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A Study of Weighted Polynomial Approximations with Several Variables (I)

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Abstract

In this paper, we investigate the weighted polynomial approximations with several variables. Our study relates to the approximation for $Wf \in L^p(\mathbb{R}^s)$ by weighted polynomials. Then we will estimate the degree of approximation.

Keywords

Weighted Polynomial Approximations with Several Variables, the Degree of Approximations

1. Introduction

Let $\mathbb{R}^s := \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (s times, $s \geq 1$ integer) be the direct product space, and let $W(x_1, x_2, \cdots, x_s) := w_1(x_1) w_2(x_2) \cdots w_s(x_s)$, where $w_i(x_i) \geq 0$ are even weight functions. We suppose that for every nonnegative integer n,

$$\int_0^\infty x^n w_i(x) \mathrm{d}x < \infty, \ n = 0, 1, 2, \cdots, \ i = 1, 2, \cdots, s.$$

In this paper, we will study to approximate the real-valued weighted function $(Wf)(x_1, x_2, \cdots, x_s)$ by weighted polynomials $(WP)(x_1, x_2, \cdots, x_s)$, where $P(x_1, x_2, \cdots, x_s) \in \mathcal{P}_{n,n,\cdots,n}(\mathbb{R}^s)$. Here, $\mathcal{P}_{n,n,\cdots,n}(\mathbb{R}^s) (=: \mathcal{P}_{n;s}(\mathbb{R}^s))$ means a class of all polynomials with at most n-degree for each variable $x_i, i = 1, 2, \cdots, s$. We need to define the norms. Let $0 < p \leq \infty$, and let $f : \mathbb{R}^s \to \mathbb{R}$ be measurable. Then we define

$$\|Wf\|_{L^p(\mathbb{R}^s)} := \begin{cases} \left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| (Wf)(x_1, \cdots, x_s) \right|^p \mathrm{d}x_1 \cdots \mathrm{d}x_s \right]^{1/p}, & \text{if } 0 < p < \infty; \\ \sup_{(x_1, \cdots, x_s) \in \mathbb{R}^s} \left| (Wf)(x_1, \cdots, x_s) \right|, & \text{if } p = \infty. \end{cases}$$

We assume that for $0 < p \le \infty$ the integral is independent of the order of integration with respect to each $x_i, i = 1, 2, \cdots, s$. When $\|Wf\|_{L^p(\mathbb{R}^s)} < \infty$, we write $Wf \in L^p(\mathbb{R}^s)$. If $p = \infty$, we require that f is continuous and $\lim_{|X|\to\infty}(Wf)(X) = 0$, where $|X| = |(x_1, \cdots, x_s)| = \max|x_i|; i = 1, 2, \cdots, s$.

Our purpose in this paper is to approximate the weighted function $Wf \in L^p(\mathbb{R}^s)$ by weighted polynomials $WP; P \in \mathcal{P}_{n;s}(\mathbb{R}^s)$. The paper is arranged as the following. In Section 2, we give the definition of the weights which are treated in this paper. In Section 3, we consider the approximation for the functions in $L^p(\mathbb{R}^s)$. In Section 4, we consider a property of higher order derivatives. In Section 5, we estimate the degree of approximations. In Section 6, we consider the approximation for the functions with bounded variation. In Section 7, we consider the approximation of the Lipschitz-type functions. In Section 8, we treat the functions with higher order derivatives.

2. Class of Weight Functions and Preliminaries

Throughout the paper $C, C_1, C_2, \cdots$ denote positive constants independent of n, x, t or polynomials $P(x)$. The same symbol does not necessarily denote the same constant in different occurrences. Let $f(x) \sim g(x)$ mean that there exists a constant $C > 0$ such that $C^{-1}f(x) \le g(x) \le Cf(x)$ holds for all $x \in I$, where $I \subset \mathbb{R}$ is a subset.

We say that $f : \mathbb{R} \to [0, \infty)$ is quasi-increasing if there exists $C > 0$ such that $f(x) \le Cf(y)$ for $0 < x < y$. Hereafter we consider following weights.

Definition 2.1. Let $Q : \mathbb{R} \to [0, \infty)$ be a continuous and even function, and satisfy the following properties:

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.

(b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.

(c) $\lim_{x\to\infty} Q(x) = \infty$.

(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \; x \ne 0$$

is quasi-increasing in $(0, \infty)$, with

$$T(x) \ge \Lambda > 1, \; x \in \mathbb{R} \setminus \{0\}.$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \le C_1 \frac{|Q'(x)|}{Q(x)}, \; a.e.\, x \in \mathbb{R}.$$

Then we write $w = \exp(-Q) \in \mathcal{F}(C^2)$.

Moreover, if there also exists a compact subinterval $J(\ni 0)$ of $\mathbb{R}$, and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \ge C_2 \frac{|Q'(x)|}{Q(x)}, \; a.e.\, x \in \mathbb{R} \setminus J,$$

then we write $w=\exp(-Q)\in\mathcal{F}(C^2+)$. If $T(x)$ is bounded, then the weight $w=\exp(-Q)\in\mathcal{F}(C^2+)$ is called a Freud-type weight, and if $T(x)$ is unbounded, then w is called an Erdös-type weight.

For $w(x)=\exp(-Q(x))\in\mathcal{F}(C^2+), Q\in C^3(\mathbb{R}\setminus\{0\})$, if there exists $K>0$ such that for $|x|\geq K$,

$$\left|\frac{Q'''(x)}{Q''(x)}\right|\leq C\left|\frac{Q''(x)}{Q'(x)}\right|, \tag{2.1}$$

and there exist $\lambda, C>0$ such that for $0<\lambda<\frac{3}{2}$,

$$\frac{|Q'(x)|}{Q(x)^{\lambda}}\leq C, \tag{2.2}$$

then we write $w\in\mathcal{F}_\lambda(C^3+)$. Furthermore, if

$$\left|\frac{Q^{(4)}(x)}{Q^{(3)}(x)}\right|\leq C\left|\frac{Q'''(x)}{Q''(x)}\right|\sim\left|\frac{Q''(x)}{Q'(x)}\right| \tag{2.3}$$

and the inequality (2.2) with $0<\lambda<\frac{4}{3}$ hold, then we write $w\in\mathcal{F}_\lambda(C^4+)$.

We have some examples satisfying Definition 2.1.

Example 2.2 (cf. [1] [2]). (1) If an exponential $Q(x)$ satisfies

$$1<\Lambda_1\leq\frac{(xQ'(x))'}{Q'(x)}\leq\Lambda_2,$$

where $\Lambda_i, i=1,2$ are constants, then we call $w=\exp(-Q(x))$ the Freud weight. The class $\mathcal{F}(C^2+)$ contains the Freud weights.

(2) For $\alpha>1, l\geq 1$ we define

$$Q(x)=Q_{l;\alpha}(x)=\exp_l(|x|^\alpha)-\exp_l(0),$$

where $\exp_l(x)=\exp(\exp(\exp\cdots\exp x)\cdots)$ (l times). Moreover, we define

$$Q_{l;\alpha,m}(x)=|x|^m\left\{\exp_l(|x|^\alpha)-\alpha^*\exp_l(0)\right\},\ \alpha+m>1, m\geq 0, \alpha\geq 0,$$

where $\alpha^*=0$ if $\alpha=0$, and otherwise $\alpha^*=1$. We note that $Q_{l;0,m}$ gives a Freud-type weight, that is, $T(x)$ is bounded..

(3) We define

$$Q_\alpha(x)=(1+|x|)^{|x|^\alpha}-1,\ \alpha>1.$$

(4) Let $w=\exp(-Q)\in\mathcal{F}(C^2+)$, and let us define

$$\mu_+:=\limsup_{x\to\infty}\frac{Q''(x)}{Q'(x)}\bigg/\frac{Q'(x)}{Q(x)},\ \mu_-:=\liminf_{x\to\infty}\frac{Q''(x)}{Q'(x)}\bigg/\frac{Q'(x)}{Q(x)}.$$

If $\mu_+=\mu_-$, then we say that the weight w is regular. All weights in examples (1), (2) and (3) are regular.

(5) More generally we can give the examples of weights $w \in \mathcal{F}_\lambda\left(C^3+\right)$. If the weight w is regular and if $Q \in \mathbf{C}^3\left(\mathbb{R}\setminus\{0\}\right)$ satisfies (2.1), then for the regular weights we have $w \in \mathcal{F}_\lambda\left(C^3+\right)$ (see [3], Corollary 5.5 (5.8)).

The following fact is very important for our study.

Proposition 2.3 ([3], Theorem 4.1 and (4.11)). Let $0<\lambda<3/2$ and $\alpha \in \mathbb{R}$. Then for $w=\exp(-Q) \in \mathcal{F}_\lambda\left(C^3+\right)$, we can construct a new weight $w_\alpha \in \mathcal{F}\left(C^2+\right)$ such that

$$T_w(x)^\alpha w(x) \sim w_\alpha(x) \text{ on } \mathbb{R},$$

and for some $C \geq 1$,

$$a_{n/C}\left(w_\alpha\right) \leq a_n(w) \leq a_{Cn}\left(w_\alpha\right) \text{ and } T_{w_\alpha}(x) \sim T_w(x)=T(x),$$

where $a_n\left(w_\alpha\right)$ and $a_n(w)$ are MRS-numbers for the weight w_α and w, respectively, and T_{w_α} and T_w are correspond for w_α or w, respectively.

Let $\{p_n\}$ be orthonormal polynomials with respect to a weight *w*, that is, p_n is the polynomial of degree *n* such that

$$\int_{-\infty}^{\infty} p_n(x) p_m(x) w^2(x) \mathrm{d}x=\delta_{mn} \quad (\text{the Kronecker delta}).$$

For $1 \leq p \leq \infty$, we denote by $L^p(\mathbb{R})$ the usual L^p space on $\mathbb{R}$ (here for $p=\infty$, if $wf \in L^\infty(\mathbb{R})$, then we require f to be continuous, and wf to have limit 0 at $\pm\infty$). For $wf \in L^p(\mathbb{R})$, we set

$$\begin{gathered} s_n(f, x):=\sum_{k=0}^{n-1} b_k(f) p_k(x), \\ \text{where } b_k(f)=\int_{-\infty}^{\infty} f(t) p_k(t) w^2(t) \mathrm{d}t \end{gathered} \tag{2.4}$$

for $n \in \mathbb{N}$ (the partial sum of Fourier-type series). The de la Vallée Poussin mean of order *n* is defined by

$$v_n(f, x):=\sum_{j=n+1}^{2n} s_j(f, x). \tag{2.5}$$

Let $w \in \mathcal{F}\left(C^2+\right)$. We need the Mhaskar-Rakhmanov-Saff numbers (MRS-numbers) a_x;

$$x=\frac{2}{\pi} \int_0^1 \frac{a_x u Q'\left(a_x u\right)}{\left(1-u^2\right)^{1/2}} \mathrm{d}u, x>0.$$

We easily see

$$\lim_{x \rightarrow \infty} a_x=\infty \text{ and } \lim_{x \rightarrow +0} a_x=0$$

and

$$\lim_{x \rightarrow \infty} \frac{a_x}{x}=0 \text{ and } \lim_{x \rightarrow +0} \frac{a_x}{x}=\infty.$$

For $wf \in L_p(\mathbb{R})(1 \leq p \leq \infty)$, the degree of weighted polynomial approximation is defined by

$$E_{n,p}(w;f) := \inf_{P\in\mathcal{P}_n} \left\| w(f-P) \right\|_{L^p(\mathbb{R})}.$$

3. Approximations for L_p-Functions

In this section, we treat the function such as $Wf \in L^p(\mathbb{R}^s)$, where $1 \le p \le \infty$), and if $p=\infty$, then we suppose that Wf is continuous and $\lim_{|X|\to\infty}(Wf)(X)=0$. For any multivariate point $X=(x_1,\cdots,x_s)\in\mathbb{R}^s$, we consider the weights;

$$W(X) := \prod_{j=1}^{s} w_j(x_j) = \prod_{j=1}^{s} \exp\left(-Q_j(x_j)\right).$$

