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Complete Convergence of Weighted Sums for Asymptotically Almost Negatively Associated Sequences

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Abstract

For weighted sums of asymptotically almost negatively associated (AANA) random variables sequences, we use the Rosenthal type moment inequalities and prove the Marcinkiewicz-Zygmund type complete convergence and obtain the complete convergence rates. Our results extend some known ones.

Keywords

Asymptotically Almost Negatively Associated, Weighted Sums, Complete Convergence

1. Introduction

A sequence $\{X_n, n \ge 1\}$ of random variables is said to be asymptotically almost negatively associated (AANA, in short) if there exists a nonnegative sequence $q(n) \to 0$ as $n \to \infty$ such that

$$\begin{aligned}&\operatorname{Cov}\left(f(X_n), g(X_{n+1}, X_{n+2}, \cdots, X_{n+k})\right)\\ &\le q(n)\left[\operatorname{Var} f(X_n)\operatorname{Var}\left(g(X_{n+1}, X_{n+2}, \cdots, X_{n+k})\right)\right]^{1/2}\end{aligned} \tag{1.1}$$

for all $n, k \ge 1$ and for all coordinate wise nondecreasing continuous functions f and g whenever the variances exist. $\{q(n), n \ge 1\}$ is said to be the mixing coefficients of $\{X_n, n \ge 1\}$.

Chandra and Ghosal [1] firstly introduced this concept, and gave a following example. Let $X_n = (1 + a_n)^{-1/2}(Y_n + a_n Y_{n+1}), n \ge 1$, where $\{Y_n, n \ge 1\}$ are independent random variables with common distribution $N(0,1)$, then $\{X_n, n \ge 1\}$ is an AANA sequence. At the same time, the Kolmogorov type inequality and strong law of large numbers (SLLN) were proved.

From then on, many authors have studied the various limit properties for AANA sequences. For example, Chandra and Ghosal [2] [3] obtained the almost sure convergence of weighted average, Kim, Ko and Lee [4] established the Hajek-Renyi type inequalities and Marcinkiewicz-Zygmund type SLLN, Cai [5] investigated the complete convergence of weighted sums, Yuan and An [6] got the Rosenthal type inequalities, L_p convergence, complete convergence and Marcinkiewicz-Zygmund type SLLN, Wang, Hu and Yang [7] obtained the complete convergence and SLLN, etc. and so on. We see the following theorems.

Theorem A. (Kim, Ko and Lee [4]) Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be a sequence of real numbers with $\sup_{n\ge1}\sum_{i=1}^{n}|a_{ni}| < \infty$ and let $\{X_n, n \ge 1\}$ be a sequence of identically distributed, mean zero AANA random variables with $E|X_1|^p < \infty, 0 < p < 2$. If $\sum_{n=2}^{\infty} q^2(n) < \infty$, then

$$\frac{1}{n^{1/p}}\sum_{i=1}^{n} a_{ni}X_i \to 0 \quad \text{a.s.} \tag{1.2}$$

Theorem A generalizes the Marcinkiewicz-Zygmund SLLN (Chow and Teicher [1], or Gut [8]) for the independent identically distributed (i.i.d.) sequences to the weighted sums of AANA sequence.

Theorem B. (Cai [5]) Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be a sequence of real numbers with $\sum_{i=1}^{n} a_{ni}^2 = O(n)$, and let $\{X_n, n \ge 1\}$ be a sequence of mean zero AANA random variables. Let $\sum_{n=1}^{\infty} q^2(n) < \infty$. If $E|X|^p < \infty$, for $0 < p < 2$ and $P(|X_i| > x) \le P(|X| > x), x > 0$. Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{1/p}\right) < \infty. \tag{1.3}$$

