Applied Mathematics, 2018, 9, 207-209
http://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

A Generalized Wallis Formula

Javad Namazi

Fairleigh Dickinson University, Madison, NJ, USA
Email: namazi@fdu.edu

How to cite this paper: Namazi, J. (2018) A Generalized Wallis Formula. *Applied Mathematics*, **9**, 207-209.
https://doi.org/10.4236/am.2018.93015

Received: February 7, 2018
Accepted: March 19, 2018
Published: March 22, 2018

Abstract

This article generalizes the famous Wallis's formula $\int_0^{2\pi} \sin^{2k}\theta \mathrm{d}\theta = \int_0^{2\pi} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!2\pi}{2^{2k}(k!)^2}$ for $k \geq 0$, to an integral over the unit sphere S^{n-1}. An application to the integral of polynomials over S^{n-1} is discussed.

Keywords

Willis Formula, Unit Sphere

One of Wallis formulas is

$$\int_0^{2\pi} \sin^{2k}\theta \mathrm{d}\theta = \int_0^{2\pi} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!2\pi}{2^{2k}(k!)^2}$$

for $k \geq 0$. This formula can be proved by various methods [1] [2] [3] [4] including a repeated application of a reduction formula such as $\int_0^{2\pi} \sin^{k}\theta \mathrm{d}\theta = \frac{k-1}{k}\int_0^{2\pi} \sin^{k-2}\theta \mathrm{d}\theta$. Note that $\sin\theta$ and $\cos\theta$ are coordinates of a point on the unit sphere in R^2. Since the above formula involves an integration over the unit circle in R^2, its extension to higher dimensions is of interest.

For each $x = (x_1, x_2, \cdots, x_n) \in R^n$, let $|x| = \left(\sum x_i^2\right)^{1/2}$ be its Euclidean norm. We call $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, where $\alpha_i \geq 0$ are non-negative integers, a multi-index, and $|\alpha| = \sum |\alpha_i|$ its degree. Set $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$ and $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$. Let $S^{n-1} = \{\xi \in R^n : |\xi| = 1\}$ be the unit sphere in R^n and $\mathrm{d}\sigma$ be its surface measure. Let $B_r(a) = \{x \in R^n : |x-a| \leq r\}$ stand for the ball of radius r centered at a. The gamma function is defined as $\Gamma(s) = \int_0^\infty \mathrm{e}^{-t}t^{s-1}\mathrm{d}t$, for $s > 0$. The generalized Wallis's formula is a special case of the following theorem.

Theorem 1 (i) $\int_{S^{n-1}} \xi^{\alpha} \mathrm{d}\sigma = 0$, *if any* α_i *is odd. In particular, the integral equals zero if* $|\alpha|$ *is odd.*

(ii) $\int_{S^{n-1}} \xi^{2\alpha} \mathrm{d}\sigma = \dfrac{(2\alpha)!2\pi^{n/2}}{2^{2|\alpha|}\alpha!\Gamma(n/2+|\alpha|)}, |\alpha| \geq 0.$

Setting $\alpha_i = k$ and $\alpha_j = 0$ for $j \neq i$ in the theorem, the generalized Wallis's formula follows

$$\int_{S^{n-1}} \xi_i^{2k} \mathrm{d}\sigma = \frac{(2k)!2\pi^{n/2}}{2^{2k} k!\Gamma(n/2+k)}, k \geq 0.$$

Note that for $|\alpha| = 0$, (ii) is equivalent to the well-known formula

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{1}$$

where ω_{n-1} is the surface area of the unit sphere in R^n. Theorem 1 is interesting in its own right and has further applications. For example, for a polynomial $p(x) = \sum_{|\alpha| \leq m} b_\alpha x^\alpha$ of degree *m*, one may express $\int_{B_r(0)} p(x)\mathrm{d}x$ as a simple polynomial of degree $n+m$ in *r*. In the following we use polar coordinates $x = \rho\xi, \rho = |x|, \xi \in S^{n-1}$.

$$\begin{aligned}\int_{B_r(0)} p(x)\mathrm{d}x &= \sum_{|\alpha| \leq m} b_\alpha \int_{B_r(0)} x^\alpha \mathrm{d}x = \sum_{|\alpha| \leq m} b_\alpha \int_0^r \rho^{|\alpha|+n-1} \mathrm{d}\rho \int_{S^{n-1}} \xi^\alpha \mathrm{d}\sigma \\ &= \sum_{2|\alpha| \leq m} \frac{b_{2\alpha} r^{2|\alpha|+n}}{2|\alpha|+n} \int_{S^{n-1}} \xi^{2\alpha} \mathrm{d}\sigma = \sum_{|\alpha| \leq [m/2]} \frac{b_{2\alpha} d_\alpha}{2|\alpha|+n} r^{2k+n} \\ &= \sum_{k=0}^{[m/2]} \left(\sum_{|\alpha|=k} \frac{b_{2\alpha} d_\alpha}{2k+n} \right) r^{2k+n} = \sum_{k=0}^{[m/2]} c_k r^{2k+n}.\end{aligned}$$

Here $d_\alpha = \int_{S^{n-1}} \xi^{2\alpha} \mathrm{d}\sigma$ as given by (ii), and [.] is the bracket function.

Proof of Theorem 1. (i) The proof is by induction on $|\alpha|$.

If $|\alpha| = 1$ then $\xi = \xi_i$ for some *i*. Therefore, $\int_{S^{n-1}} \xi^\alpha \mathrm{d}\sigma = \int_{S^{n-1}} \xi_i \mathrm{d}\sigma = 0$ by the symmetry of the sphere.