As shown under, we will also use $X(u) := (u_1,u_2,\cdots,u_s)$. Let $w_i = \exp(-Q_i) \in \mathcal{F}_\lambda(C^3+)$, $0<\lambda<3/2$, $i=1,2,\cdots,s$. From Proposition 2.3 we see $T_i^{1/4}w_i \sim w_{i,1/4} \in \mathcal{F}(C^2+)$, $i=1,2,\cdots,s$. Then we admit to write $T_i^{1/4}w_i \in \mathcal{F}(C^2+)$. For the weight W we construct the modulus of continuity of f. It involves the function

$$\Phi_{t,i}(x_i) := \sqrt{1-\frac{|x_i|}{\sigma_i(t)}} + \frac{1}{\sqrt{T_i(\sigma_i(t))}},\ i=1,2,\cdots,s,$$

where $\sigma_i(t)$ is defined by

$$\sigma_i(t) := \inf\left\{ a_n^{(i)} : \frac{a_n^{(i)}}{n} \le t \right\},\ t>0,$$

where $a_n^{(i)}$ is the MRS-number for the weight $w_i(x)$. If $a_n^{(i)}/n = t$, then we have $\sigma_i(t) = a_n^{(i)}$. In the sequel, if $1\le j\le s$ is an integer, then $\|f\|_{p;j}$ will denote the L^p norm of f taken with respect to the j-th variable. This is a function of the remaining $(s-1)$ variables. For each fixed $\hat{X}_j := (x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_s) \in \mathbb{R}_j^{s-1}$, we write

$$f_{\hat{X}_j}(x) := f(x_1,\cdots,x_{j-1},x,x_{j+1},\cdots,x_s),\ j=1,2,\cdots,s. \tag{3.1}$$

Using

$$\Delta_h f_{\hat{X}_j}(x) := f_{\hat{X}_j}\left(x+\frac{h}{2}\right) - f_{\hat{X}_j}\left(x-\frac{h}{2}\right),$$

we define the modulus of continuity. For the Freud-type weight, we define

$$\begin{aligned}&\bar{\omega}_{p,j}\left(f_{\hat{X}_j}, w_j; t\right)\\ &:= \left(\frac{1}{t}\int_0^t \left\| w_j(x)\left(\Delta_h f_{\hat{X}_j}(x)\right)\right\|_{L^p(|x|\le\sigma_j(2t))}^p \mathrm{d}h\right)^{1/p}\\ &\quad + \inf_{c_j(\text{constant})} \left\| \left(f_{\hat{X}_j}(x) - c_j\right) w_j(x)\right\|_{L^p(|x|\le\sigma_j(4t))}.\end{aligned}$$

If w_j is Erdös-type, then we define

$$\bar{\omega}_{p,j}\left(f_{\hat{X}_j}, w_j; t\right)$$
$$:= \left(\frac{1}{t}\int_0^t \left\| w_j(x)\left(\Delta_{h\Phi_{t,j}(x)} f_{\hat{X}_j}(x)\right)\right\|_{L^p(|x|\le\sigma_j(2t))}^p \mathrm{d}h\right)^{1/p}$$
$$+ \inf_{c_j(\text{constant})} \left\| \left(f_{\hat{X}_j}(x) - c_j\right) w_j(x)\right\|_{L^p(|x|\le\sigma_j(4t))}.$$

We remark that if $T_j(x)$ is bounded, then we see $\Phi_{t,j}(x) \sim 1$, so we do not need the definition for the Freud-type weight.

Let v_n be the de la Vallée Poussin mean opetator, and let $v_{n,j}(f), j = 1,2,\cdots,s$ denote the operation to f with respect to j-th co-ordinate, and $v_n^{[j]}$ will denote the operator v_n applied to f with respect to each of the first j co-ordinates. Clearly,

$$v_n^{[1]}(f) = v_{n,1}(f),\ v_n^{[j]}(f) = v_n^{[j-1]}\left(v_{n,j}(f)\right),\ j = 2,3,\cdots,s. \tag{3.2}$$

Let $a_n^{(j)}$ be the MRS-number for the weight $w_j = \exp(-Q_j)$.

First, we consider the following Proposition.

Proposition 3.1 ([4], Theorem 3.14). For $1 \le p < \infty$, $C_c(\mathbb{R}^s)$ is dence in $L^p(\mathbb{R}^s)$, where $C_c(\mathbb{R}^s)$ is a set of all continuous functions with a compact support on $\mathbb{R}^s$.

From this proposition, for any $\varepsilon > 0$ there exist a constant $K > 0$ and a continuous function f_K with a compact support $[-K,K]^s$ such that

$$\left\| W(X)\left(f(X) - f_K(X)\right)\right\|_{L^p(|X|\le K)} < \varepsilon. \tag{3.3}$$

Then we give the following assumption:

Assumption 3.2. In (3.3) we suppose that for every co-ordinate $x_j, j = 1,2,\cdots,s$

$$\left\| w_j(x)\left(f_{\hat{X}_j}(x) - \left(f_K\right)_{\hat{X}_j}(x)\right)\right\|_{L^p(|x|\le K)} < \varepsilon \tag{3.4}$$

holds.

We define a new class of functions $L_p^*(\mathbb{R}^s), 1 \le p \le \infty$ as follows:

$$L_W^{p*}(\mathbb{R}^s) := \left\{ f \,\middle|\, Wf \in L_p(\mathbb{R}^s) \text{ holds } (3.4)\right\}, \tag{3.5}$$

where if $p = \infty$, then $L_W^{p*}(\mathbb{R}^s) = L_W^p(\mathbb{R}^s)$ and we suppose that f is continuous and

$$\lim_{|X|\to\infty} W(X) f(X) = 0$$

(we write this fact as $Wf \in C_0(\mathbb{R}^s)$). We state the theorem in this section.

Theorem 3.3. (1) We suppose
$w_j = \exp(-Q_j) \in \mathcal{F}_\lambda(C^3+)(0 < \lambda < 3/2)$, $j = 1,2,\cdots,s$, and let

$$T_j\left(a_n^{(j)}\right) \le c\left(\frac{n}{a_n^{(j)}}\right)^{2/3},\ j = 1,2,\cdots,s. \tag{3.6}$$

Let $n \ge 1, 1 \le p \le \infty$. Then we have

$$\left\| W\left(f - v_n^{[s]}(f)\right)\right\|_{L^p(\mathbb{R}^s)} \le \sum_{j=1}^{s} C_j \left\| \left(\prod_{1\le i\le s,\, i\ne j} w_i\right)\left(\prod_{k=1}^{j-1} T_k^{1/4}\right)\bar{\omega}_{p,j}\left(f_{\hat{X}_j}, T_j^{1/4}w_j; c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}, \tag{3.7}$$

where

$$\mathbb{R}_j^{s-1} := \left\{\left(x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_s\right)\right\}, \tag{3.8}$$

and $\prod_{k=1}^{0} T_k^{1/4} = 1$. Especially $f \in L^{p*}_{T^{\langle s\rangle}W}\left(\mathbb{R}^s\right)$, then we have

$$\sum_{j=1}^{s} C_j \left\| \left(\prod_{1\le i\le s,\, i\ne j} w_i\right)\left(\prod_{k=1}^{j-1} T_k^{1/4}\right)\bar{\omega}_{p,j}\left(f_{\hat{X}_j}, T_j^{1/4}w_j; c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)} \to 0 \tag{3.9}$$

as $n \to \infty$.

(2) We suppose $w_j = \exp\left(-Q_j\right) \in \mathcal{F}\left(C^2+\right)$, $j = 1,2,\cdots,s$, and let (3.6) holds. Let $n \ge 1, 1 \le p \le \infty$. Then we have

$$\left\| \frac{W\left(f - v_n^{[s]}(f)\right)}{\prod_{k=1}^{s} T_k^{1/4}}\right\|_{L^p(\mathbb{R}^s)} \le \sum_{j=1}^{s} C_j \left\| \left(\prod_{1\le i\le s,\, i\ne j} w_i\right)\bar{\omega}_{p,j}\left(f_{\hat{X}_j}, w_j; c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}. \tag{3.10}$$

Especially $f \in L^{p*}_{W}\left(\mathbb{R}^s\right)$, then we have

$$\sum_{j=1}^{s} C_j \left\| \left(\prod_{1\le i\le s,\, i\ne j} w_i\right)\bar{\omega}_{p,j}\left(f_{\hat{X}_j}, w_j; c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)} \to 0 \text{ as } n \to \infty. \tag{3.11}$$

First we will show (3.7). We need some preliminaries.

Proposition 3.4 ([5], Theorem 1). Let $1 \le p \le \infty$.

(1) We assume that $w \in \mathcal{F}\left(C^2+\right)$ satisfies $T\left(a_n\right) \le C\left(n/a_n\right)^{2/3}$. Then there exists a constant $C > 0$ such that when $wg \in L_p\left(\mathbb{R}\right)$, then

$$\left\| \frac{wv_n(g)}{T^{1/4}}\right\|_{L^p(\mathbb{R})} \le C\left\|wg\right\|_{L^p(\mathbb{R})},$$

and so,

$$\left\|wv_n(g)\right\|_{L^p(\mathbb{R})} \le CT\left(a_n\right)^{1/4}\left\|wg\right\|_{L^p(\mathbb{R})}.$$

(2) We assume that $w \in \mathcal{F}_\lambda\left(C^3+\right)\left(0 < \lambda < 3/2\right)$ satisfies $T\left(a_n\right) \le C\left(n/a_n\right)^{2/3}$. Then there exists a constant $C > 0$ such that if $T^{1/4}wg \in L^p\left(\mathbb{R}\right)$, then

$$\left\|wv_n(g)\right\|_{L^p(\mathbb{R})} \le C\left\|T^{1/4}wg\right\|_{L^p(\mathbb{R})}.$$

Proposition 3.5 ([5], Corollary 6.2 (6.5)). Let $1 \le p \le \infty$.

(1) Let $w \in \mathcal{F}\left(C^2+\right)$, and $n \ge 1$ be an integer. Then

$$\left\| \frac{w\left(g - v_n(g)\right)}{T^{1/4}}\right\|_{L^p(\mathbb{R})} \le CE_{n,p}\left(w;g\right),$$

where C do not depend on g and n.

(2) Let $w \in \mathcal{F}_\lambda\left(C^3+\right)(0<\lambda<3/2)$, and $n \ge 1$ be an integer. Then

$$\left\|w\left(g-v_n(g)\right)\right\|_{L^p(\mathbb{R})} \le CE_{n,p}\left(T^{1/4}w;g\right),$$

where C do not depend on g and n.

Proposition 3.6. ([6]) Let $w=\exp(-Q)\in \mathcal{F}\left(C^2+\right)$, and let $0<p\le\infty$. Then there exist $n_0 \in \mathbb{N}$ and positive constants C_1, C_2 such that for every $wg \in L^p(\mathbb{R})$ (and for $p=\infty$, we require g to be continuous, and wf to vanish at $\pm\infty$) and every $n \ge n_0$,

$$E_{n,p}(g;w) \le C_1 \bar{\omega}_p\left(g,w;C_2\frac{a_n}{n}\right),$$

where n_0 and C_1, C_2 do not depend on g and n, and $\omega_p^*(g,w,t)$ will be defined in Section 6.

We set

$$T^{\langle j\rangle} := \prod_{i=1}^{j} T_i^{1/4}, \quad \mathbb{R}_{(j)} := \left\{x_j \in \mathbb{R}\right\},$$

$$\mathbb{R}_{\le j}^{j} := \left\{\left(x_1,\cdots,x_j\right)\in\mathbb{R}^j\right\}, \quad \mathbb{R}_{j\le}^{s-j+1} := \left\{\left(x_j,\cdots,x_s\right)\in\mathbb{R}^{s-j+1}\right\},$$

$$\mathbb{R}_j^{s-1} := \left\{\left(x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_s\right)\in\mathbb{R}^{s-1}\right\},$$

$$W := \prod_{i=1}^{s} w_i,\ W_j := \prod_{i=1, i\ne j}^{s} w_i, \quad j=1,2,\cdots,s.$$

We need the infinite-finite inequality.

Theorem 3.7 (Infinite-finite inequality). Let $0<p\le\infty, L>0$, and let $P(X)\in\mathcal{P}_{n,\cdots,n}\left(\mathbb{R}^s\right)\left(=:\mathcal{P}_{n;s}\left(\mathbb{R}^s\right)\right)$. Then

$$\left\|W(X)P(X)\right\|_{L^p\left(\mathbb{R}^s\right)} \le C\left\|W(X)P(X)\right\|_{L^p\left(|x_i|\le a_n^{(i)}\left(1-L\delta_n^{(i)}\right),\, i=1,2,\cdots,s\right)}. \tag{3.12}$$

If $r>1$, then there exists $\varepsilon>0$ such that

$$\left\|W(X)P(X)\right\|_{L^p\left(\mathbb{R}_{i,r}^s\right)} \le C\exp\left(-n^{\varepsilon}\right)\left\|W(X)P(X)\right\|_{L^p\left(\mathbb{R}^s\right)}, \tag{3.13}$$

where $\mathbb{R}_{i,r}^s := \left\{x_i; |x_i| \ge a_{rn}^{(i)}\right\}\times\mathbb{R}_i^{s-1}$.

