathfrak{A}_1 = \left\{\left\{Z_{n+q}, Z_q, \breve{D}\right\}\right\}, \text{where } q = 1, 2, \cdots, k;$$

$$\mathfrak{A}_2=\left\{\left\{Z_j,\breve{D}\right\}\right\}, \text{ where } j=1,2,\cdots,n+k;$$

$$\mathfrak{A}_3=\left\{\left\{Z_{n+j},Z_j\right\}\right\}, \text{where } j=1,2,\cdots,k;$$

$$\mathfrak{A}_4=\left\{\left\{Z_j\right\},\left\{\breve{D}\right\}\right\}, \text{ where } j=1,2,\cdots,n+k;$$

$$\mathfrak{A}_0=\left\{V(D,\alpha)\subset D\,|\,V(D,\alpha)\notin\mathfrak{A}_1\cup\mathfrak{A}_2\cup\mathfrak{A}_3\cup\mathfrak{A}_4\right\},$$

$$B(\mathfrak{A}_0)=\left\{\alpha\in B_X(D)\,|\,V(D,\alpha)\in\mathfrak{A}_0\right\},$$

$$B_0=\left\{X\times T\,|\,T\notin D_2\cup D_3\right\}$$

Then the following statements are true:

1) *If* $\left|X\setminus\breve{D}\right|\geq 1$*, then the* $S_0=B(\mathfrak{A}_o)$ *is irreducible generating set for the semigroup* $B_X(D)$*;*

2) *If* $X=\breve{D}$*, then the* $S_1=B_0\cup B(\mathfrak{A}_o)$ *is irreducible generating set for the semigroup* $B_X(D)$*.*

Proof. Let $D\in\Sigma_8(X,n+k+1)$, $k\geq 3$ and $\left|X\setminus\breve{D}\right|\geq 1$. First, we proved that every element of the semigroup $B_X(D)$ is generated by elements of the set S_0. Indeed, let α be an arbitrary element of the semigroup $B_X(D)$. Then quasinormal representation of a binary relation α has a form

$$\alpha=\left(Y_0^\alpha\times\breve{D}\right)\cup\bigcup_{i=1}^{n+k}\left(Y_i^\alpha\times Z_i\right),$$

where $\bigcup_{i=0}^{n+k}Y_i^\alpha=X$ and $Y_i^\alpha\cap Y_j^\alpha=\varnothing$ $(0\leq i\neq j\leq n+k)$. For the $V\left(X^*,\alpha\right)$ we consider the following cases:

1) If $V\left(X^*,\alpha\right)\notin\mathfrak{A}_1\cup\mathfrak{A}_2\cup\mathfrak{A}_3\cup\mathfrak{A}_4$, then $\alpha\in B(\mathfrak{A}_o)\subseteq S_0$ by definition of a set S_0.

Now, let $V\left(X^*,\alpha\right)\in\mathfrak{A}_1\cup\mathfrak{A}_2\cup\mathfrak{A}_3\cup\mathfrak{A}_4$.

2) If $V\left(X^*,\alpha\right)\in\mathfrak{A}_1$, then quasinormal representation of a binary relation α has a form $\alpha=\left(Y_{n+j}^\alpha\times Z_{n+j}\right)\cup\left(Y_j^\alpha\times Z_j\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Y_{n+j}^\alpha,Y_j^\alpha,Y_0^\alpha\notin\{\varnothing\}$ $(j=1,2,\cdots,k)$ and from the Lemma 2.3 follows that α is generated by elements of the elements of set $B(\mathfrak{A}_0)\subseteq S_0$ by definition of a set S_0.

3) If $V\left(X^*,\alpha\right)\in\mathfrak{A}_2$, then quasinormal representation of a binary relation α has a form $\alpha=\left(Y_j^\alpha\times Z_j\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Y_j^\alpha,Y_0^\alpha\notin\{\varnothing\}$, $j=1,2,\cdots,n+k$ and from the Lemma 2.4 follows that α is generated by elements of the elements of set $B(\mathfrak{A}_0)\subseteq S_0$ by definition of a set S_0.

4) If $V\left(X^*,\alpha\right)\in\mathfrak{A}_3$, then quasinormal representation of a binary relation α has a form $\alpha=\left(Y_{n+j}^\alpha\times Z_{n+j}\right)\cup\left(Y_j^\alpha\times Z_j\right)$, where $Y_{n+j}^\alpha,Y_j^\alpha\notin\{\varnothing\}$, $j=1,2,\cdots,k$ and from the Lemma 2.5 follows that α is generated by elements of the elements of set $B(\mathfrak{A}_0)\subseteq S_0$ by definition of a set S_0.

Now, let $V\left(X^*,\alpha\right)\in\mathfrak{A}_4$, then quasinormal representation of a binary relation α has a form $\alpha=X\times\breve{D}$, or $\alpha=X\times Z_j$, where $j=1,2,\cdots,n+k$.

