

A Comparison between Modified Sumudu Decomposition Method and Homotopy Perturbation Method

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How to cite this paper: Ahmed, S.A. (2018) A Comparison between Modified Sumudu Decomposition Method and Homotopy Perturbation Method. *Applied Mathematics*, 9, 199-206.

<https://doi.org/10.4236/am.2018.93014>

Received: January 27, 2018

Accepted: March 16, 2018

Published: March 19, 2018

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Abstract

In this paper, we present a comparative study between the modified Sumudu decomposition method (MSDM) and homotopy perturbation method (HPM). The study outlines the important features of the two methods. The analysis will be explained by discussing the nonhomogeneous Kortewege-de Vries (KdV) problems.

Keywords

The Sumudu Decomposition Method, The Homotopy Perturbation Method, Kortewege-de Vries (KdV) Problems

1. Introduction

This paper outlines effective comparison between two powerful methods that were newly advanced. The first method is modified form of the Sumudu decomposition method (MSDM) developed by Devedra Kumar [1] and used in [2] [3] [4] [5] [6] among many others. The second method is homotopy perturbation method (HPM) developed by He [7] and used from many researchers [8] [9] [10]. The two methods give quickly convergent series and are successfully used to solve a large number of linear and nonlinear problems.

In the present study, we consider the nonhomogeneous KdV equation of the following type:

$$U_t + aUU_x + bU_{xxx} = f(x, t), \quad (1)$$

with the initial condition;

$$U(x, 0) = g(x). \quad (2)$$

where a and b are constants.

The nonlinear KdV Equation (1) is an important mathematical model with wide applications in quantum mechanics and nonlinear optics. The KdV equation has several applications to physical problems. It approximately describes the evolution of long water waves. In addition, it is used in various fields such as, shallow water waves, acoustic waves in plasma, and long internal waves in a density.

The main objective of this contribution is to introduce a comparative study to solve KdV equation by using two of the most recently developed methods, the (MSDM), and the (HPM). The two methods accurately compute the solution in a series form or in an exact form.

2. Analysis of Numerical Methods

2.1. Basic Idea of the MSDM

To illustrate the basic idea of this method, we consider a general non-homogeneous partial differential equation with the initial conditions of the form:

$$\begin{aligned} DU(x,t) + RU(x,t) + NU(x,t) &= g(x,t) \\ U(x,0) &= h(x), \quad U_t(x,0) = f(x). \end{aligned} \quad (3)$$

where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, R is other linear differential operator of less order than D , N represent the general nonlinear operator and $g(x,t)$ is the source term.

Taking the Sumudu transform of both sides of Equation (3), we get:

$$S[LU(x,t)] + S[RU(x,t)] = S[g(x,t)]; \quad (4)$$

Using the differentiation property of the Sumudu transform and given initial conditions, we have:

$$S[U(x,t)] = u^2 S[g(x,t)] + h(x) + uf(x) - u^2 S[RU(x,t) + NU(x,t)]. \quad (5)$$

If we apply the inverse operator S^{-1} to both sides of the Equation (5), we obtain:

$$U(x,t) = G(x,t) - S^{-1}[u^2 S[RU(x,t) + NU(x,t)]]. \quad (6)$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, apply the Adomain decomposition method [11] [12] [13] [14]:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t), \quad (7)$$

The nonlinear term can be decomposed as:

$$NU(x,t) = \sum_{n=0}^{\infty} A_n(U), \quad (8)$$

For some Adomian polynomials $A_n(U)$ that are given by:

$$A_n(U_0, U_1, U_2, \dots, U_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{n=0}^{\infty} \lambda^n U_n \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots$$

Substituting Equation (7) and Equation (8) in Equation (6), we get:

$$\sum_{n=0}^{\infty} U_n(x, t) = G(x, t) - S^{-1} \left[u^2 S \left[R \sum_{n=0}^{\infty} U_n(x, t) + \sum_{n=0}^{\infty} A_n(U) \right] \right]. \quad (9)$$

Accordingly, the formal recursive relation is defined by:

$$\begin{aligned} U_0(x, t) &= G(x, t), \\ U_{k+1}(x, t) &= -S^{-1} \left[u^2 S [RU_k + A_k] \right], \quad k \geq 0. \end{aligned} \quad (10)$$

The modified Sumudu decomposition method introduces a slight variation to the recursive relation (10) that will lead to the determination of the components of U in a faster and easier way [15] [16]. For specific cases, the function $G(x, t)$ can be set as the sum of two partial functions, namely $f_1(x, t)$ and $f_2(x, t)$. In other words, we can set:

$$G(x, t) = f_1 + f_2; \quad (11)$$

Using (11), we introduce a qualitative change in the formation of recursive relation (10). The modified recursive relation can be identified by:

$$U_0 = f_1, \quad (12)$$

$$U_1 = f_2 - S^{-1} \left[u^2 S [RU_0 + A_0] \right], \quad (13)$$

$$U_{k+1} = -S^{-1} \left[u^2 S [RU_k + A_k] \right]; \quad k \geq 1 \quad (14)$$

The solution through the modified Sumudu decomposition method is highly depend upon the choice of $f_1(x, t)$ and $f_2(x, t)$.