Theorem C. (Yuan and An [6]) Let $\{X_n, n \ge 1\}$ be an AANA sequence of identically distributed random variables with mixing coefficients $\{q(n), n \ge 1\}, \alpha p > 1, \alpha > 1/2$, and suppose that $EX_1 = 0$ for $\alpha \le 1$. If $\sum_{n=1}^{\infty} q^{2/(3r)}(n) < \infty$ where $r = (\alpha p - 1)/(\alpha - 1/2) + p + 2$, then $E|X_1|^p < \infty$ is equivalent to

$$\sum_{n=1}^{\infty} n^{\alpha p - 2}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{1.4}$$

The main purpose of this paper is to further investigate the complete convergence, almost sure convergence and complete convergence rate of weighted sums for AANA random variable sequences. In the following sections, theorem 2.1 (Section 2) extends theorem A to some more relaxed conditions and gets a more general result. Theorem 2.2 is about complete convergence rates which extends theorem B and theorem C to the cases of weighted sums.

2. Main Results

Throughout this paper we use the following notations: $I(\cdot)$ denotes the indicator function, C stands for a positive constant its value may be different on different places, $\ll$ represents the Vinogradov symbol O, $=:$ means defined as and $\|\cdot\|_p$ denotes the L_p norm.

Theorem 2.1. Let $\{X_n, n\geq 1\}$ be a sequence of mean zero, identically distributed AANA random variables with $\sum_{n=1}^{\infty} q^2(n) < \infty$. Let $\{a_{ni}, i\geq 1, n\geq 1\}$ be a sequence of real numbers satisfying $\sup_{n\geq 1}\sum_{i=1}^{n} a_{ni}^2 < \infty$. If $E|X_1|^p < \infty, 0 < p < 2$, then

$$\frac{1}{n^{1/p}} \max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| \to 0 \text{ completely.} \tag{2.1}$$

Remark 2.1. As we known, complete convergence leads to the almost sure convergence but its converse does not hold. So the result of theorem 2.1 is stronger than theorem A. On the other hand, $\sum_{i=1}^{n} a_{ni}^2 \leq \left(\sum_{i=1}^{n}|a_{ni}|\right)^2 \leq C < \infty$ under the condition of $\sup_{n\geq 1}\sum_{i=1}^{n}|a_{ni}| < \infty$. Thus theorem A is a corollary of theorem 2.1.

Theorem 2.2. Let $\{X_n, n\geq 1\}$ be a sequence of centered identically distributed AANA random variables with mixing coefficients $\{q(n), n\geq 1\}$, $E|X_1|^p < \infty, \alpha p > 1, \alpha > 1/2$. Suppose that $EX_1 = 0$ for $p > 1$. Let $\{a_{ni}, i\geq 1, n\geq 1\}$ be a sequence of real numbers with $\sum_{i=1}^{n}|a_{ni}|^r = O(n)$, $r > (\alpha p - 1)/(\alpha - 1/2)$ if $p\geq 2$; or $r = 2$ if $0 < p < 2$: Take $\tilde{r} = \left(1/2^{k-1} - 2/r\right)r/(r-1)$, where k is a positive integer number satisfying $2^k < r \leq 2^{k+1}$. If $\sum_{n=1}^{\infty} q^{\tilde{r}}(n) < \infty$, then for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right) < \infty \tag{2.2}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\sup_{k\geq n}\left|k^{-\alpha}\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon\right) < \infty. \tag{2.3}$$

3. Proofs

To prove our results we need the following two lemmas.

Lemma 3.1. (Yuan and An [6]) Let $\{X_n, n\geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n\geq 1\}$. Let $f_1, f_2, \cdots$ be all nondecreasing (or all nonincreasing) functions, then $\{f_n(X_n), n\geq 1\}$ is still a sequence of AANA random variables with mixing coefficients $\{q(n), n\geq 1\}$.