To prove Theorem 3.7 we use the following proposition with $s=1$.

Proposition 3.8 ([2], Theorem 1.9). Let $0<p\le\infty, L>0$, and let $P(x)\in\mathcal{P}_n(\mathbb{R})$. Then

$$\left\|w(x)P(x)\right\|_{L^p(\mathbb{R})} \le C\left\|w(x)P(x)\right\|_{L^p\left(|x|\le a_n\left(1-L\delta_n\right)\right)}. \tag{3.14}$$

If $r>1$, then there exists $\varepsilon>0$ such that

$$\left\|w(x)P(x)\right\|_{L^p\left(a_{rn}\le|x|\right)} \le C\exp\left(-n^{\varepsilon}\right)\left\|w(x)P(x)\right\|_{L^p\left(|x|\le a_n\right)}. \tag{3.15}$$

Proof of Theorem 3.7. For the proof of (3.12) we use (3.14). We put A for the left side of the above equation. Let $0<p<\infty$. By repeatedly applying Proposition

3.8 (3.14), we have

$$A^p = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|w_2(x_2)\cdots w_s(x_s)\right|^p \times\left\{\int_{-\infty}^{\infty}\left|w_1(x_1)P(x_1,\cdots,x_s)\right|^p \mathrm{d}x_1\right\}\mathrm{d}x_2\cdots\mathrm{d}x_s$$

$$\le C_1\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|w_2(x_2)\cdots w_s(x_s)\right|^p \int_{-a_n^{(1)}\left(1-L\delta_n^{(1)}\right)}^{a_n^{(1)}\left(1-L\delta_n^{(1)}\right)}\left|w_1(x_1)P(x_1,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s$$

$$= C_1\int_{-a_n^{(1)}\left(1-L\delta_n^{(1)}\right)}^{a_n^{(1)}\left(1-L\delta_n^{(1)}\right)} w_1^p(x_1)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|w_2(x_2)\cdots w_s(x_s)P(x_1,\cdots,x_s)\right|^p \mathrm{d}x_2\cdots\mathrm{d}x_s\mathrm{d}x_1$$

$$\le C_2\int_{-a_n^{(1)}\left(1-L\delta_n^{(1)}\right)}^{a_n^{(1)}\left(1-L\delta_n^{(1)}\right)}\int_{-a_n^{(2)}\left(1-L\delta_n^{(2)}\right)}^{a_n^{(2)}\left(1-L\delta_n^{(2)}\right)} w_1^p(x_1)w_2^p(x_2)$$

$$\times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|w_3(x_2)\cdots w_s(x_s)P(x_1,\cdots,x_s)\right|^p \mathrm{d}x_3\cdots\mathrm{d}x_d\mathrm{d}x_1\mathrm{d}x_2$$

$$\le\cdots$$

$$\le C_s\int_{-a_n^{(1)}\left(1-L\delta_n^{(1)}\right)}^{a_n^{(1)}\left(1-L\delta_n^{(1)}\right)}\cdots\int_{-a_n^{(s)}\left(1-L\delta_n^{(s)}\right)}^{a_n^{(s)}\left(1-L\delta_n^{(s)}\right)}\left|w_1(x_1)\cdots w_s(x_s)P(x_1,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s$$

$$= C_s\left\|W(X)P(X)\right\|_{L^p\left(|x_i|\le a_n^{(i)}\left(1-L\delta_n^{(i)}\right),\, i=1,2,\cdots,s\right)}^p.$$

Next, we show the case of $p=\infty$.

$$A=\sup_{x_s\in\mathbb{R}}\cdots\sup_{x_2\in\mathbb{R}}\left|w_2(x_2)\cdots w_s(x_s)\right|\sup_{x_1\in\mathbb{R}}\left|w_1(x_1)P(x_1,\cdots,x_s)\right|$$

$$\le C_1\sup_{x_s\in\mathbb{R}}\cdots\sup_{x_2\in\mathbb{R}}\left|w_2(x_2)\cdots w_s(x_s)\right|\sup_{|x_1|\le a_n^{(1)}\left(1-L\delta_n^{(1)}\right)}\left|w_1(x_1)P(x_1,\cdots,x_s)\right|$$

$$= C_1\sup_{x_s\in\mathbb{R}}\cdots\sup_{x_3\in\mathbb{R}}\left|w_3(x_3)\cdots w_s(x_s)\right|\sup_{|x_1|\le a_n^{(1)}\left(1-L\delta_n^{(1)}\right)}\sup_{x_2\in\mathbb{R}}\left|w_1(x_1)w_2(x_2)P(x_1,\cdots,x_s)\right|$$

$$\le C_1C_2\sup_{x_s\in\mathbb{R}}\cdots\sup_{x_3\in\mathbb{R}}\left|w_3(x_3)\cdots w_s(x_s)\right|$$

$$\times\sup_{|x_1|\le a_n^{(1)}\left(1-L\delta_n^{(1)}\right)}\sup_{|x_2|\le a_n^{(2)}\left(1-L\delta_n^{(2)}\right)}\left|w_1(x_1)w_2(x_2)P(x_1,\cdots,x_s)\right|$$

$$\le\cdots$$

$$\le C_1C_2\cdots C_s\sup_{|x_1|\le a_n^{(1)}\left(1-L\delta_n^{(1)}\right)}\sup_{|x_2|\le a_n^{(2)}\left(1-L\delta_n^{(2)}\right)}\cdots\sup_{|x_2|\le a_n^{(s)}\left(1-L\delta_n^{(s)}\right)}\left|w_1(x_1)w_2(x_2)\right.$$

$$\left.\times\cdots\times w_s(x_s)P(x_1,\cdots,x_s)\right|$$

$$= C\sup_{|x_i|\le a_n^{(i)}\left(1-L\delta_n^{(i)}\right),\, i=1,\cdots,s}\left|w_1(x_1)w_2(x_2)\cdots w_s(x_s)P(x_1,\cdots,x_s)\right|.$$

Similarly, using Proposition 3.8 (3.15), we easily have (3.13). #

Lemma 3.9. Let $1\le p\le\infty$.

(1) We assume that $w_i\in\mathcal{F}\left(C^2+\right), i=1,2,\cdots,s$ satisfies (3.6). Then there exists a constant $C>0$ such that when $Wh\in L_p\left(\mathbb{R}^s\right)$,

$$\left\| \frac{Wv_n^{[j]}(h)}{T^{\langle j \rangle}} \right\|_{L^p(\mathbb{R}^s)} \le C \|Wh\|_{L^p(\mathbb{R}^s)}, \quad j = 1, 2, \cdots, s,$$

and

$$\left\| Wv_n^{[s]}(f) \right\|_{L^p(\mathbb{R}^s)} \le C \prod_{i=1}^{s} T_i^{1/4}\left(a_n^{(i)}\right) \|Wf\|_{L^p(\mathbb{R}^s)}.$$

(2) We assume that $w_i \in \mathcal{F}_\lambda\left(C^3+\right)(0<\lambda<3/2), i=1,2,\cdots,s$. Let $T^{\langle s \rangle}Wh \in L^p\left(\mathbb{R}^s\right)$, then

$$\left\| Wv_n^{[j]}(h) \right\|_{L^p(\mathbb{R}^s)} \le C \left\| T^{\langle j \rangle} Wh \right\|_{L^p(\mathbb{R}^s)}, \quad j = 1, 2, \cdots, s.$$

Proof. (1) From Theorem 3.4 (1), for $j=1$

$$\left\| \frac{Wv_n^{[1]}(h)}{T^{\langle 1 \rangle}} \right\|_{L^p(\mathbb{R}^s)} = \left\| \left\| \frac{Wv_{n,1}\left(h_{\hat{X}_1}\right)}{T_1^{1/4}} \right\|_{L^p\left(\mathbb{R}^1_{\le 1}\right)} \right\|_{L^p\left(\mathbb{R}^{s-1}_{2\le}\right)} \le C \|Wh\|_{L^p(\mathbb{R}^s)}.$$

Inductively,

$$\left\| \frac{Wv_n^{[j]}(h)}{T^{\langle j \rangle}} \right\|_{L^p(\mathbb{R}^s)} = \left\| \left\| \frac{Wv_n^{[j-1]}\left(v_{n,j}(h)\right)}{T^{\langle j \rangle}} \right\|_{L^p\left(\mathbb{R}^{j-1}_{\le j-1}\right)} \right\|_{L^p\left(\mathbb{R}^{s-j+1}_{j\le}\right)}$$

$$\le C \left\| \frac{Wv_{n,j}\left(h_{\hat{X}_j}\right)}{T_j^{1/4}} \right\|_{L^p(\mathbb{R}^s)} \le C \|Wh\|_{L^p(\mathbb{R}^s)}.$$

For the second formula, using Theorem 3.7 and the above inequality, we have

$$\left\| Wv_n^{[j]}(h) \right\|_{L^p(\mathbb{R}^s)}$$

$$\le C \prod_{i=1}^{j} T_i^{1/4}\left(a_n^{(i)}\right) \left\| \left(\prod_{k=j+1}^{s} w_k \right) \left\| \frac{\left(\prod_{i=1}^{j} w_i\right) v_n^{[j]}(h)}{T^{\langle j \rangle}} \right\|_{L^p\left(|x_i| \le a_{2n}^{(i)}, 1\le i \le j\right)} \right\|_{L^p\left(\mathbb{R}^{s-j}_{j+1\le}\right)}$$

$$\le C \prod_{i=1}^{j} T_i^{1/4}\left(a_n^{(i)}\right) \|Wh\|_{L^p(\mathbb{R}^s)},$$

$$j = 1, 2, \cdots, s.$$

Similarly we have the following:

(2) From Theorem 3.4 (2) for $j=1$,

$$\left\| Wv_n^{[1]}(h) \right\|_{L^p(\mathbb{R}^s)} = \left\| Wv_{n,1}(h) \right\|_{L^p(\mathbb{R}^s)} \le C \left\| WT_1^{1/4} h \right\|_{L^p(\mathbb{R}^s)},$$

$$j = 1, 2, \cdots, s.$$

Inductively,

$$\left\| Wv_n^{[j]}(h) \right\|_{L^p(\mathbb{R}^s)} = \left\| Wv_n^{[j-1]}\left(v_{n,j}(h)\right) \right\|_{L^p(\mathbb{R}^s)} \le C \left\| WT^{\langle j-1\rangle} v_{n,j}(h) \right\|_{L^p(\mathbb{R}^s)}$$
$$\le C \left\| WT^{\langle j\rangle} h \right\|_{L^p(\mathbb{R}^s)}. \qquad \#$$

Proof of (3.7) in Theorem 3.3. By Proposition 3.5 (2) and Proposition 3.6, we get

$$\left\| w_1(x)\left(f_{\hat{X}_1} - v_{n,1}\left(f_{\hat{X}_1}\right)\right) \right\|_{L^p(\mathbb{R})} \le CE_{n,p}\left(f_{\hat{X}_1}; T_1^{1/4} w_1 \right) \le C_1 \bar{\omega}_{p,1}\left(f_{\hat{X}_1}, T_1^{1/4} w_1; c_1 \frac{a_n^{(1)}}{n} \right),$$

where the constant C_1 and c_1 is independent of $\hat{X}_1$. Similarly, for $j = 1, 2, \cdots, s$,

$$\left\| w_j(x)\left(f_{\hat{X}_j} - v_{n,j}\left(f_{\hat{X}_j}\right)\right) \right\|_{L^p(\mathbb{R})} \le C_j \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, T_j^{1/4} w_j; c_j \frac{a_n^{(j)}}{n} \right). \tag{3.16}$$