Assume now the assertion is true for $|\alpha| \leq m$ for some $m \geq 1$. Let $|\alpha| = m+1$ and assume, without loss of generality, that α_1 is odd. Applying the divergence theorem results in

$$\begin{aligned}\int_{S^{n-1}} \xi^\alpha \mathrm{d}\sigma &= \int_{S^{n-1}} \xi_1 \left(\xi_1^{\alpha_1 - 1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n} \right) \mathrm{d}\sigma \\ &= \int_{B_1(0)} \frac{\partial}{\partial x_1} \left(x_1^{\alpha_1 - 1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \right) \mathrm{d}x.\end{aligned} \tag{2}$$

If $\alpha_1 = 1$, the last integral in (2) is zero. Otherwise, a conversion to polar coordinates in (2), yields,

$$\begin{aligned}\int_{S^{n-1}} \xi^\alpha \mathrm{d}\sigma &= (\alpha_1 - 1) \int_{B_1(0)} x_1^{\alpha_1 - 2} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mathrm{d}x \\ &= (\alpha_1 - 1) \int_0^1 \rho^{m+n-2} \mathrm{d}\rho \int_{S^{n-1}} \xi^\beta \mathrm{d}\sigma = \frac{\alpha_1 - 1}{m+n-1} \int_{S^{n-1}} \xi^\beta \mathrm{d}\sigma,\end{aligned}$$

where $\beta = (\alpha_1 - 2, \alpha_2, \cdots, \alpha_n)$. The last integral is now zero, by the induction hypothesis.

ii) The proof is by induction on $|\alpha|$.

For $|\alpha|=0$, we must establish (1). Let $e_n=\int_{R^n}\mathrm{e}^{-\pi|x|^2}\mathrm{d}x$. Writing e_n as a product of integrals and using polar coordinates in R^2 followed by a change of variables, one obtains

$$\begin{aligned}e_n&=\prod_{i=1}^{n}\int_R \mathrm{e}^{-\pi x_i^2}\mathrm{d}x_i=(e_1)^n=(e_2)^{n/2}\\&=\left(\int_0^{2\pi}\mathrm{d}\theta\int_0^\infty r\mathrm{e}^{-\pi r^2}\mathrm{d}r\right)^{n/2}=\left(\int_0^\infty \mathrm{e}^{-u}\mathrm{d}u\right)^{n/2}=1.\end{aligned}$$

We used a change of variable $u=\pi r^2$ in the previous integral. Converting to polar coordinates for R^n results in

$$\begin{aligned}1=e_n&=\int_{R^n}\mathrm{e}^{-\pi|x|^2}\mathrm{d}x=\omega_{n-1}\int_0^\infty r^{n-1}\mathrm{e}^{-\pi r^2}\mathrm{d}r\\&=\frac{\omega_{n-1}}{2\pi^{n/2}}\int_0^\infty u^{n/2-1}\mathrm{e}^{-u}\mathrm{d}u=\frac{\Gamma(n/2)}{2\pi^{n/2}}\omega_{n-1}.\end{aligned}$$

Identity (1) follows immediately from the last equation.

Now suppose the claim is true for $|\alpha|=m$. Let $|\alpha|=m+1$. We may assume, without loss of generality, that $\alpha_1\ge 1$. Applying the divergence theorem followed by a conversion to polar coordinates leads to

$$\begin{aligned}\int_{S^{n-1}}\xi^{2\alpha}\mathrm{d}\sigma&=\int_{S^{n-1}}\xi_1\left(\xi_1^{2\alpha_1-1}\xi_2^{2\alpha_2}\cdots\xi_n^{2\alpha_n}\right)\mathrm{d}\sigma=\int_{B_1(0)}\frac{\partial}{\partial x_1}\left(x_1^{2\alpha_1-1}x_2^{2\alpha_2}\cdots x_n^{2\alpha_n}\right)\mathrm{d}x\\&=(2\alpha_1-1)\int_{B_1(0)}x_1^{2\alpha_1-2}x_2^{2\alpha_2}\cdots x_n^{2\alpha_n}\mathrm{d}x=\frac{2\alpha_1-1}{n+2m}\int_{S^{n-1}}\xi^{2\beta}\mathrm{d}\sigma,\end{aligned}$$

where $\beta=(\alpha_1-1,\alpha_2,\cdots,\alpha_n)$. Since $|\beta|=m$, and using the fact that $\Gamma(s+1)=s\Gamma(s)$ along with the induction hypothesis, we get

$$\int_{S^{n-1}}\xi^{2\alpha}\mathrm{d}\sigma=\frac{2\alpha_1-1}{n+2m}\cdot\frac{(2\beta)!2\pi^{n/2}}{2^{2|\beta|}\beta!\Gamma(n/2+m)}=\frac{(2\alpha)!2\pi^{n/2}}{2^{2|\alpha|}\alpha!\Gamma(n/2+|\alpha|)}.$$

□

References

[1] Muller, C. (1966) Spherical Harmonics. Springer-Verlag Berlin Heidelberg. https://doi.org/10.1007/BFb0094775

[2] Macrobert, T.M. (1948) Spherical Harmonics: An Elementary Treatise on Harmonic Functions with Applications. Dover Publications, New York.

[3] Stein, E. and Weiss, G. (1971) Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton.

[4] Jeffrey, H. and Jeffrey, B. (1999) Methods of Mathematical Physics. 3rd Edition, Cambridge University Press, Cambridge.