2.2. Basic Idea of HPM

Consider the following general nonlinear differential equation,

$$u - N(u) = f, \quad (15)$$

where N is a nonlinear operator from Hilbert space H to H , u is an unknown function, and f is a known function in H .

The homotopy perturbation method u as a series with components u_n , and $N(u)$ as a series with components H_n , homotopy polynomials, which can be calculated using the formula:

$$H_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N \left(\sum_{i=0}^{\infty} u_i \lambda^i \right) \Big|_{\lambda=0} \quad (16)$$

To illustrate the homotopy perturbation method (HPM), we consider (15) as;

$$L(v) = v(x) - f(x) - N(v) = 0 \quad (17)$$

with solution $u(x)$. As a possible remedy, we can define homotopy $H(v, p)$ as follows:

$$H(v, 0) = F(v), \quad H(v, 1) = L(v)$$

where $F(v)$ is an integral operator with known solution v_0 , which can be ob-

tained easily. Typically, we may choose a convex homotopy in the form;

$$H(v, p) = (1 - p)F(v) + pL(v) = 0 \tag{18}$$

and continuously trace an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution function $H(u, 0)$. The embedding parameter p monotonically increase from zero to unit as the trivial problem $F(v) = 0$ is continuously deformed to original problem $L(v) = 0$.

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots \tag{19}$$

when $p \rightarrow 1$, Equation (18) corresponds to Equation (17) and Equation (19) becomes the approximate solution of Equation (17), *i.e.*

$$v = \lim_{p \rightarrow 1} v_0 + v_1 + v_2 + v_3 + \dots \tag{20}$$

3. Application

In this section, we demonstrate the analysis of two methods by applying two methods to the following two Kortewege-de Vries (KdV) partial differential equations.

Example 1: Consider the following inhomogeneous nonlinear KdV equation [17]:

$$u_t + uu_x + u_{xxx} = \sin x + t \cos x(t \sin x - 1); \tag{21}$$

With the initial condition:

$$u(x, 0) = 0. \tag{22}$$

1) Using HPM

To solve Equations (21)-(22) by homotopy perturbation method, we construct the following homotopy:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} = p \left(-u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} + \sin x + t \cos x(t \sin x - 1) - \frac{\partial u_0}{\partial t} \right) \tag{23}$$

Assume the solution of Equation (23) to be in the form:

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots \tag{24}$$

Substituting (24) into (23) and comparing coefficients of terms with identical powers of p , leads to:

$$p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0. \tag{25}$$

$$p^1 : \frac{\partial v_1}{\partial t} = -u_0 \frac{\partial u_0}{\partial x} - \frac{\partial^3 u_0}{\partial x^3} + \sin x + t^2 \cos x \sin x - t \cos x - \frac{\partial u_0}{\partial t} \tag{26}$$

⋮

The given initial value admits the use of:

$$u_0(x, 0) = 0 \tag{27}$$

The solution reads:

$$u_0(x, t) = 0 \tag{28}$$

$$u_1(x, t) = t \sin x + \frac{t^3}{3} \sin x \cos x - \frac{t^2}{2} \cos x, \quad (29)$$

$$u_2(x, y) = -\frac{t^3}{3} \sin x \cos x + \frac{t^2}{2} \cos x - \frac{t^4}{3} \sin^2 x - \frac{t^4}{3} \sin^3 x + \dots \quad (30)$$

Examining the components u_1 and u_2 in Equation (29) and Equation (30), we can easily observe that the last two terms in u_1 and the first two terms in u_2 are the self-canceling (noise terms) [18]. Hence, the non-noise terms in u_1 yields the exact solution of Equations (21)-(22), given by:

$$u(x, t) = t \sin x. \quad (31)$$

Notes on HPM:

From the previous analysis, we can observe that:

- **HPM** can be applied it to various nonlinear problems. The main disadvantage is that we should suitably choose an initial guess.
- **HPM** needs some modification to the rapid convergence of the series solution.

To overcome these disadvantages of **HPM**, the following **ADSTM** method is suggested.