Lemma 3.2. (Yuan and An [6]) Let $\{X_n, n\geq 1\}$ be a sequence of AANA random variables with mean zero and mixing coefficients $\{q(n), n\geq 1\}$, then there exists a positive constant C_p depending only on p such that

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leq C_p\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n-1} q^{2-2/p}(i)\|X_i\|_p\right)^p\right\} \tag{3.1}$$

for all $n\geq 1$ and $1 < p < 2$, and such that

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_i\right|^p\right)\le C_p\left\{\sum_{i=1}^{n}E\left|X_i\right|^p+\left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}+\left(\sum_{i=1}^{n-1}q^{2/p}(i)\left\|X_i\right\|_p\right)^p\right.$$
$$\left.+\left(\sum_{i=1}^{n-1}q^{1/2^{k-1}-2/p}(i)\left\|X_i\right\|_p\right)^p\right\} \tag{3.2}$$

for all $n\ge 1$ and $2^k<p\le 2^{k+1}$ where integer number $k\ge 1$.

In particular, if $\sum_{n=1}^{\infty}q^2(n)<\infty$, then

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_i\right|^p\right)\le C_p\sum_{i=1}^{n}E\left|X_i\right|^p \tag{3.3}$$

for all $n\ge 1$ and $1<p\le 2$.

Remark 3.1. It's obvious that if $p=2$, taking $k=0$ on the right hand of (3.2), the two inequalities (3.1) and (3.2) are the same.

Corollary 3.1. Under the conditions of Lemma 3.2, we have the following moment inequality

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_i\right|^p\right)\le C_p\left\{\left[1+C\left(\sum_{i=1}^{n-1}q^{\tilde r}(i)\right)^{p-1}\right]\sum_{i=1}^{n}E\left|X_i\right|^p+\left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}\right\} \tag{3.4}$$

for all $n\ge 1$ and $2^k<p\le 2^{k+1}$, where integer number $k\ge 0$,
$\tilde r=\left(1/2^{k-1}-2/p\right)p/(p-1)$.

Proof of Corollary 3.1. For $2^k<p\le 2^{k+1}$, we know $1/2^{k-1}-2/p\le 2/p$. Since $q(n)\to 0(n\to\infty)$, for n large enough, there exists a positive constant C such that

$$\left(\sum_{i=1}^{n-1}q^{2/p}(i)\left\|X_i\right\|_p\right)^p\le C\left(\sum_{i=1}^{n-1}q^{1/2^{k-1}-2/p}(i)\left\|X_i\right\|_p\right)^p. \tag{3.5}$$

Applying the Holder inequality on the right hand of (3.5) we get

$$\left(\sum_{i=1}^{n-1}q^{1/2^{k-1}-2/p}(i)\left\|X_i\right\|_p\right)^p\le\left(\sum_{i=1}^{n-1}q^{\tilde r}(i)\right)^{p-1}\left(\sum_{i=1}^{n}E\left|X_i\right|^p\right). \tag{3.6}$$

Thus (3.4) follows from (3.2), (3.5) and (3.6).

Proof of Theorem 2.1. Without loss of generality we may assume $a_{ni}\ge 0$ for all $n\ge 1, i\ge 1$. Let

$$Y_{ni}=-n^{1/p}I\left(X_i<-n^{1/p}\right)+X_iI\left(\left|X_i\right|\le n^{1/p}\right)+n^{1/p}I\left(X_i>n^{1/p}\right),$$
$$Y'_{ni}=X_i-Y_{ni}=\left(X_i+n^{1/p}\right)I\left(X_i<-n^{1/p}\right)+\left(X_i-n^{1/p}\right)I\left(X_i>n^{1/p}\right).$$

Since Lemma 3.1, $\{Y_{ni},i\ge 1\}$ and $\{a_{ni}Y_{ni},i\ge 1\}$ are AANA for all $n\ge 1$. It's easy to see that

$$n^{-1/p}\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}X_i\right|\le n^{-1/p}\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}Y_{ni}\right|+n^{-1/p}\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}Y'_{ni}\right|=:I_1+I_2.$$

To prove (2.1) it suffices to prove $I_1\to 0$ completely, and $I_2\to 0$ completely.