Using $f - v_n^{[s]} = \left(f - v_n^{[1]}(f)\right) + \left(v_n^{[1]}(f) - v_n^{[2]}(f)\right) + \cdots + \left(v_n^{[s-1]}(f) - v_n^{[s]}(f)\right)$, we get from Lemma 3.9 (2) and (3.2),

$$\begin{aligned}
&\left\| W\left(f - v_n^{[s]}(f)\right) \right\|_{L^p(\mathbb{R}^s)} \\
&\le \left\| W\left(f - v_n^{[1]}(f)\right) \right\|_{L^p(\mathbb{R}^s)} + \sum_{j=2}^{s} \left\| W\left(v_n^{[j-1]}(f) - v_n^{[j]}(f)\right) \right\|_{L^p(\mathbb{R}^s)} \\
&\le C\left[\left\| W\left(f - v_{n,1}(f)\right) \right\|_{L^p(\mathbb{R}^s)} + \sum_{j=2}^{s} \left\| Wv_n^{[j-1]}\left(f - v_{n,j}(f)\right) \right\|_{L^p(\mathbb{R}^s)} \right] \\
&\le C\left[\left\| W\left(f - v_{n,1}(f)\right) \right\|_{L^p(\mathbb{R}^s)} + \sum_{j=2}^{s} \left\| W\left(\prod_{k=1}^{j-1} T_k^{1/4}\right)\left(f - v_{n,j}(f)\right) \right\|_{L^p(\mathbb{R}^s)} \right] \\
&\le C_1 \left\| W_1 \bar{\omega}_{p,1}\left(f_{\hat{X}_1}, T_1^{1/4} w_1; c_1 \frac{a_n^{(1)}}{n} \right) \right\|_{L^p\left(\mathbb{R}_1^{s-1}\right)} \\
&\quad + \sum_{j=2}^{s} C_j \left\| W_j \left(\prod_{k=1}^{j-1} T_k^{1/4}\right) \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, T_j^{1/4} w_j; c_j \frac{a_n^{(j)}}{n} \right) \right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}
\end{aligned}$$

by (3.16)

$$\le \sum_{j=1}^{s} C_j \left\| W_j \left(\prod_{k=1}^{j-1} T_k^{1/4}\right) \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, T_j^{1/4} w_j; c_j \frac{a_n^{(j)}}{n} \right) \right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)},$$

where $\prod_{k=1}^{j-1} T_k^{1/4} = 1$ for $j = 1$. Hence we obtain (3.7).

Proof of (3.10) in Theorem 3.3. By Proposition 3.5 (1) and Proposition 3.6, we get

$$\left\| \frac{w_1(x)\left(f_{\hat{X}_1} - v_{n,1}\left(f_{\hat{X}_1}\right)\right)}{T_1^{1/4}(x)} \right\|_{L^p(\mathbb{R})} \le CE_{n,p}\left(f_{\hat{X}_1}; w_1 \right) \le C_1 \bar{\omega}_{p,1}\left(f_{\hat{X}_1}, w_1; c_1 \frac{a_n^{(1)}}{n} \right),$$

where the constant C_1 and c_1 is independent of $\hat{X}_1$. Similarly, for $j = 1, 2, \cdots, s$,

$$\left\| \frac{w_j(x)\left(f_{\hat{X}_j} - v_{n,j}\left(f_{\hat{X}_j}\right)\right)}{T_j^{1/4}(x)} \right\|_{L^p(\mathbb{R})} \le C_j \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, w_j; c_j \frac{a_n^{(j)}}{n}\right). \tag{3.17}$$

Using $f - v_n^{[s]} = \left(f - v_n^{[1]}(f)\right) + \left(v_n^{[1]}(f) - v_n^{[2]}(f)\right) + \cdots + \left(v_n^{[s-1]}(f) - v_n^{[s]}(f)\right)$, we get from Lemma 3.9 (1) and (3.2),

$$\begin{aligned}
&\left\| \frac{W\left(f - v_n^{[s]}(f)\right)}{T^{\langle s \rangle}} \right\|_{L^p(\mathbb{R}^s)} \\
&\le \left\| \frac{W\left(f - v_n^{[1]}(f)\right)}{T^{\langle 1 \rangle}} \right\|_{L^p(\mathbb{R}^s)} + \sum_{j=2}^{s} \left\| \frac{W\left(v_n^{[j-1]}(f) - v_n^{[j]}(f)\right)}{T^{\langle j \rangle}} \right\|_{L^p(\mathbb{R}^s)} \\
&\le C\left[\left\| \frac{W\left(f - v_{n,1}(f)\right)}{T_1^{1/4}} \right\|_{L^p(\mathbb{R}^s)} + \sum_{j=2}^{s} \left\| \frac{W v_n^{[j-1]}\left(\left\{f - v_{n,j}(f)\right\}\right) / T_j^{1/4}}{T^{\langle j-1 \rangle}} \right\|_{L^p(\mathbb{R}^s)} \right] \\
&\le C\left[\left\| \frac{W\left(f - v_{n,1}(f)\right)}{T_1^{1/4}} \right\|_{L^p(\mathbb{R}^s)} + \sum_{j=2}^{s} \left\| \frac{W\left(f - v_{n,j}(f)\right)}{T_j^{1/4}} \right\|_{L^p(\mathbb{R}^s)} \right] \\
&\le C_1 \left\| \left(\prod_{2 \le i \le s} w_i\right) \bar{\omega}_{p,1}\left(f_{\hat{X}_1}, w_1; c_1 \frac{a_n^{(1)}}{n}\right) \right\|_{L^p(\mathbb{R}_1^{s-1})} \\
&\quad + \sum_{j=2}^{s} C_j \left\| \left(\prod_{1 \le i \le s, i \ne j} w_i\right) \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, w_j; c_j \frac{a_n^{(j)}}{n}\right) \right\|_{L^p(\mathbb{R}_j^{s-1})}
\end{aligned}$$

by (3.17)

$$\le \sum_{j=1}^{s} C_j \left\| \left(\prod_{1 \le i \le s, i \ne j} w_i\right) \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, w_j; c_j \frac{a_n^{(j)}}{n}\right) \right\|_{L^p(\mathbb{R}_j^{s-1})}.$$

Hence we obtain (3.10).

To prove (3.9) and (3.11) we need a lemma:

Lemma 3.10. Let $0 < h \le 1$, $\sigma(t) \ge 1$ and $|x| \le \sigma(2t)$. We have

$$\left\| (wf)\left(x \pm h\Phi_t(x)\right) \right\|_{L^p(\mathbb{R})} \sim \left\| (wf)(x) \right\|_{L^p(\mathbb{R})}. \tag{3.18}$$

Proof. We may show

$$\left\| (wf)\left(x \pm h\sqrt{1 - \frac{|x|}{\sigma(t)}}\right) \right\|_{L^p(\mathbb{R})} \sim \left\| (wf)(x) \right\|_{L^p(\mathbb{R})}. \tag{3.19}$$

Let $x > 0$. If we put

$$x \pm h\sqrt{1 - \frac{x}{\sigma(t)}} =: y, \quad \frac{a_{2u}}{2u} = t, \quad \frac{a_v}{v} = 2t, \tag{3.20}$$

we will see

$$\frac{1}{2} \le \frac{\mathrm{d}y}{\mathrm{d}x} \le \frac{3}{2}. \tag{3.21}$$

Then we conclude (3.19). Now, from (3.20)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \mp \frac{h}{2\sqrt{\sigma(t)}\sqrt{\sigma(t)-x}}.$$

Since $a_{2u}/u = 2t$, we see

$$\frac{a_v}{v} = \frac{a_{2u}}{u} > \frac{a_u}{u},$$

that is, we have

$$\sigma(2t) = a_v < a_u < \sigma(t) = a_{2u}.$$

Then, using ([2], Lemmas 3.6, 3.7), we see

$$\frac{h}{2\sqrt{\sigma(t)}\sqrt{\sigma(t)-x}} \le \frac{t}{2\sqrt{\sigma(t)}\sqrt{\sigma(t)-\sigma(2t)}} \le \frac{\sqrt{a_{2u}}}{4u}\frac{1}{\sqrt{a_{2u}-a_u}}$$
$$\le \frac{\sqrt{a_{2u}}}{4u}\sqrt{C_1\frac{T(a_{2u})}{a_{2u}}} \le \frac{1}{4u}\sqrt{C_1C_2}u^{1-\delta} = \frac{\sqrt{C_1C_2}}{4}\frac{1}{u^{\delta}} \le \frac{1}{2}$$

for some $0 < \delta \le 1$ and u large enough. We have (3.21). So we conclude (3.19). #

Proof of (3.9). We will estimate

$$\left\| W_j T^{\langle j-1\rangle} \left\{ \frac{1}{t}\int_0^t \left\| T_j^{1/4}(x) w_j(x) \left(\Delta_{h\Phi_{t,j}(x)} f_{\hat{X}_j}(x) \right) \right\|^p_{L^p(|x|\le\sigma_j(2t))} \mathrm{d}h \right\}^{1/p} \right\|_{L^p(\mathbb{R}_j^{s-1})}.$$

To do so we may estimate

$$I_1 := \left\{ \frac{1}{t}\int_0^t \left\| T_j^{1/4}(x) w_j(x) \left(\Delta_{h\Phi_{t,j}(x)} f_{\hat{X}_j}(x) \right) \right\|^p_{L^p(|x|\le\sigma_j(2t))} \mathrm{d}h \right\}^{1/p}.$$

For $\varepsilon > 0$ we take $K > 0$ large enough, and then by $f \in L^{p*}_{T^{\langle s\rangle}w}(\mathbb{R})$ we can select a continuous function f_K such that

$$I_1 \le \left\{ \frac{1}{t}\int_0^t \left\| T_j^{1/4}(x) w_j(x) \left(\Delta_{h\Phi_{t,j}(x)} (f - f_K)_{\hat{X}_j}(x) \right) \right\|^p_{L^p(|x|\le\sigma_j(2t))} \mathrm{d}h \right\}^{1/p}$$
$$+ \left\{ \frac{1}{t}\int_0^t \left\| T_j^{1/4}(x) w_j(x) \left(\Delta_{h\Phi_{t,j}(x)} (f_K)_{\hat{X}_j}(x) \right) \right\|^p_{L^p(|x|\le\sigma_j(2t))} \mathrm{d}h \right\}^{1/p}$$
$$= A + B.$$

We note $w_j(x) \sim w\left(x + (h/2)\Phi_{t,j}(x)\right) \sim w\left(x - (h/2)\Phi_{t,j}(x)\right)$ (see [7], Lemma 7). If $\sigma_j(2t) \le K$ from our assumption and Lemma 3.10 we have

$$A \le C\left\{\frac{1}{t}\int_0^t \left\| T_j^{1/4}(x) w_j(x)\left(f - f_K\right)_{\hat{X}_j}(x)\right\|_{L^p(|x| \le \sigma_j(2t))}^p \mathrm{d}h\right\}^{1/p}$$
$$\le C\varepsilon\left\{\frac{1}{t}\int_0^t \mathrm{d}h\right\}^{1/p} \le C\varepsilon.$$

If $\sigma_j(2t) > K$, then by Lemma 3.10 we see

$$A \le C\left\{\frac{1}{t}\int_0^t \left\| T_j^{1/4}(x) w_j(x)\left(f - f_K\right)_{\hat{X}_j}(x)\right\|_{L^p(|x| \le K)}^p \mathrm{d}h\right\}^{1/p}$$
$$+\left\{\frac{1}{t}\int_0^t \left\| T_j^{1/4}(x) w_j(x) f_{\hat{X}_j}(x)\right\|_{L^p(K < |x| \le \sigma_j(2t))}^p \mathrm{d}h\right\}^{1/p}$$
$$\le C\varepsilon + C_1\varepsilon \le C_2\varepsilon.$$

When we take $t > 0$ small enough, we see

$$B = \left\{\frac{1}{t}\int_0^t \left\| T_j^{1/4}(x) w_j(x)\left(\Delta_{h\Phi_{t,j}(x)}\left(f_K\right)_{\hat{X}_j}(x)\right)\right\|_{L^p(|x| \le K)}^p \mathrm{d}h\right\}^{1/p}$$
$$\le \left\{\frac{1}{t}\int_0^t \varepsilon \mathrm{d}h\right\}^{1/p} \le C\varepsilon,$$

because of the continuity of f_K. Therefore we have $I_1 < \varepsilon$. Consequently, we have

$$\left\| W_j T^{\langle j-1\rangle}\left\{\frac{1}{t}\int_0^t \left\| T_j^{1/4}(x) w_j(x)\left(\Delta_{h\Phi_{t,j}(x)} f_{\hat{X}_j}(x)\right)\right\|_{L^p(|x| \le \sigma_j(2t))}^p \mathrm{d}h\right\}^{1/p}\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)} \tag{3.22}$$
$$\le C\varepsilon\left\| W_j T^{\langle j-1\rangle}\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)} \le C\varepsilon.$$