2) Using MSDM

By taking Sumudu transform for (21) and using (22) we obtain:

$$S[U(x, t)] = u \sin x + 2u^3 \cos x \sin x - u^2 \cos x - uS[UU_x + U_{xxx}]. \quad (32)$$

Applying S^{-1} to both sides of (32) we obtain;

$$U(x, t) = t \sin x + \frac{t^3}{3} \cos x \sin x - \frac{t^2}{2} \cos x - S^{-1}[uS[UU_x + U_{xxx}]]. \quad (33)$$

Substituting;

$$U(x, t) = \sum_{n=0}^{\infty} U_n(x, t); \quad (34)$$

And the nonlinear terms of;

$$UU_x = \sum_{n=0}^{\infty} A_n. \quad (35)$$

Into (33) gives;

$$\sum_{n=0}^{\infty} U_n(x, t) = t \sin x + \frac{t^3}{3} \cos x \sin x - \frac{t^2}{2} \cos x - S^{-1} \left[uS \left[\sum_{n=0}^{\infty} A_n + \left(\sum_{n=0}^{\infty} U_n(x, t) \right)_{xxx} \right] \right] \quad (36)$$

This gives the modified recursive relation;

$$U_0(x, t) = t \sin x, \quad (37)$$

$$U_1(x, t) = \frac{t^3}{3} \cos x \sin x - \frac{t^2}{2} \cos x - S^{-1} \left[uS \left[A_0 + (U_0)_{xxx} \right] \right], \quad (38)$$

$$U_{k+1}(x, t) = -S^{-1} (A_k + U_k), \quad k \geq 1. \quad (39)$$

The first few of the components are given by;

$$U_0(x, t) = t \sin x, \quad (40)$$

$$U_1(x, t) = \frac{t^3}{3} \cos x \sin x - \frac{t^2}{2} \cos x - S^{-1} \left[uS \left[A_0 + (U_0)_{xxx} \right] \right] = 0, \quad (41)$$

$$U_{k+1}(x, t) = 0, \quad k \geq 1. \quad (42)$$

The solution in a closed form is given by;

$$U(x, t) = t \sin x. \quad (43)$$

Example 2: Consider the following inhomogeneous nonlinear KdV equation [17]:

$$u_t + uu_x + u_{xxx} = (xt + \sin x)(t + \cos x) + x + \cos x; \quad (44)$$

With the initial condition:

$$u(x, 0) = \sin x. \quad (45)$$

1) Using HPM

Using homotopy perturbation method like in **Example 1**, we obtain the following components:

$$u_0(x, t) = \sin x \quad (46)$$

$$u_1(x, t) = xt + \frac{1}{3}xt^3 + \frac{1}{2}xt^2 \cos x + t \cos x + t \sin x \cos x + \frac{1}{2}t^2 \sin x, \quad (47)$$

$$u_2(x, t) = -t \cos x - t \sin x \cos x - \frac{1}{2}t^2 \sin x - \dots. \quad (48)$$

It is obvious that the last three terms in u_1 and the first three terms in u_2 are the self-canceling (noise terms). Keeping the remaining non-noise terms in u_1 leads to the exact solution of Equations (44)-(45), given by:

$$u(x, t) = xt + \sin x. \quad (49)$$

2) Using ADSTM

Proceeding as in **Example 1**, Equation (44) becomes:

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x, t) &= xt + \sin x + \frac{1}{3}xt^3 + \frac{1}{2}xt^2 \cos x + t \cos x + t \sin x \cos x \\ &+ \frac{1}{2}t^2 \sin x - S^{-1} \left[uS \left[\sum_{n=0}^{\infty} A_n + \left(\sum_{n=0}^{\infty} U_n(x, t) \right)_{xxx} \right] \right] \end{aligned} \quad (50)$$

The modified Sumudu decomposition method admits the of a modified recursive relation given by:

$$U_0(x, t) = xt + \sin x, \quad (51)$$

$$\begin{aligned} U_1(x, t) &= \frac{1}{3}xt + \frac{1}{2}xt^2 \cos x + t \cos x + t \sin x \cos x + \frac{1}{2}t^2 \sin x \\ &- S^{-1} \left[uS \left[A_0 + (U_0(x, t))_{xxx} \right] \right], \end{aligned} \quad (52)$$

$$U_{k+1}(x, t) = -S^{-1} \left[uS \left[A_k + (U_k(x, t))_{xxx} \right] \right], \quad k \geq 1. \quad (53)$$

Consequently, we obtain:

$$U_0(x, t) = xt + \sin x, \quad (54)$$

$$U_1(x, t) = 0, \quad (55)$$

$$U_{k+1}(x, t) = 0, \quad k \geq 1. \quad (56)$$

The exact solution is given by:

$$U(x, t) = xt + \sin x. \quad (57)$$

4. Conclusions

The main objective of this paper is to introduce a comparative study between modified Sumudu decomposition method and Homotopy perturbation method. The two methods are strong and effective methods that both hand approximations of higher reliability of series solution. A remarkable conclusion can made here. Modified Sumudu decomposition method gives rapid convergence of the series solution without showing noise terms. However, homotopy perturbation method provides the components of the series solution, where these components should have noise terms.

More importantly, the modified Sumudu decomposition method (MSDM) gives better efficiency in many cases, and this implies that the modified Sumudu decomposition method has an advantage over the homotopy perturbation method.

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