By assumption $\sup_{n\ge 1}\sum_{i=1}^{n}a_{ni}^2<\infty$ and the C_r inequality we have

$\sum_{i=1}^{n} a_{ni} \ll n^{1/2}$. For $EX_1 = 0$, considering two cases $0 < p \le 1$ and $1 < p < 2$ we can easily get

$$n^{-1/p}\left|\sum_{i=1}^{n} a_{ni}EY_{ni}\right| = n^{-1/p}\left|\sum_{i=1}^{n} a_{ni}EY'_{ni}\right| \to 0. \tag{3.7}$$

Thus, to prove $I_1 \to 0$ completely it suffices to prove

$$n^{-1/p}\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}\left(Y_{ni} - EY_{ni}\right)\right| \to 0 \text{ completely.} \tag{3.8}$$

By the Chebyseve inequality, $\sum_{n=1}^{\infty} q^2(n) < \infty$ and (3.3) of Lemma 3.2 we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}\left(Y_{ni} - EY_{ni}\right)\right| > \varepsilon n^{1/p}\right) \\
&\ll \sum_{n=1}^{\infty} n^{-2/p} E\max_{1\le k\le n}\left(\sum_{i=1}^{k} a_{ni}\left(Y_{ni} - EY_{ni}\right)\right)^2 \\
&\ll \sum_{n=1}^{\infty} n^{-2/p}\sum_{i=1}^{n} a_{ni}^2 EY_{ni}^2 \\
&\ll \sum_{n=1}^{\infty} n^{-2/p}\sum_{i=1}^{n} a_{ni}^2\left(EX_i^2 I\left(\left|X_i\right| \le n^{1/p}\right) + n^{2/p}P\left(\left|X_i\right| > n^{1/p}\right)\right) \\
&\ll \sum_{n=1}^{\infty} n^{-2/p} EX_1^2 I\left(\left|X_1\right| \le n^{1/p}\right) + n^{2/p}P\left(\left|X_1\right| > n^{1/p}\right) =: I'_1 + I''_1.
\end{aligned} \tag{3.9}$$

It's easy to see that

$$I''_1 = \sum_{n=1}^{\infty} P\left(\left|X_1\right| > n^{1/p}\right) \le E\left|X_1\right|^p < \infty. \tag{3.10}$$

For I'_1 we have

$$\begin{aligned}
I'_1 &= \sum_{n=1}^{\infty} n^{-2/p} EX_1^2 I\left(\left|X_1\right| \le n^{1/p}\right) \\
&= \sum_{n=1}^{\infty} n^{-2/p}\sum_{j=1}^{n} EX_1^2 I\left((j-1)^{1/p} < \left|X_1\right| \le j^{1/p}\right) \\
&= \sum_{j=1}^{\infty} EX_1^2 I\left((j-1)^{1/p} < \left|X_1\right| \le j^{1/p}\right)\sum_{n=j}^{\infty} n^{-2/p} \\
&\ll \sum_{j=1}^{\infty} j^{-2/p+1} EX_1^2 I\left((j-1)^{1/p} < \left|X_1\right| \le j^{1/p}\right) \\
&\le \sum_{j=1}^{\infty} E\left|X_1\right|^p I\left((j-1)^{1/p} < \left|X_1\right| \le j^{1/p}\right) = E\left|X_1\right|^p < \infty.
\end{aligned} \tag{3.11}$$

Thus (3.8) follows from (3.9), (3.10) and (3.11). Consequently $I_1 < \infty$. Since (3.7), to prove $I_2 \to 0$ completely it suffices to prove

$$n^{-1/p}\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}\left(Y'_{ni} - EY'_{ni}\right)\right| \to 0 \text{ completely.} \tag{3.12}$$

