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On Finding Geodesic Equation of Normal Distribution and Gaussian Curvature

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Abstract

In this paper, we apply two different algorithms to find the geodesic equation of the normal distribution. The first algorithm consists of solving a triply partial differential equation where these equations originated from the normal distribution. While the second algorithm applies the well-known Darboux Theory. These two algorithms draw the same geodesic equation. Finally, we applied Baltzer R.'s finding to compute the Gaussian Curvature.

Keywords

Darboux Theory, Differential Geometry, Geodesic Equation, Partial Differential Equation, Normal Distribution

1. Introduction

The importance of the normal distribution has changed throughout history. Earlier authors only referenced this distribution as a convenient approximation to the binomial distribution. Laplace and Gauss helped spread the theoretical importance of the distribution at the beginning of the nineteenth century. The normal theory became widely accepted as the basis of statistical work, especially in astronomy. The beginning of the twentieth century led to another major development in the systems of non-normal frequency curves. In both theory and practice, the normal distribution has a unique position in probability theory, and it can be used as an approximation for other distributions. In practice, the normal theory can be applied, with small risk of serious error, when a substantially non-normal distribution corresponds more closely to observed data values. This allows us to take advantage of the elegant nature and extensive supporting numerical tables of the normal theory. A more detailed historical development of the normal theory can be found in Kendall, M. and Stuart, A. [1] or Johnson, N.L., Kotz, S. and Balakishnan, N. [2]. Chen, W. W. S. [3] [4] recently found the geodesic equations of gamma and the logistic distributions which is similar to the content in the current paper. In 1997, Kass, R.E. and Vos, P. W. [5] provided the book that covers the differential geometrical course related to statistics exponential family. However, in this paper, we focus on the development of the curved distance of the normal theory and apply two different algorithms to find the shortest distance between two points on a curved surface that has a non-zero Gaussian Curvature.

2. List the Fundamental Tensor

The probability density function for the normal distribution is given by

$$g(x,u,v)=\frac{1}{\sqrt{2\pi v^2}}\exp\left(-\frac{(x-u)^2}{2v^2}\right),\qquad 0\le x\le\infty$$
$$\ln g(x)=-\frac{1}{2}\left(\ln 2\pi v^2\right)-\frac{(x-u)^2}{2v^2} \tag{2.1}$$

where u is the location parameter, and v is the scale parameter. Then it is simple to derive the following second order derivative:

$$\frac{\partial^2 \ln g(x)}{\partial u^2}=\frac{-1}{v^2};\qquad \frac{\partial^2 \ln g(x)}{\partial v\partial u}=\frac{-2(x-u)}{v^3};$$
$$\frac{\partial^2 \ln g(x)}{\partial v^2}=\frac{1}{v^2}-\frac{3(x-u)^2}{v^4}. \tag{2.2}$$

From the above Equation (2.2), we can define the metric tensor components for the normal distribution as follows:

$$E=-E\left(\frac{\partial^2 \ln g(x)}{\partial u^2}\right)=\frac{1}{v^2},\qquad F=-E\left(\frac{\partial^2 \ln g(x)}{\partial v\partial u}\right)=0,$$
$$G=-E\left(\frac{\partial^2 \ln g(x)}{\partial v^2}\right)=\frac{2}{v^2} \tag{2.3}$$

where E, F, and G are usually called the coefficient of the first fundamental forms. Using the above results (2.3), we can easily derive the following results:

$$E_u=0,\quad E_v=\frac{-2}{v^3},\quad G_u=0,\quad G_v=\frac{-4}{v^3},\ F_u=0,\ F_v=0,\ F_{uv}=0,$$
$$EG=\frac{2}{v^4},\quad \sqrt{EG}=\frac{\sqrt{2}}{v^2};\quad \frac{1}{\sqrt{EG}}=\frac{v^2}{\sqrt{2}}, \tag{2.4}$$

$$\Gamma_{11}^1=\frac{E_u}{2E}=0,\qquad \Gamma_{12}^2=\frac{G_u}{2G}=0,\qquad \Gamma_{11}^2=\frac{-E_v}{2G}=\frac{1}{2v},$$
$$\Gamma_{22}^1=\frac{-G_u}{2E}=0,\quad \Gamma_{12}^1=\frac{E_v}{2E}=\frac{-1}{v},\quad \Gamma_{22}^2=\frac{G_v}{2G}=\frac{-1}{v} \tag{2.5}$$