Finally, we see

$$\left\| W_j T^{\langle j-1\rangle} \inf_{c_j(\text{constant})}\left\|\left(f_{\hat{X}_j}(x) - c_j\right) T_j^{1/4}(x) w_j(x)\right\|_{L^p\left(\mathbb{R}\setminus[-\sigma_j(4t),\sigma_j(4t)]\right)}\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}$$
$$\le C\left\| W_j T^{\langle j-1\rangle}\left\| f_{\hat{X}_j}(x) T_j^{1/4}(x) w_j(x)\right\|_{L^p\left(\mathbb{R}\setminus[-\sigma_j(4t),\sigma_j(4t)]\right)}\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}$$
$$\le C w_j^{1/4}\left(\sigma_j(4t)\right)\left\| Wf\right\|_{L^p\left(\mathbb{R}_j^{s}\right)}.$$

Here, if we set $4t = a_u/u$, then we see

$$w_j^{1/4}\left(\sigma_j(4t)\right) = \exp\left(-\frac{1}{4}Q_j(a_u)\right) \sim \exp\left(-\frac{u}{2\sqrt{T(a_u)}}\right) \le \mathrm{e}^{-u^\delta}$$

for some $0 < \delta < 1$, that is,

$$w_j^{1/4}\left(\sigma_j(4t)\right) \le C\mathrm{e}^{-u^\delta} \le C\frac{a_u}{4u} = Ct.$$

Therefore

$$\left\| W_j T^{\langle j-1\rangle} \inf_{c_j(\text{constant})} \left\| \left(f_{\hat{X}_j}(x) - c_j \right) T_j^{1/4}(x) w_j(x) \right\|_{L^p(\mathbb{R}\setminus[-\sigma_j(4t),\sigma_j(4t)])} \right\|_{L^p(\mathbb{R}_j^{s-1})} \tag{3.23}$$

$$\leq Ct.$$

For given $\varepsilon > 0$ if we take $K > 0$ large enough and then $t > 0$ small enough, then by (3.22) and (3.23) we have

$$\left\| W_j T^{\langle j-1\rangle} \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, T_j^{1/4} w_j, t \right) \right\|_{L^p(\mathbb{R}_j^{s-1})} < \varepsilon.$$

Consequently, we have (3.9).

Proof of (3.11). If in the proof of (3.9) we set as $T = 1$ (constant), then we obtain (3.11). #

Corollary 3.11. We suppose that $w_j = \exp(-Q_j) \in \mathcal{F}(C^2+)$, $j = 1,2,\cdots,s$ are the Freud-type weights. Let $1 \leq p \leq \infty$. Then we have

$$\left\| W\left(f - v_n^{[s]}(f) \right) \right\|_{L^p(\mathbb{R}^s)}$$

$$\leq \sum_{j=1}^{s} C_j \left\| \left(\prod_{1\leq i\leq s, i\neq j} w_i \right) \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, w_j; c_j \frac{a_n^{(j)}}{n} \right) \right\|_{L^p(\mathbb{R}_j^{s-1})}.$$

Especially $f \in L_W^{p*}(\mathbb{R}^s)$, then we have

$$\sum_{j=1}^{s} C_j \left\| \left(\prod_{1\leq i\leq s, i\neq j} w_i \right) \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, w_j; c_j \frac{a_n^{(j)}}{n} \right) \right\|_{L^p(\mathbb{R}_j^{s-1})} \to 0 \quad \text{as } n \to \infty.$$

Corollary 3.12. We suppose $w_j = \exp(-Q_j) \in \mathcal{F}_\lambda(C^3+)(0 < \lambda < 3/2)$, $j = 1,2,\cdots,s$, and let (3.6) hold. Let $1 \leq p \leq \infty$. If $T^{\langle s\rangle} Wf \in C_0(\mathbb{R}^s) \cap L^p(\mathbb{R}^s)$), then we have

$$\left\| W\left(f - v_n^{[s]}(f) \right) \right\|_{L^p(\mathbb{R}^s)} \to 0 \text{ as } n \to \infty.$$

Moreover, we suppose $w_j = \exp(-Q_j) \in \mathcal{F}(C^2+)$, $j = 1,2,\cdots,s$, and let (3.6) hold. If $Wf \in C_0(\mathbb{R}^s) \cap L^p(\mathbb{R}^s)$, then we have

$$\left\| \frac{W\left(f - v_n^{[s]}(f) \right)}{T^{\langle s\rangle}} \right\|_{L^p(\mathbb{R}^s)} \to 0 \text{ as } n \to \infty.$$

4. A Property of Higher Order Derivatives

In this section we show an important theorem which is useful in approximation theory. We use the following notations for $w_i(x_i) = \exp(-Q_i(x_i)) \in \mathcal{F}(C^2+), i = 1,2,\cdots,s$. Let r be a positive integer, and $|x_i| \geq \gamma > 0$.

$$W_0 := \prod_{i=1}^{s} w_{i,0};\ Q_i'(x_i)^r w_i(x_i) \sim w_{i,0}(x_i) = \exp(-Q_{i,0}(x_i)) \in \mathcal{F}(C^2+),$$

$$W_\nu := \prod_{i=1}^{s} w_{i,\nu};\ Q_i'(x_i)^{r-\nu} w_i(x_i) \sim w_{i,\nu}(x_i) = \exp(-Q_{i,\nu}(x_i)) \in \mathcal{F}(C^2+),$$
$\nu = 1, 2, \cdots, r$.

Then we see

$$Q_i'(x_i)^{r-\nu+1} w_i(x_i) \sim Q_{i,\nu}'(x_i) w_{i,\nu}(x_i).$$

Especially, if $\nu = r$, then

$$w_{i,r}(x_i) = w_i(x_i).$$

Theorem 4.1. Let $w_i = \exp(-Q_i) \in \mathcal{F}(C^2+), i = 1, 2, \cdots, s$, and let $1 \le p \le \infty$. Let a constant $\gamma \ge 0$ be fixed. We suppose that $g := g(x_1, x_2, \cdots, x_s)$ is absolutely continuous on $\mathbb{R}^s$ and $Wg^{(1,1,\cdots,1)} \in L^p(\mathbb{R}^s)$. Then we have

$$\begin{aligned} &\left\| \prod_{i=1}^{s} (Q_i' w_i) g \right\|_{L^p(|x_i| \le \gamma, i=1,\cdots,s)} \\ &\le C \left\{ \int_{|x_1| \le \gamma} \cdots \int_{|x_s| \le \gamma} \sum_{0 \le j_1 \le 1} \left| w_{1,j_1}(x_1) \right|^p \cdots \sum_{0 \le j_s \le 1} \left| w_{s,j_s}(x_s) \right|^p \right. \\ &\quad \left. \times \left| g^{(j_1, j_2, \cdots, j_s)}(y_1, y_2, \cdots, y_s) \right|^p \mathrm{d}x_s \cdots \mathrm{d}x_1 \right\}^{1/p}, \end{aligned} \tag{4.1}$$

where we set for each $j_i = 0$ or 1, $i = 1, 2, \cdots, s$,

$$y_i = \begin{cases} \gamma, & j_i = 0; \\ x_i, & j_i = 1. \end{cases}$$

Furthermore, let r be a positive integer, and let for each $i = 1, 2, \cdots, s$, $w_i = \exp(-Q_i) \in \mathcal{F}_\lambda(C^3+) \subset \mathcal{F}(C^2+)$ $(0 < \lambda < 3/2)$. We suppose that $g^{(r-1, r-1, \cdots, r-1)}$ is absolutely continuous and $Wg^{(r,r,\cdots,r)} \in L^p(\mathbb{R}^s)$. Then we have

$$\left\{ \int_{\mathbb{R}^s} \left| w_{1,j_1}(x_1) \right|^p \left| w_{s,j_s}(x_s) \right|^p \left| g^{(j_1, j_2, \cdots, j_s)}(y_1, y_2, \cdots, y_s) \right|^p \mathrm{d}x_s \cdots \mathrm{d}x_1 \right\}^{1/p} < \infty, \tag{4.2}$$

for each $0 \le j_i \le r, i = 1, 2, \cdots, s$ with

$$y_i = \begin{cases} \gamma, & 0 \le j_i \le r-1; \\ x_i, & j_i = r, \end{cases} \tag{4.3}$$

and

$$\begin{aligned} &\left\| \left(\prod_{i=1}^{s} Q_i' \right)^r Wg \right\|_{L^p(|x_i| \ge \gamma, i=1,\cdots,s)} \\ &\le C \left\{ \int_{|x_1| \ge \gamma} \cdots \int_{|x_s| \ge \gamma} \sum_{0 \le j_1 \le r} \left| w_{1,j_1}(x_1) \right|^p \cdots \sum_{0 \le j_s \le r} \left| w_{s,j_s}(x_s) \right|^p \right. \\ &\quad \left. \times \left| g^{(j_1, j_2, \cdots, j_s)}(y_1, y_2, \cdots, y_s) \right|^p \mathrm{d}x_s \cdots \mathrm{d}x_1 \right\}^{1/p} < \infty. \end{aligned} \tag{4.4}$$

Proposition 4.2 ([8], Theorem 9, cf. [9], Lemma 3.4.4). Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$ and a constant $\gamma \ge 0$ be fixed.

(a) We have

$$\left| Q'(x) w(x) \int_\gamma^x w^{-1}(t) \mathrm{d}t \right| \le C, \quad |x| \ge \gamma.$$

(b) Let $1 \le p \le \infty$, and let r be a positive integer. If g is absolutely continuous, $g(\gamma)=0$ and $wg' \in L^p(\mathbb{R})$, then

$$\|Q'wg\|_{L^p(|x|\ge\gamma)} \le C\|wg'\|_{L^p(|x|\ge\gamma)}.$$

When $w=\exp(-Q)\in\mathcal{F}_\lambda(C^{r+1}+)\subset\mathcal{F}(C^2+)$ ($0<\lambda<(r+1)/r$), and $g^{(r-1)}$ is absolutely continuous, $g^{(j)}(\gamma)=0, j=0,1,\cdots,r-1$ with $wg^{(r)}\in L^p(\mathbb{R})$, we see

$$\left\|(Q')^r wg\right\|_{L^p(|x|\ge\gamma)} \le C\left\|wg^{(r)}\right\|_{L^p(|x|\ge\gamma)}.$$

Proposition 4.3 ([3], Theorem 4.2). Let $w=\exp(-Q)\in\mathcal{F}_\lambda(C^3+)\subset\mathcal{F}(C^2+)$. Then for $\alpha\in\mathbb{R}$, we can construct a new weight $w_\alpha\in\mathcal{F}(C^2+)$ such that

$$(1+|Q'(x)|)^\alpha w(x)\sim w_\alpha(x)=\exp(-Q_\alpha)$$

on $\mathbb{R}$, $(1/c)a_n(w)\le a_n(w_\alpha)\le ca_n(w)$ (c is an absolutely constant) on $\mathbb{N}$ and $T_{w_\alpha}(x)\sim T_w(x)$ hold on $\mathbb{R}$. Furthermore, we see

$$Q_\alpha^{(j)}(x)\sim Q^{(j)}(x)\,(j=0,1) \text{ for } |x|\ge\gamma>0.$$

Proof of Theorem 4.1. For the proof of (4.1) we may put $r=1$ with $w_i=\exp(-Q_i)\in\mathcal{F}(C^2+), i=1,2,\cdots,s$ in the proof of (4.4) below. So we prove only (4.4). We use Proposition 4.2 and 4.3 repeatedly.