In fact, according to the Chebyshev inequality, (3.3) and assumption

$$\sup_{n\ge1}\sum_{i=1}^{n} a_{ni}^2 < \infty,$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}\left(Y'_{ni}-EY'_{ni}\right)\right|>\varepsilon n^{1/p}\right)\\
&\ll \sum_{n=1}^{\infty} n^{-2/p} E\max_{1\le k\le n}\left(\sum_{i=1}^{k} a_{ni}\left(Y'_{ni}-EY'_{ni}\right)\right)^2\\
&\ll \sum_{n=1}^{\infty} n^{-2/p}\sum_{i=1}^{n} a_{ni}^2 EY'^2_{ni}\\
&\ll \sum_{n=1}^{\infty} n^{-2/p} EX_1^2 I\left(\left|X_1\right|>n^{1/p}\right)\\
&= \sum_{n=1}^{\infty} n^{-2/p}\sum_{k=n}^{\infty} EX_1^2 I\left(k^{1/p}<\left|X_1\right|\le (k+1)^{1/p}\right)\\
&\ll \sum_{k=1}^{\infty} EX_1^2 I\left(k^{1/p}<\left|X_1\right|\le (k+1)^{1/p}\right)\sum_{n=1}^{k} n^{-2/p}\\
&\ll \sum_{k=1}^{\infty} k^{-2/p+1}\cdot (k+1)^{(2-p)/p} E\left|X_1\right|^p I\left(k^{1/p}<\left|X_1\right|\le (k+1)^{1/p}\right)\\
&\ll \sum_{k=1}^{\infty} E\left|X_1\right|^p I\left(k^{1/p}<\left|X_1\right|\le (k+1)^{1/p}\right)\\
&\ll E\left|X_1\right|^p<\infty.
\end{aligned}
\tag{3.13}
$$

From (3.12) and (3.13) we know $I_2 \to 0$ completely. The proof of Theorem 2.1 is complete.

Proof of Theorem 2.2. Without loss of generality we assume $a_{ni} \ge 0$ for all $n \ge 1, i \ge 1$. Let

$$Y_{ni} = -n^{\alpha} I\left(X_i < -n^{\alpha}\right) + X_i I\left(\left|X_i\right| \le n^{\alpha}\right) + n^{\alpha} I\left(X_i > n^{\alpha}\right).$$

By Lemma 2.1 we see that $\{Y_{ni}, i \ge 1\}$ and $\{a_{ni}Y_{ni}, i \ge 1\}$ are AANA for all $n \ge 1$. So

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right|>\varepsilon n^{\alpha}\right)\\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\bigcup_{i=1}^{n}\left\{\left|X_i\right|>n^{\alpha}\right\}\right)+\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}Y_{ni}\right|>\varepsilon n^{\alpha}\right)\\
&=: I_1 + I_2.
\end{aligned}
\tag{3.14}
$$

To prove (2.2) it suffices to prove $I_1 < \infty$ and $I_2 < \infty$.

Since $\alpha p > 1$ we have

$$I_1 = \sum_{n=1}^{\infty} n^{\alpha p-1} P\left(\left|X_1\right| > n^{\alpha}\right) \ll E\left|X_1\right|^p < \infty. \tag{3.15}$$

By the C_r inequality and assumption $\sum_{i=1}^{n} a_{ni}^r = O(n)$, we have $\sum_{i=1}^{n} a_{ni} = O(n)$. Under the condition $E\left|X_1\right|^p < \infty$, we consider two cases $0 < p \le 1$ and $p > 1$ respectively, and we can easily get

$$n^{-\alpha}\left|\sum_{i=1}^{n} a_{ni} EY_{ni}\right| \to 0. \tag{3.16}$$

From (3.16), the Chebyshev inequality and Corollary 2.1 we know

$$
\begin{aligned}
I_2 &\ll \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}\left(Y_{ni}-EY_{ni}\right)\right| > \varepsilon n^{\alpha}/2\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} E\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}\left(Y_{ni}-EY_{ni}\right)\right|^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\left\{\left[1+C\left(\sum_{i=1}^{n} q^{\tilde r}(i)\right)^{r-1}\right]\sum_{i=1}^{n} E\left|a_{ni}Y_{ni}\right|^{r}+\left(\sum_{i=1}^{n} E\left(a_{ni}Y_{ni}\right)^{2}\right)^{r/2}\right\} \\
&=: I_{21}+I_{22}.
\end{aligned} \tag{3.17}
$$