3. The Geodesic Equation

To find the geodesic equation of the normal distribution we must solve a triply of partial differential equations, which is provided in the **Appendix I**. We will seek its solution in this section.

$$\mathrm{d}s^2 = \frac{1}{v^2}\mathrm{d}u^2 + \frac{2}{v^2}\mathrm{d}v^2 \tag{3.1}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} - \frac{2}{v}\frac{\mathrm{d}u\mathrm{d}v}{\mathrm{d}s\mathrm{d}s} = 0, \tag{3.2}$$

$$\frac{\mathrm{d}^2 v}{\mathrm{d}s^2} + \frac{1}{2v}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2 - \frac{1}{v}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)^2 = 0, \tag{3.3}$$

The Equation (3.1) is a well-known distance function. It will only need two out of above three equations to find normal distribution geodesic equation. We will choose the first and second equations, *i.e.* (3.1) and (3.2). To simplify the notation, from (3.2) we let

$$p = \frac{\mathrm{d}u}{\mathrm{d}s}, \quad \text{then } \frac{\mathrm{d}p}{\mathrm{d}s} - \frac{2}{v}p\frac{\mathrm{d}v}{\mathrm{d}s} = 0 \tag{3.4}$$

Then divided the above Equation (3.4) by p.

$$\frac{\frac{\mathrm{d}p}{\mathrm{d}s}}{p} - \frac{2}{v}\frac{\mathrm{d}v}{\mathrm{d}s} = 0 \tag{3.5}$$

Integration on both sides of (3.5) with respect to p, we get

$$\ln p - 2\ln v = C_1$$

$$\ln pv^{-2} = C_1 \quad \text{or} \quad pv^{-2} = \mathrm{e}^{C_1} = A \tag{3.6}$$

$$\frac{\mathrm{d}u}{\mathrm{d}s} = Av^2, \quad \mathrm{d}s^2 = \frac{\mathrm{d}u^2}{A^2v^4} \tag{3.7}$$

Substitute (3.7) into (3.1)

$$\frac{\mathrm{d}u^2}{A^2v^4} = \frac{1}{v^2}\mathrm{d}u^2 + \frac{2}{v^2}\mathrm{d}v^2$$

$$\mathrm{d}u^2 = A^2v^2\left(\mathrm{d}u^2 + 2\mathrm{d}v^2\right)$$

$$\left(1 - A^2v^2\right)\mathrm{d}u^2 = 2A^2v^2\mathrm{d}v^2$$

$$\pm\mathrm{d}u = \frac{\pm\sqrt{2}Av\mathrm{d}v}{\sqrt{1 - A^2v^2}} \tag{3.8}$$

Integrate both side of (3.8) to get

$$\pm u = \pm\int\frac{\sqrt{2}Av\mathrm{d}v}{\sqrt{1 - A^2v^2}} + B$$

$$\pm u \pm \int\frac{\sqrt{2}Av\mathrm{d}v}{\sqrt{1 - A^2v^2}} = B \tag{3.9}$$

where A and B are arbitrary constants.