$$\begin{aligned}
&\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_1\|\ge\gamma}\left|\prod_{i=1}^s\left(Q_i'^r(x_i)w_i(x_i)\right)g(x_1,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}\\
&=\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_2\|\ge\gamma}\left|\prod_{i=2}^s\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.\\
&\quad\left.\times\int_{\|x_1\|\ge\gamma}\left|Q_1'^r(x_1)w_1(x_1)g(x_1,\cdots,x_s)\right|^p\mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}\\
&\le C_{1,0}\left[\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_2\|\ge\gamma}\left|\prod_{i=2}^s\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.\right.\\
&\quad\left.\left.\times\int_{\|x_1\|\ge\gamma}\left|Q_{1,1}'(x_1)w_{1,1}(x_1)g(x_1,\cdots,x_s)\right|^p\mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}\right],
\end{aligned}$$

where $(Q_1')^r w_1 = Q_1'(Q_1')^{r-1}w_1\sim Q_1'w_{1,1}\sim Q_{1,1}'w_{1,1}, w_{1,1}=\mathrm{e}^{-Q_{1,1}}$,

$$\begin{aligned}
&\le C_{1,0}\left[\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_2\|\ge\gamma}\left|\prod_{i=2}^s\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.\right.\\
&\quad\left.\times\int_{\|x_1\|\ge\gamma}\left|Q_{1,1}'(x_1)w_{1,1}(x_1)\left(g(x_1,\cdots,x_s)-g(\gamma,x_2,\cdots,x_s)\right)\right|^p\mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}\\
&\quad+\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_2\|\ge\gamma}\left|\prod_{i=2}^s\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.\\
&\quad\left.\left.\times\int_{\|x_1\|\ge\gamma}\left|Q_{1,1}'(x_1)w_{1,1}(x_1)g(\gamma,x_2,\cdots,x_s)\right|^p\mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}\right]\\
&\le C_{1,1}\left[\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_2\|\ge\gamma}\left|\prod_{i=2}^s\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.\right.\\
&\quad\left.\times\int_{\|x_1\|\ge\gamma}\left|w_{1,1}(x_1)g^{(1,0,\cdots,0)}(x_1,\cdots,x_s)\right|^p\mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}\\
&\quad\left.+\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_1\|\ge\gamma}\left|\prod_{i=1}^s\left(Q_i'^r(x_i)w_i(x_i)\right)g(\gamma,x_2,\cdots,x_s)\right|^p\mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}\right]
\end{aligned}$$

by $Q'_{1,1}w_{1,1} \sim (Q'_1)^r w_1$,

$$\le C'_{1,1}\left[\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_2\|\ge\gamma}\left|\prod_{i=2}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.\right.$$

$$\left.\times\int_{\|x_1\|\ge\gamma}\left|Q'_{1,2}(x_1)w_{1,2}(x_1)g^{(1,0,\cdots,0)}(x_1,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}$$

where $w_{1,1} \sim (Q'_1)^{r-1} w \sim Q'_{1,2}w_{1,2},\ w_{1,2} = \mathrm{e}^{-Q_{1,2}}$,

$$\left.+\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_1\|\ge\gamma}\left|\prod_{i=1}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)g(\gamma,x_2,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}\right]$$

$$\le C_{1,2}\left[\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_2\|\ge\gamma}\left|\prod_{i=2}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.\right.$$

$$\left.\times\int_{\|x_1\|\ge\gamma}\left|w_{1,2}(x_1)g^{(2,0,\cdots,0)}(x_1,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}$$

$$+\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_1\|\ge\gamma}\left|\prod_{i=2}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.$$

$$\left.\times\left|Q_1'^{r-1}(x_1)w_1(x_1)g^{(1,0,\cdots,0)}(\gamma,x_2,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}$$

$$\left.+\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_1\|\ge\gamma}\left|\prod_{i=1}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)g(\gamma,x_2,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}\right]$$

$$\le\cdots$$

$$\le C_{1,r}\left[\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_2\|\ge\gamma}\left|\prod_{i=2}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.\right.$$

$$\left.\times\int_{\|x_1\|\ge\gamma}\left|w_1(x_1)g^{(r,0,\cdots,0)}(x_1,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}$$

$$+\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_1\|\ge\gamma}\left|\prod_{i=2}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.$$

$$\left.\times\left|Q_1'(x_1)w_1(x_1)g^{(1,0,\cdots,0)}(\gamma,x_2,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}$$

$$+\cdots$$

$$+\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_1\|\ge\gamma}\left|\prod_{i=2}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.$$

$$\left.\times\left|Q_1'^{r-1}(x_1)w_1(x_1)g^{(1,0,\cdots,0)}(\gamma,x_2,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}$$

$$\left.+\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_1\|\ge\gamma}\left|\prod_{i=1}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)g(\gamma,x_2,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}\right]$$

$$= C_{1,r}\left\{\int_{\|x_s\|\ge\gamma}\cdots\int_{\|x_2\|\ge\gamma}\left|\prod_{i=2}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.$$

$$\left.\times\int_{\|x_1\|\ge\gamma}\sum_{0\le j_1\le r}\left|w_{1,j_1}(x_1)g^{(j_1,0,\cdots,0)}(y_1,x_2,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p},$$

where

$$y_1=\begin{cases}\gamma, & 0\le j_1\le r-1;\\ x_1, & j_1=r.\end{cases}$$

We continue this manner with respect to $x_2, x_3, \cdots, x_s$. Then we can easily obtain as follows:

$$
\begin{aligned}
&\left\{\int_{|x_s|\ge\gamma}\cdots\int_{|x_1|\ge\gamma}\left|\prod_{i=1}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)g(x_1,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}\\
&\le C_{1,r}\left\{\int_{|x_s|\ge\gamma}\cdots\int_{|x_2|\ge\gamma}\left|\prod_{i=2}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.\\
&\quad\left.\times\int_{|x_1|\ge\gamma}\sum_{0\le j_1\le r}\left|w_{1,j_1}(x_1)g^{(j_1,0,\cdots,0)}(y_1,x_2,\cdots,x_s)\right|^p \mathrm{d}x_1\cdots\mathrm{d}x_s\right\}^{1/p}\\
&\le C_2\left\{\int_{|x_1|\ge\gamma}\sum_{0\le j_1\le r}\left|w_{1,j_1}(x_1)\right|^p\int_{|x_s|\ge\gamma}\cdots\int_{|x_3|\ge\gamma}\left|\prod_{i=3}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.\\
&\quad\left.\times\int_{|x_2|\ge\gamma}\sum_{0\le j_2\le r}\left|w_{2,j_2}(x_2)g^{(j_1,j_2,0,\cdots,0)}(y_1,y_2,x_3,\cdots,x_s)\right|^p \mathrm{d}x_2\cdots\mathrm{d}x_s\mathrm{d}x_1\right\}^{1/p}\\
&= C_2\left\{\int_{|x_1|\ge\gamma}\int_{|x_2|\ge\gamma}\sum_{0\le j_1\le r}\left|w_{1,j_1}(x_1)\right|^p\sum_{0\le j_2\le r}\left|w_{2,j_2}(x_2)\right|^p\int_{|x_s|\ge\gamma}\cdots\int_{|x_4|\ge\gamma}\left|\prod_{i=4}^{s}\left(Q_i'^r(x_i)w_i(x_i)\right)\right|^p\right.\\
&\quad\left.\times\int_{|x_3|\ge\gamma}\left|w_{3,j_3}(x_3)g^{(j_1,j_2,j_3,0,\cdots,0)}(y_1,y_2,y_3,x_4,\cdots,x_s)\right|^p\times\mathrm{d}x_3\cdots\mathrm{d}x_s\mathrm{d}x_2\mathrm{d}x_1\right\}^{1/p}\\
&\le\cdots\\
&\le C_r\left\{\int_{|x_1|\ge\gamma}\cdots\int_{|x_s|\ge\gamma}\sum_{0\le j_1\le r}\left|w_{1,j_1}(x_1)\right|^p\cdots\sum_{0\le j_s\le r}\left|w_{s,j_s}(x_s)\right|^p\right.\\
&\quad\left.\times g^{(j_1,j_2,\cdots,j_s)}(y_1,y_2,\cdots,y_s)|^p \mathrm{d}x_s\cdots\mathrm{d}x_1\right\}^{1/p},
\end{aligned}
$$

where for each $0\le j_i\le r, i=1,2,\cdots,s$. We set (4.3).

Let $Wg^{(r,\cdots,r)}\in L^p(\mathbb{R}^s)$. Then we need to show

$$
\begin{aligned}
A:=&\left\{\int_{|x_1|\ge\gamma}\cdots\int_{|x_s|\ge\gamma}\sum_{0\le j_1\le r}\left|w_{1,j_1}(x_1)\right|^p\cdots\sum_{0\le j_s\le r}\left|w_{s,j_s}(x_s)\right|^p\right.\\
&\left.\times\left|g^{(j_1,j_2,\cdots,j_s)}(y_1,y_2,\cdots,y_s)\right|^p \mathrm{d}x_s\cdots\mathrm{d}x_1\right\}^{1/p}<\infty.
\end{aligned}
$$

We rearrange $(x_1,x_2,\cdots,x_s)$ as $x_{k_i}=y_i=x_i, t+1\le k_i\le s$ if $j_i=r$, and as $x_{k_i}=y_i=\gamma<1\le k_i\le t$ if $0\le j_i\le r-1$, where $0\le t\le s$. Then we set $f\left(x_{k_1},x_{k_2},\cdots,x_{k_s}\right):=g(x_1,x_2,\cdots,x_s)$. We see

$$
g^{(j_1,j_2,\cdots,j_s)}(y_1,y_2,\cdots,y_s)=f^{\left(j_{k_1},j_{k_2},\cdots,j_{k_s}\right)}(\gamma,\cdots,\gamma,x_{t+1},x_{t+2},\cdots,x_s).
$$

Then we have

$$
\begin{aligned}
A&=\left\|\prod_{i=1}^{t}w_{k_i}\left\|\prod_{i=t+1}^{s}w_{k_i}f^{\left(j_{k_1},\cdots,j_{k_s}\right)}(\gamma,\cdots,\gamma,x_{t+1},\cdots,x_s)\right\|_{L^p\left(\mathbb{R}^{s-t}_{t+1\le}\right)}\right\|_{L^p\left(\mathbb{R}^{t}_{\le t}\right)}\\
&=:\left\|\prod_{i=1}^{t}w_{k_i}\left|h^{\left(j_{k_1},\cdots,j_{k_t}\right)}(\gamma,\cdots,\gamma)\right|\right\|_{L^p\left(\mathbb{R}^{t}_{\le t}\right)} \qquad \#\\
&\le C\left|h^{\left(j_{k_1},\cdots,j_{k_t}\right)}(\gamma,\cdots,\gamma)\right|\left\|\prod_{i=1}^{t}w_{k_i}\right\|_{L^p\left(\mathbb{R}^{t}_{\le t}\right)}<\infty.
\end{aligned}
$$

We can generalize Theorem 4.1 easily. We give a class of nonnegative integers $(j_1, j_2, \cdots, j_s)$, and set $J_s := (j_1, j_2, \cdots, j_s)$. For $r_i \ge 1, i = 1, 2, \cdots, s$ we set $R_s := (r_1, r_2, \cdots, r_s)$. Then we consider the order as follows:

$$K_s := (k_1, k_2, \cdots, k_s) \le R_s := (r_1, r_2, \cdots, r_s)$$

means

$$k_i \le r_i \ (i = 1, 2, \cdots, s).$$

Corollary 4.4. Let $K_s = (k_1, k_2, \cdots, k_s) \le R_s = (r_1, r_2, \cdots, r_s)$ be classes of nonnegative integers, where $r_i \ge 1, i = 1, 2, \cdots, s$. For each $i = 1, 2, \cdots, s$, we suppose $w_i = \exp(-Q_i) \in \mathcal{F}_\lambda(C^{r+1}+) \subset \mathcal{F}(C^2+)$ $(0 < \lambda < (r+1)/r)$. If $g^{(r_1-1, r_2-1, \cdots, r_s-1)}$ is absolutely continuous, and $Wg^{(r_1, r_2, \cdots, r_s)} \in L^p(\mathbb{R}^s)$, then we see

$$\begin{aligned}
&\left\| \left(\prod_{i=1}^s Q_i' \right)^{r_i - k_i} Wg^{(k_1, \cdots, k_s)} \right\|_{L^p(|x_i| \ge \gamma, i = 1, \cdots, s)} \\
&\le C \left\{ \int_{|x_1| \ge \gamma} \cdots \int_{|x_s| \ge \gamma} \sum_{k_1 \le j_1 \le r_1} \left| w_{1, j_1 - k_1}(x_1) \right|^p \cdots \sum_{k_s \le j_s \le r_s} \left| w_{s, j_s - k_s}(x_s) \right|^p \right. \\
&\quad \left. \times \left| g^{(j_1, j_2, \cdots, j_s)}(y_1, y_2, \cdots, y_s) \right|^p \mathrm{d}x_s \cdots \mathrm{d}x_1 \right\}^{1/p} < \infty,
\end{aligned}$$

where for each $i = 1, 2, \cdots, s$ we set

$$y_i = \begin{cases} \gamma, & k_i \le j_i \le r_i - 1; \\ x_i, & j_i = r_i. \end{cases}$$

We remark that $Wg^{(r_1, \cdots, r_s)} \in L^p(\mathbb{R}^s)$ means $Wg^{(k_1, \cdots, k_s)} \in L^p(\mathbb{R}^s)$ for $0 \le K_s = (k_1, \cdots, k_s) \le R_s = (r_1, \cdots, r_s)$.