By condition $\sum_{n=1}^{\infty} q^{\tilde r}(n)<\infty$ and the C_r inequality

$$
\begin{aligned}
I_{21} &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\left[1+C\left(\sum_{i=1}^{n} q^{\tilde r}(i)\right)^{r-1}\right]\sum_{i=1}^{n} E\left|a_{ni}Y_{ni}\right|^{r} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\sum_{i=1}^{n} a_{ni}^{r}\left\{E\left|X_1\right|^{r} I\left(\left|X_1\right|\le n^{\alpha}\right)+n^{\alpha r}P\left(\left|X_1\right|>n^{\alpha}\right)\right\} \\
&=: I_{211}+I_{212}.
\end{aligned} \tag{3.18}
$$

No matter $0<p<2$ or $p\ge 2$ we have $r>p$. Using assumption $\sum_{i=1}^{n} a_{ni}^{r}=O(n)$ and the method of (3.11) we get

$$
I_{211}=\sum_{n=1}^{\infty} n^{\alpha(p-r)-2}\sum_{i=1}^{n} a_{ni}^{r} E\left|X_1\right|^{r} I\left(\left|X_1\right|\le n^{\alpha}\right)\ll E\left|X_1\right|^{p}<\infty, \tag{3.19}
$$

and

$$
I_{212}=\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n} a_{ni}^{r} P\left(\left|X_1\right|>n^{\alpha}\right)\ll E\left|X_1\right|^{p}<\infty \ \left(\text{by } (3.15)\right). \tag{3.20}
$$

From (3.18), (3.19) and (3.20) we know $I_{21}<\infty$.

$$
\begin{aligned}
I_{22} &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\left(\sum_{i=1}^{n} E\left(a_{ni}Y_{ni}\right)^{2}\right)^{r/2} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\left\{\sum_{i=1}^{n} a_{ni}^{2}\left[EX_1^{2} I\left(\left|X_1\right|\le n^{\alpha}\right)+n^{2\alpha}P\left(\left|X_1\right|>n^{\alpha}\right)\right]\right\}^{r/2} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\left\{\left(\sum_{i=1}^{n} a_{ni}^{2} EX_1^{2} I\left(\left|X_1\right|\le n^{\alpha}\right)\right)^{r/2}+\left(\sum_{i=1}^{n} a_{ni}^{2} n^{2\alpha}P\left(\left|X_1\right|>n^{\alpha}\right)\right)^{r/2}\right\} \\
&=: I_{221}+I_{222}.
\end{aligned} \tag{3.21}
$$

Since $\sum_{i=1}^{n} a_{ni}^{r}=O(n), r\ge 2$ and the C_r inequality, we know $\sum_{i=1}^{n} a_{ni}^{2}\ll n$. Therefore

$$
\begin{aligned}
I_{221} &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\left(\sum_{i=1}^{n} a_{ni}^{2} EX_1^{2} I\left(\left|X_1\right|\right)\le n^{\alpha}\right)^{r/2} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha r+r/2-2}\left(EX_1^{2} I\left(\left|X_1\right|\right)\le n^{\alpha}\right)^{r/2}.
\end{aligned} \tag{3.22}
$$

We consider the following two cases:

1) If $0<p<2$, then $r=2$, taking the method of (3.11) we have

$$
I_{221}\ll \sum_{n=1}^{\infty} n^{\alpha p-2\alpha-1} EX_1^{2} I\left(\left|X_1\right|\le n^{\alpha}\right)\ll E\left|X_1\right|^{p}<\infty. \tag{3.23}
$$

and

$$
\begin{aligned}
I_{222} &= \sum_{n=1}^{\infty} n^{\alpha p-2\alpha-2} \sum_{i=1}^{n} a_{ni}^2 n^{2\alpha} P\left(\left|X_1\right| > n^{\alpha}\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1} P\left(\left|X_1\right| > n^{\alpha}\right) \\
&\ll E\left|X_1\right|^p < \infty \ \left(\text{by } (3.15)\right).
\end{aligned} \tag{3.24}
$$