Alternatively, we can find the geodesic equation of the normal distribution by solving one partial differential equation. This idea originated from Darboux's [6] theory. In Section 2, Equation (2.3) we know that the coefficient of the first fundamental form is given as,

$$E=\frac{1}{v^2},\ \ F=0,\ \ G=\frac{2}{v^2}\ \text{ or }\ EG-F^2=\frac{2}{v^4}.$$

The equation $\nabla\theta=1$; $\dfrac{E\theta_v^2-2F\theta_u\theta_v+G\theta_u^2}{EG-F}=1$ became

$$\frac{1}{v^2}\left(\theta_v^2+2\theta_u^2\right)=\frac{2}{v^4} \tag{3.10}$$

To solve the above partial differential Equation (3.10), we use the separable variable method as follows:

$$\theta_v^2+2\theta_u^2=\frac{2}{v^2}\ \text{ or }\ 2\theta_u^2=\frac{2}{v^2}\left(1-\frac{v^2}{2}\theta_v^2\right)$$

$$\theta_u^2=\frac{1}{v^2}\left(1-\frac{v^2}{2}\theta_v^2\right)=A^2 \tag{3.11}$$

or

$$\theta_u=\pm A\ \text{ and }\ \theta=\pm Au \tag{3.12}$$

We also use Equation (3.11)

$$\frac{1}{v^2}\left(1-\frac{v^2}{2}\theta_v^2\right)=A^2$$

$$\left(\frac{\partial\theta}{\partial v}\right)^2=\frac{2}{v^2}\left(1-A^2v^2\right)$$

$$\theta=\pm\int\frac{\sqrt{2}\sqrt{1-A^2v^2}}{v}\mathrm{d}v \tag{3.13}$$

Combining solution (3.12) and (3.13), we finally find the general solution of normal distribution θ as follows

$$\theta=\pm Au\pm\int\frac{\sqrt{2}\sqrt{1-A^2v^2}}{v}\mathrm{d}v$$

Thus, by applying the Darboux Theory, we can find the geodesic equation of the normal distribution by taking a partial derivative with respect to A and equal to constant B, *i.e.* $\dfrac{\partial\theta}{\partial A}=B$.

$$\pm u\pm\int\frac{\sqrt{2}Av\mathrm{d}v}{\sqrt{1-A^2v^2}}=B \tag{3.14}$$

This solution (3.14) coincides with the result of the previous (3.9)

4. Computing the Gaussian Curvature

From **Appendix II**, we use Baltzer, R's formula, to compute the first part of the determinant.

$$\begin{vmatrix} -\frac{1}{2}E_{vv}+F_{uv}-\frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u-\frac{1}{2}E_v \\ F_v-\frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} = \begin{vmatrix} -\frac{3}{v^4} & 0 & \frac{1}{v^3} \\ 0 & \frac{1}{v^2} & 0 \\ \frac{-2}{v^3} & 0 & \frac{2}{v^2} \end{vmatrix} = \frac{-6}{v^8}+\frac{2}{v^8}=\frac{-4}{v^8} \quad (4.1)$$

Here is the second part of the determinant:

$$\begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix} = \begin{vmatrix} 0 & \frac{-1}{v^3} & 0 \\ \frac{-1}{v^3} & \frac{1}{v^2} & 0 \\ 0 & 0 & \frac{2}{v^2} \end{vmatrix} = \frac{-2}{v^8} \quad (4.2)$$

Combine (4.1) and (4.2) to calculate the Gaussian Curvature of the normal distribution as below.

$$K=\left[\frac{v^4}{2}\right]^2\left\{\left(\frac{-4}{v^8}\right)-\left(\frac{-2}{v^8}\right)\right\}=\left(\frac{v^8}{4}\right)\left(\frac{-2}{v^8}\right)=\frac{-1}{2}$$