5) If $\alpha=X\times\breve{D}$, then from the statement b) of the Lemma 2.6 follows that binary relation α is generated by elements of the set $B(\mathfrak{A}_0)$.

6) If $\alpha = X \times Z_j$, where $j = 1, 2, \cdots, n+k$, then from the statement a) of the Lemma 2.6 and 2.7 follows that binary relation α is generated by elements of the set $B(\mathfrak{A}_0)$.

Thus, we have that S_0 is a generating set for the semigroup $B_X(D)$.

If $|X \setminus \breve{D}| \geq 1$, then the set S_0 is an irreducible generating set for the semigroup $B_X(D)$ since, S_0 is a set external elements of the semigroup $B_X(D)$.

The statement a) of the Theorem 2.1 is proved.

Now, let $X = \breve{D}$. First, we proved that every element of the semigroup $B_X(D)$ is generated by elements of the set S_1. The cases 1), 2), 3), 4) and 5) are proved analogously of the cases 1), 2), 3), 4) and 5 given above and consider case, when $V(X^*, \alpha) \in \mathfrak{A}_1$.

If $V(X^*, \alpha) = Z_j$, where $j = 1, 2, \cdots, k$, then from the statement a) of the Lemma 2.7 follows that binary relation α is generated by elements of the set $B(\mathfrak{A}_0)$.

If $V(X^*, \alpha) = Z_j$, where $Z_j \in D_2 \cup D_3$, then from the statement b) of the Lemma 2.6 follows that binary relation $\alpha = X \times T$ is external element for the semigroup $B_X(D)$.

Thus, we have that S_1 is a generating set for the semigroup $B_X(D)$.

If $X = \breve{D}$, then the set S_1 is an irreducible generating set for the semigroup $B_X(D)$ since S_1 is a set external elements of the semigroup $B_X(D)$.

The statement b) of the Theorem 2.1 is proved.

Theorem 2.1 is proved.

Corollary 2.1. *Let* $D \in \Sigma_8(X, n+k+1)$ $(k \geq 3)$ *and*

$$D_1 = \{Z_1, Z_2, \cdots, Z_k\}, D_2 = \{Z_{k+1}, Z_{k+2}, \cdots, Z_n\}, D_3 = \{Z_{n+1}, Z_{n+2}, \cdots, Z_{n+k}\};$$

$$\mathfrak{A}_1 = \{\{Z_{n+q}, Z_q, \breve{D}\}\}, \text{where } q = 1, 2, \cdots, k;$$

$$\mathfrak{A}_2 = \{\{Z_j, \breve{D}\}\}, \text{where } j = 1, 2, \cdots, n+k;$$

$$\mathfrak{A}_3 = \{\{Z_{n+j}, Z_j\}\}, \text{where } j = 1, 2, \cdots, k;$$

$$\mathfrak{A}_4 = \{\{Z_j\}, \{\breve{D}\}\}, \text{where } j = 1, 2, \cdots, n+k;$$

$$\mathfrak{A}_0 = \{V(D, \alpha) \subset D \mid V(D, \alpha) \notin \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3 \cup \mathfrak{A}_4\},$$

$$B(\mathfrak{A}_0) = \{\alpha \in B_X(D) \mid V(D, \alpha) \in \mathfrak{A}_0\},$$

$$B_0 = \{X \times T \mid T \notin D_2 \cup D_3\}$$

Then the following statements are true:

1) *If* $|X \setminus \breve{D}| \geq 1$, *then* $S_0 = B(\mathfrak{A}_o)$ *is the uniquely defined generating set for the semigroup* $B_X(D)$;

2) *If* $X = \breve{D}$, *then* $S_1 = B_0 \cup B(\mathfrak{A}_o)$ *is the uniquely defined generating set for the semigroup* $B_X(D)$.

Proof. It is well known, that if B is all external elements of the semigroup $B_X(D)$ and B' is any generated set for the $B_X(D)$, then $B \subseteq B'$ (see [1] [2] Lemma 1.15.1). From this follows that the sets $S_0 = B(\mathfrak{A}_o)$ and

$S_1 = B_0 \cup B(\mathfrak{A}_\circ)$ are defined uniquely, since they are sets external elements of the semigroup $B_X(D)$.

Corollary 2.1 is proved.

It is well-known, that if B is all external elements of the semigroup $B_X(D)$ and B' is any generated set for the $B_X(D)$, then $B \subseteq B'$ (Definition 1.1).

In this article, we find irredusible generating set for the complete semigroups of binary relations defined by X-semilattices of unions of the class $\Sigma_8(X, n+k+1)$ $(k \geq 3)$. This generating set is uniquely defined, since they are defined by elements of the external elements of the semigroup $B_X(D)$.

References

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