2) If $p \geq 2$, then $r > (\alpha p - 1)/(\alpha - 1/2) > p$. From $EX_1^2 I\left(\left|X_1\right| \leq n^{\alpha}\right) < \infty$ and (3.22) we get

$$
I_{221} = \sum_{n=1}^{\infty} n^{(\alpha p-1)-(\alpha-1/2)r-1} < \infty, \tag{3.25}
$$

and

$$
\begin{aligned}
I_{222} &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} \left(\sum_{i=1}^{n} a_{ni}^2 n^{2\alpha} P\left(\left|X_1\right| > n^{\alpha}\right) \right)^{r/2} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha pr/2+r/2-2} \left(E\left|X_1\right|^p \right)^{r/2} \\
&\ll \sum_{n=1}^{\infty} n^{(\alpha p-1)(1-r/2)-1} < \infty \ \left(\text{since } \alpha p > 1, r > 2\right).
\end{aligned} \tag{3.26}
$$

Thus $I_{221} < \infty$ follows from (3.23) and (3.26), $I_{221} < \infty$ from (3.24) and (3.26). So $I_{22} < \infty$ by (3.21). Hence $I_2 < \infty$ by (3.17). (2.2) is proved.

As for (2.3), inspired by Gut [7] (page 318_319), we have

$$
\begin{aligned}
& \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\sup_{k \geq n} \left| k^{-\alpha} \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon \right) \\
&= \sum_{j=1}^{\infty} \sum_{n=2^{j-1}}^{2^j-1} n^{\alpha p-2} P\left(\sup_{k \geq n} \left| k^{-\alpha} \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon \right) \\
&\ll \sum_{j=1}^{\infty} P\left(\sup_{k \geq 2^{j-1}} \left| k^{-\alpha} \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon \right) \sum_{n=2^{j-1}}^{2^j-1} 2^{j(\alpha p-2)} \\
&\ll \sum_{j=1}^{\infty} 2^{j(\alpha p-1)} P\left(\sup_{l \geq j} \max_{2^{l-1} \leq k < 2^l} \left| k^{-\alpha} \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon \right) \\
&\ll \sum_{j=1}^{\infty} 2^{j(\alpha p-1)} \sum_{l=j}^{\infty} P\left(\max_{1 \leq k < 2^l} \left| \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon 2^{(l-1)\alpha} \right). \\
&= \sum_{l=1}^{\infty} P\left(\max_{1 \leq k < 2^l} \left| \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon 2^{(l-1)\alpha} \right) \sum_{j=1}^{l} 2^{j(\alpha p-1)} \\
&\ll \sum_{l=1}^{\infty} 2^{l(\alpha p-1)} P\left(\max_{1 \leq k < 2^l} \left| \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon 2^{(l-1)\alpha} \right) \\
&\ll \sum_{l=1}^{\infty} \sum_{n=2^{l-1}}^{2^l-1} 2^{l(\alpha p-2)} P\left(\max_{1 \leq k < 2^l} \left| \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon 2^{(l-1)\alpha} \right) \\
&\ll \sum_{l=1}^{\infty} \sum_{n=2^{l-1}}^{2^l-1} n^{\alpha p-2} P\left(\max_{1 \leq k < 2^l} \left| \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon' n^{\alpha} \right) \quad \left(\text{where } \varepsilon' = 2^{-\alpha}\varepsilon\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon' n^{\alpha} \right).
\end{aligned} \tag{3.27}
$$

Thus (2.3) follows from (2.2) and (3.27). The proof of Theorem 2.2 is completed.

Fund

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