5. Degree of Approximation

We define the degree of approximation for $Wf \in L^p(\mathbb{R}^s)$ as follows:

$$E_{n,p;s}(W, f) := \inf_{P \in \mathcal{P}_{n;s}(\mathbb{R}^s)} \left\| W(f - P) \right\|_{L^p(\mathbb{R}^s)}.$$

Using this $E_{n,p;s}(W, f)$, we can estimate the degree of approximation of $Wf \in L^p(\mathbb{R}^s)$ from $\mathcal{P}_{n;s}(\mathbb{R}^s)$.

Theorem 5.1. (1) Let $w_i \in \mathcal{F}(C^2+)(i = 1, 2, \cdots, s)$ and let $1 \le p \le \infty, Wf \in L^p(\mathbb{R}^s)$. Furthermore, we suppose (3.3). Then we have

$$\left\| \frac{W}{\prod_{i=1}^s T_i(x_i)^{1/4}} \left(f - v_n^{[s]}(f) \right) \right\|_{L^p(\mathbb{R}^s)} \le C E_{n,p;s}(W, f).$$

(2) If $w_i \in \mathcal{F}_\lambda(C^3+)(i = 1, 2, \cdots, s), 0 < \lambda < 3/2$, and let $\left\| \left(\prod_{i=1}^s T_i^{1/4} \right) Wf \right\|_{L^p(\mathbb{R}^s)} < \infty$, then we have

$$\left\| W\left(f - v_n^{[s]}(f) \right) \right\|_{L^p(\mathbb{R}^s)} \le C E_{n,p;s}\left(\left(\prod_{i=1}^s T_i^{1/4} \right) W, f \right).$$

(3) Let $w_i \in \mathcal{F}\left(C^2+\right)(i=1,2,\cdots,s)$ and let $1 \le p \le \infty, Wf \in L^p\left(\mathbb{R}^s\right)$. Then we have

$$\left\|W\left(f-v_n^{[s]}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)} \le C\left(\prod_{i=1}^s T_i^{1/4}\left(a_n^{(i)}\right)\right)E_{n,p;s}(W,f).$$

(4) Furthermore, let $w_i \in \mathcal{F}_\lambda\left(C^3+\right)(0<\lambda<3/2), i=1,2,\cdots,s$. If $f \in L_{W^\delta}^{p^*}\left(\mathbb{R}^s\right)$ for some $0<\delta<1$, then we have

$$E_{n,p;s}(W,f) \to 0 \text{ as } n \to \infty.$$

Proof. (1) There exists $P \in \mathcal{P}_n$ such that $\|W(f-P)\|_{L^p\left(\mathbb{R}^s\right)} \le CE_{n,p;s}(W,f)$. Therefore, by Lemma 3.9 (1)

$$\begin{aligned}
&\left\|\frac{W}{\prod_{i=1}^s T_i^{1/4}}\left(f-v_n^{[s]}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)} \\
&= \left\|\frac{W}{\prod_{i=1}^s T_i^{1/4}}(f-P)\right\|_{L^p\left(\mathbb{R}^s\right)} + \left\|\frac{W}{\prod_{i=1}^s T_i^{1/4}}v_n^{[s]}(f-P)\right\|_{L^p\left(\mathbb{R}^s\right)} \\
&\le C\|W(f-P)\|_{L^p\left(\mathbb{R}^s\right)} \le CE_{n,p;s}(W,f).
\end{aligned}$$

(2) We see $W(X)\prod_{i=1}^s T_i^{1/4}(x_i) \sim \tilde{W}(X)=\prod_{i=1}^s \tilde{w}_i(x_i)$, $T_i^{1/4}w \sim \tilde{w}_i \in \mathcal{F}\left(C^2+\right), i=1,2,\cdots,s$. Then, there exists $P \in \mathcal{P}_n$ such that $\left\|\tilde{W}(f-P)\right\|_{L^p\left(\mathbb{R}^s\right)} \le CE_{n-1,p;s}\left(\tilde{W},f\right)$. Therefore, by Lemma 3.9 (2)

$$\begin{aligned}
&\left\|W\left(f-v_n^{[s]}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)} \\
&= \|W(f-P)\|_{L^p\left(\mathbb{R}^s\right)} + \left\|Wv_n^{[s]}(f-P)\right\|_{L^p\left(\mathbb{R}^s\right)} \\
&\le CE_{n,p;s}\left(\tilde{W},f\right) \le CE_{n,p;s}\left(\left(\prod_{i=1}^s T_i^{1/4}\right)W,f\right).
\end{aligned}$$

(3) Similarly, we have (3).

(4) It follows from Theorem 3.3. #

Theorem 5.2. Let $w_i \in \mathcal{F}_\lambda\left(C^3+\right)(i=1,2,\cdots,s), 0<\lambda<3/2$, and let $1 \le p \le \infty$. Then if $Wf \in L_p\left(\mathbb{R}^s\right)$, we have

$$\left\|\frac{W}{\prod_{i=1}^s T_i^{(2j_i+1)/4}}v_n^{[s]}(f)^{(j_1,\cdots,j_s)}\right\|_{L^p\left(\mathbb{R}^s\right)} \le C\prod_{i=1}^s\left(\frac{n}{a_n^{(i)}}\right)^{j_i}\|Wf\|_{L^p\left(\mathbb{R}^s\right)},$$

and

$$\left\|Wv_n^{[s]}(f)^{(j_1,\cdots,j_s)}\right\|_{L^p\left(\mathbb{R}^s\right)} \le C\prod_{i=1}^s\left(\frac{n}{a_n^{(i)}}\right)^{j_i}\left(\prod_{i=1}^s T_i^{(2j_i+1)/4}\left(a_n^{(i)}\right)\right)\|Wf\|_{L^p\left(\mathbb{R}^s\right)}.$$

Proposition 5.3 ([10], Lemma 2.5, [2], Corollary 10.2). Let $1 \le p \le \infty$ and $w \in \mathcal{F}_\lambda\left(C^3+\right)(0<\lambda<3/2)$. Then there exists a constant $C_1=C_1(w,p)>0$ such that, if $P \in \mathcal{P}_n(\mathbb{R})$ ($n \in \mathbb{N}$),

$$\left\| \frac{w}{T^{j/2}} P^{(j)} \right\|_{L^p(\mathbb{R}^s)} \le C_1 \left(\frac{n}{a_n} \right)^j \|wP\|_{L^p(\mathbb{R}^s)}, \ j \in \mathbb{N},$$

and

$$\left\| wP^{(j)} \right\|_{L^p(\mathbb{R}^s)} \le C_1 \left(\frac{nT(a_n)^{1/2}}{a_n} \right)^j \|wP\|_{L^p(\mathbb{R}^s)}, \ j \in \mathbb{N}.$$

Proof of Theorem 5.2. We use Proposition 5.3 and Lemma 3.7 (1).

$$\begin{aligned}
&\left\| \frac{W}{\prod_{i=1}^s T_i^{(2j_i+1)/4}} v_n^{[s]}(f)^{(j_1,\cdots,j_s)} \right\|_{L^p(\mathbb{R}^s)} \\
&\le C \prod_{i=1}^s \left(\frac{n}{a_n^{(i)}} \right)^{j_i} \left\| \frac{W}{\prod_{i=1}^s T_i^{-1/4}} v_n^{[s]}(f) \right\|_{L^p(\mathbb{R}^s)} \\
&\le C \prod_{i=1}^s \left(\frac{n}{a_n^{(i)}} \right)^{j_i} \|Wf\|_{L^p(\mathbb{R}^s)},
\end{aligned}$$

and further, using Theorem A_1 (the Markov-Bernstein inequality) in Appendix,

$$\begin{aligned}
&\left\| Wv_n^{[s]}(f)^{(j_1,\cdots,j_s)} \right\|_{L^p(\mathbb{R}^s)} \le C \prod_{i=1}^s \left(\frac{nT_i^{1/2}\left(a_n^{(i)}\right)}{a_n^{(i)}} \right)^{j_i} \left\| Wv_n^{[s]}(f) \right\|_{L^p(\mathbb{R}^s)} \\
&\le C \prod_{i=1}^s \left(\frac{nT_i^{1/2}\left(a_n^{(i)}\right)}{a_n^{(i)}} \right)^{j_i} \left\| Wv_n^{[s]}(f) \right\|_{L^p\left(|x_i| \le a_n^{(i)}, i=1,2,\cdots,s\right)} \\
&\le C \prod_{i=1}^s \left(\frac{n}{a_n^{(i)}} \right)^{j_i} \prod_{i=1}^s T_i^{(2j_i+1)/4}\left(a_n^{(i)}\right) \left\| \frac{W}{\prod_{i=1}^s T_i^{1/4}} v_n^{[s]}(f) \right\|_{L^p(\mathbb{R}^s)} \\
&\le C \prod_{i=1}^s \left(\frac{n}{a_n^{(i)}} \right)^{j_i} \prod_{i=1}^s T_i^{(2j_i+1)/4}\left(a_n^{(i)}\right) \|Wf\|_{L^p(\mathbb{R}^s)}.
\end{aligned}$$

#

In the rest of only this section, we suppose

$$w = \exp(-Q) = w_i = \exp(-Q_i), \quad i = 1, 2, \cdots, s,$$

so

$$a_n = a_n^{(i)}, \ T = T_i, \quad i = 1, 2, \cdots, s.$$

Let

$$W_i := W_i\left(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_s\right) := \prod_{j \ne i, 1 \le j \le s} w_j\left(x_j\right), \quad i = 1, 2, \cdots, s.$$

In ([7], Corollary 8) we give the Favard-type inequalities:

Proposition 5.4 [7]. Let $w \in \mathcal{F}(C^2+)$, and let $r \ge 0$ be an integer. Let $1 \le p \le \infty$, and let $wf^{(r)} \in L_p(\mathbb{R})$. Then we have

$$E_{p,n}(f,w) \le C\left(\frac{a_n}{n}\right)^k \left\|wf^{(k)}\right\|_{L_p(\mathbb{R})}, k=1,2,\cdots,r,$$

and equivalently,

$$E_{p,n}(f,w) \le C\left(\frac{a_n}{n}\right)^k E_{p,n-k}\left(f^{(k)},w\right).$$

The following theorem is a generalization of Proposition 5.4.

Theorem 5.5. We suppose $w_j = \exp\left(-Q_j\right) \in \mathcal{F}_\lambda\left(C^3+\right)\left(0<\lambda<3/2\right)$, $j=1,2,\cdots,s$, and let (3.3) satisfy, that is,

$$T_j\left(a_n\right) \le c\left(\frac{n}{a_n}\right)^{2/3}, j=1,2,\cdots,s.$$

Let $Wf^{(r,r,\cdots,r)} \in L_p\left(\mathbb{R}^s\right)$ for some positive integer r. Then we have

$$E_{n,p;s}(W;f) \le C\left(\frac{a_n}{n}\right)^r \left\|T^{\langle s\rangle} W f^{(r,r,\cdots,r)}\right\|_{L^p\left(\mathbb{R}^s\right)}.$$

Equivalently,

$$E_{n,p;s}(W;f) \le C\left(\frac{a_n}{n}\right)^r E_{n-r,p;s}\left(T^{\langle s\rangle}W; f^{(r,r,\cdots,r)}\right).$$

Proof. Using $f-v_n^{[s]} = \left(f-v_n^{[1]}(f)\right)+\left(v_n^{[1]}(f)-v_n^{[2]}(f)\right)+\cdots+\left(v_n^{[s-1]}(f)-v_n^{[s]}(f)\right)$, we get from Lemma 3.9 (2) and (3.2),

$$\begin{aligned}
E_{n,p;s} &\le \left\|W\left(f-v_n^{[s]}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)} \\
&\le \left\|W\left(f-v_n^{[1]}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)} + \sum_{j=2}^{s}\left\|W\left(v_n^{[j-1]}(f)-v_n^{[j]}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)} \\
&\le C\left[\left\|W\left(f-v_{n,1}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)} + \sum_{j=2}^{s}\left\|Wv_n^{[j-1]}\left(f-v_{n,j}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)}\right] \\
&\le C\left[\left\|W\left(f-v_{n,1}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)} + \sum_{j=2}^{s}\left\|W\left(\prod_{k=1}^{j-1}T_i^{1/4}\right)\left(f-v_{n,j}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)}\right]
\end{aligned}$$