5. Concluding Remarks

In **Appendix II**, we defined the Gaussian Curvature, $K=\kappa_1\kappa_2$, as the product of two extreme values. If $K>0$, then we call the point as an elliptic point, and $K<0$, we say it is a hyperbolic point, and $K=0$, a parabolic point. In R^3, the plane and the cylinder are standard examples for surfaces with a constant; $K=0$. The cylinder can be unwound to a plane without changing the distances locally. The sphere is the standard example for a surface with a constant curvature. Their tangent planes never cut the surface. Hyperbolic curvature can be seen on parts of the torus, like the tube of a bicycle. The inner side, facing the spoke, shows hyperbolic curvature. The outer side, facing the street, is elliptically curved. In the neighborhood of hyperbolic points, tangent planes always cut the surfaces. We have shown that the Gaussian Curvature of a normal distribution is −0.5 and the surface of the upper real half-plane of all (u, v)-points with $v>0$ is identified with the family of all normal distributions. Finally, we want to use a real life example to demonstrate that the geodesic distance is preferable to the Euclidean distance. Suppose we stock $y\sim N\left(\mu,\sigma_0^2\right)$ with the unknown expected yield μ and known risk σ_0^2. We wish to test the hypothesis that $H_0:\mu=\mu_0$ versus $H_a:\mu\neq\mu_0$, with a sample of size of one. The optimal test in this situation with critical region is $C=\left(\left\langle\bar{x}/\left|\bar{x}-\mu_0\right|\right\rangle\delta_{1-\alpha/2}\sigma_0\right)$. The question becomes, "is the distance between $N\left(\mu_0,\sigma_0^2\right)$ and $N\left(\bar{x},\sigma_0^2\right)$ big enough for us to reject H_0?" The answer will depend on the σ^2. For $\sigma^2\to\infty$, the distance between $N\left(\mu_0,\sigma_0^2\right)$ and $N\left(\bar{x},\sigma_0^2\right)$ should converge to zero, for $\sigma^2\to 0$ it should become infinitely large. For this reason, the family of normal distribution should not be identified with a flat but with a curved surface. This demonstrates that the geodesic equation should be used instead of the Euclidean distance function.

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Appendix I

We list the six well known Christoffel Symbols as follows. For a detailed derivation see Struik [7] or Grey [8].

$$\Gamma_{11}^{1}=\frac{GE_u-2FF_u+FE_v}{2\left(EG-F^2\right)},\quad \Gamma_{12}^{2}=\frac{EG_u-FE_v}{2\left(EG-F^2\right)}$$

$$\Gamma_{11}^{2}=\frac{2EF_u-EE_v-FE_u}{2\left(EG-F^2\right)},\quad \Gamma_{22}^{1}=\frac{2GF_v-GG_u-FG_v}{2\left(EG-F^2\right)}$$

$$\Gamma_{12}^{1}=\frac{GE_v-FG_u}{2\left(EG-F^2\right)},\quad \Gamma_{22}^{2}=\frac{EG_v-2FF_v+FG_u}{2\left(EG-F^2\right)}$$

In general, the solution of the geodesic equation depends upon a pair of partial differential equations as below.

$$\frac{\mathrm{d}^2u}{\mathrm{d}s^2}+\Gamma_{11}^{1}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2+2\Gamma_{12}^{1}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\frac{\mathrm{d}v}{\mathrm{d}s}\right)+\Gamma_{22}^{1}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)^2=0$$

$$\frac{\mathrm{d}^2v}{\mathrm{d}s^2}+\Gamma_{11}^{2}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2+2\Gamma_{12}^{2}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\frac{\mathrm{d}v}{\mathrm{d}s}\right)+\Gamma_{22}^{2}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)^2=0$$

Appendix II

In 1886, R. Baltzer used algebra to prove Gauss' findings. Here are the results of Baltzer's findings:

$$K=\kappa_1\kappa_2=\frac{eg-f^2}{EG-F^2}=\frac{1}{\left(EG-F^2\right)^2}$$
$$\times\left\{\begin{vmatrix}-\frac{1}{2}E_{vv}+F_{uv}-\frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u-\frac{1}{2}E_v\\ F_v-\frac{1}{2}G_u & E & F\\ \frac{1}{2}G_v & F & G\end{vmatrix}-\begin{vmatrix}0 & \frac{1}{2}E_v & \frac{1}{2}G_u\\ \frac{1}{2}E_v & E & F\\ \frac{1}{2}G_u & F & G\end{vmatrix}\right\}$$

where all the symbols E, F, G and their first, second derivatives adopted from geometry text like references [7] [8]. The above form of the Gaussian Curvature has the coefficient of the second fundamental form then standardized by the coefficient of the first fundamental form.