We estimate each term. From Proposition 5.4 with the weight $T_j^{1/4}w_j$,

$$\begin{aligned}
&\left\|W\left(\prod_{k=1}^{j-1}T_k^{1/4}\right)\left(f-v_{n,j}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)} \\
&= \left\|\left\|W_i\left(\prod_{k=1}^{j-1}T_k^{1/4}\right)w_j\left(f_{\hat{X}_j}-v_{n,j}\left(f_{\hat{X}_j}\right)\right)\right\|_{L^p\left(\mathbb{R}_{(j)}\right)}\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)} \\
&\le C\left\|W_i\left(\prod_{k=1}^{j-1}T_k^{1/4}\right)E_{n,p}\left(T_j^{1/4}w_j, f_{\hat{X}_j}\right)\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)} \\
&\le C\left(\frac{a_n}{n}\right)^r\left\|\prod_{i\ne j}w_i\left(\prod_{k=1}^{j-1}T_k^{1/4}\right)\left\|T_j^{1/4}w_j f_{\hat{X}_j}^{(r,0,\cdots,0)}\right\|_{L^p\left(\mathbb{R}_{(j)}\right)}\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}.
\end{aligned}$$

Now, we use Theorem 4.1 and the fact

$$T_i(x_i) \le Q_i'(x_i)^r,\ i = 1,2,\cdots,s,$$

then we have

$$\left\| W\left(\prod_{k=1}^{j-1} T_k^{1/4}\right)\left(f - v_{n,j}(f)\right)\right\|_{L^p(\mathbb{R}^s)}$$

$$\le C\left(\frac{a_n}{n}\right)^r \left\| \prod_{i\ne j} Q_i'(x_i)^r\, w_i \left\| T_j^{1/4} w_j\, f_{\hat{x}_j}^{(r,0,\cdots,0)} \right\|_{L^p(\mathbb{R}_{(j)})} \right\|_{L^p(\mathbb{R}_j^{s-1})}$$

$$\le C\left(\frac{a_n}{n}\right)^r \left\| T^{\langle s\rangle} W f^{(r,r,\cdots,r)} \right\|_{L^p(\mathbb{R}^s)}.$$

Consequently, we have

$$\left\| W\left(f - v_n^{[s]}(f)\right)\right\|_{L^p(\mathbb{R}^s)} \le C\left(\frac{a_n}{n}\right)^r \left\| T^{\langle s\rangle} W f^{(r,r,\cdots,r)} \right\|_{L^p(\mathbb{R}^s)}.$$

Corollary 5.6. Under the conditions of Theorem 5.5, if w is a Freud-type weight, then

$$E_{n,p;s}(W;f) \le C\left(\frac{a_n}{n}\right)^r \left\| W f^{(r,r,\cdots,r)} \right\|_{L^p(\mathbb{R}^s)}.$$

Equivalently,

$$E_{n,p;s}(W;f) \le C\left(\frac{a_n}{n}\right)^r E_{n-r,p;s}\left(W; f^{(r,r,\cdots,r)}\right).$$

Let $1 \le p \le \infty$. For $\|Wf\|_{L_p(\mathbb{R}^s)} < \infty$ we define the K-functional $\mathcal{K}_{r,p}(W;f,\delta)$ by

$$K_{r,p;s}(W;f,\delta) := \inf_g \left\{ \left\| W(f-g)\right\|_{L_p(\mathbb{R}^s)} + \delta^r \left\| W g^{(r,r,\cdots,r)} \right\|_{L_p(\mathbb{R}^s)} \right\},$$

where the infimum is over all functions $g^{(r-1,r-1,\cdots,r-1)}$ which are absolutely continuous and $\left\| W g^{(r,r,\cdots,r)} \right\|_{L_p(\mathbb{R}^s)} < \infty$. We have the following.

Theorem 5.7. We suppose $w_j = \exp(-Q_j) \in \mathcal{F}_\lambda(C^3+)(0 < \lambda < 3/2)$, $j = 1,2,\cdots,s$, and let

$$T_j(a_n) \le c\left(\frac{n}{a_n}\right)^{2/3},\ j = 1,2,\cdots,s.$$

Let $1 \le p \le \infty$, and let $\left\| T^{\langle s\rangle} W f \right\|_{L_p(\mathbb{R}^s)} < \infty$. Then we have

$$E_{n,p;s}(W,f) \le K_{r,p;s}\left(T^{\langle s\rangle} W, f, \frac{a_n}{n}\right).$$

Proof. We take g as

$$\left\|T^{\langle s\rangle}W\left(f-g\right)\right\|_{L_p(\mathbb{R})}+\delta^r\left\|T^{\langle s\rangle}Wg^{(r,r,\cdots,r)}\right\|_{L_p(\mathbb{R}^s)}\le CK_{r,p;s}\left(T^{\langle s\rangle}W;f,\delta\right),$$

and for this g we select $P\in\mathcal{P}_n\left(\mathbb{R}^s\right)$ such that

$$\left\|W\left(g-P\right)\right\|_{L^p(\mathbb{R}^s)}\le E_{n,p;s}\left(W;g\right).$$

Then, from Theorem 5.5 we see

$$\begin{aligned}E_{n,p;s}\left(W,f\right)&\le\left\|W\left(f-P\right)\right\|_{L^p(\mathbb{R}^s)}\\&\le\left\|W\left(f-g\right)\right\|_{L^p(\mathbb{R}^s)}+\left\|W\left(g-P\right)\right\|_{L^p(\mathbb{R}^s)}\\&\le\left\|W\left(f-g\right)\right\|_{L^p(\mathbb{R}^s)}+CE_{n,p;s}\left(W,g\right)\\&\le\left\|T^{\langle s\rangle}W\left(f-g\right)\right\|_{L^p(\mathbb{R}^s)}+C\left(\frac{a_n}{n}\right)^r\left\|T^{\langle s\rangle}Wg^{(r,r,\cdots,r)}\right\|_{L^p(\mathbb{R}^s)}\\&\le CK_{r,p}\left(T^{\langle s\rangle}W,f,\frac{a_n}{n}\right).\end{aligned}$$ #

Corollary 5.8. Let $1\le p\le\infty$, and let $w\in\mathcal{F}\left(C^2+\right)$ be a Freud-type weight. If $\|Wf\|_{L_p(\mathbb{R}^s)}<\infty$, then we have

$$E_{n,p;s}\left(W,f\right)\le CK_{r,p;s}\left(W,f,\frac{a_n}{n}\right).$$

Let $0<p\le\infty$. Damelin [11] gives a K-functional as follows:

$$\bar{K}_{r,p}\left(f,w,t^r\right):=\inf_{P\in\mathcal{P}_n}\left\{\left\|w\left(f-P\right)\right\|_{L^p(\mathbb{R})}+t^r\left\|P^{(r)}\Phi_t^r w\right\|_{L^p(\mathbb{R})}\right\},$$

where $t>0$ and $r\ge1$ are chosen in advance and

$$n=n(t):=\inf\left\{k;\frac{a_k}{k}\le t\right\}.$$

We recall the r-th order of the modulus of smoothness $\omega_{r,p}\left(w;f;t\right)$, which is defined as follows (cf. [6] and [11]). Let r be a positive integer, and let $0<p\le\infty$. We set

$$\Delta_h^r\left(f,x\right):=\sum_{i=0}^{r}\binom{r}{i}(-1)^i f\left(x+\frac{rh}{2}-ih\right),\ x\in\mathbb{R}.$$

For the Freud-type weight,

$$\bar{\omega}_{r,p}\left(f,w,t\right):=\left(\frac{1}{t}\int_0^t\left\|w\Delta_h^r\left(f,x\right)\right\|_{L^p(|x|\le\sigma(2t))}\mathrm{d}h\right)^{1/p}+\inf_{P\in\mathcal{P}_{r-1}}\left\|\left(f-P\right)w\right\|_{L^p(|x|\le\sigma(4t))}.$$

For the Erdös-type weight,

$$\bar{\omega}_{r,p}\left(f,w,t\right):=\left(\frac{1}{t}\int_0^t\left\|w\Delta_{h\Phi_t(x)}^r\left(f,x\right)\right\|_{L^p(|x|\le\sigma(2t))}\mathrm{d}h\right)^{1/p}+\inf_{P\in\mathcal{P}_{r-1}}\left\|\left(f-P\right)w\right\|_{L^p(|x|\le\sigma(4t))},$$

where

$$\Phi_t(x) := \sqrt{\left|1-\frac{|x|}{\sigma(t)}\right|} + \frac{1}{\sqrt{T(\sigma(t))}}.$$

We remark that if $T(x)$ is bounded then we see $\Phi_t(x) \sim 1$. So, we may consider for only the Erdös-type weight. Then the following proposition holds.

Proposition 5.9 ([11], Theorem 1.2, 1.3). Let $0 < p \le \infty, r \ge 1$, and let $w \in \mathcal{E}_1$ (contains $\mathcal{F}(C^2+)$). Let $f: \mathbb{R} \to \mathbb{R}$ for which $wf \in L^p(\mathbb{R})$ (for $p = \infty$, we require f to be continuous, and fw to vanish at $\pm\infty$). Then we have

$$E_{n,p}(w, f) \le C_1 \bar{\omega}_{r,p}\left(f, w, C_2 \frac{a_n}{n}\right) \sim \bar{K}_{r,p}\left(f, w, t^r\right).$$

On $\mathbb{R}^s$ we define

$$\Omega_{r,p}(f, W, t) := \max_{i=1,\cdots,s} \left\| W_i \bar{\omega}_{r,p}\left(f_{\hat{X}_i}, w_i, t\right)\right\|_{L^p(\mathbb{R}^{s-1})},$$

$$\mathcal{K}_{r,p}(f, W, t^r) := \max_{i=1,\cdots,s} \left\| W_i \bar{K}_{r,p}\left(f_{\hat{X}_i}, w_i, t^r\right)\right\|_{L^p(\mathbb{R}^{s-1})}.$$

We see $\Omega_{r,p}(f, W, t) \sim \mathcal{K}_{r,p}(f, W, t^r)$. Then we have the following:

Theorem 5.10. We suppose
$w_i = \exp(-Q_i) \in \mathcal{F}_\lambda(C^3+)(0 < \lambda < 3/2), i = 1, 2, \cdots, s$, and let

$$T_i(a_n) \le c\left(\frac{n}{a_n}\right)^{2/3}, i = 1, 2, \cdots, s.$$

Then

$$E_{n,p;s}(W, f) \le C\Omega_{r,p}\left(f, T^{\langle s\rangle}W, t\right) \sim \mathcal{K}_{r,p}\left(f, T^{\langle s\rangle}W, t^r\right).$$

Proof. Using
$f - v_n^{[s]} = \left(f - v_n^{[1]}(f)\right) + \left(v_n^{[1]}(f) - v_n^{[2]}(f)\right) + \cdots + \left(v_n^{[s-1]}(f) - v_n^{[s]}(f)\right)$, we get from Proposition 3.3 (2) and (3.2),

$$E_{n,p;s}(W, f) \le \left\| W\left(f - v_n^{[s]}(f)\right)\right\|_{L^p(\mathbb{R}^s)}$$

$$\le \left\| W\left(f - v_n^{[1]}(f)\right)\right\|_{L^p(\mathbb{R}^s)} + \sum_{j=2}^{s} \left\| W\left(v_n^{[j-1]}(f) - v_n^{[j]}(f)\right)\right\|_{L^p(\mathbb{R}^s)}$$

$$\le C\left[\left\| W\left(f - v_{n,1}(f)\right)\right\|_{L^p(\mathbb{R}^s)} + \sum_{j=2}^{s} \left\| W v_n^{[j-1]}\left(f - v_{n,j}(f)\right)\right\|_{L^p(\mathbb{R}^s)}\right]$$

$$\le C\left[\left\| W\left(f - v_{n,1}(f)\right)\right\|_{L^p(\mathbb{R}^s)} + \sum_{j=2}^{s} \left\| W\left(\prod_{k=1}^{j-1} T_k^{1/4}\right)\left(f - v_{n,j}(f)\right)\right\|_{L^p(\mathbb{R}^s)}\right]$$

#

$$\le C\sum_{j=1}^{s} \left\| W_j\left(\prod_{k=1}^{j-1} T_k^{1/4}\right) E_{n,p}\left(T_j^{1/4} w_j, f_{\hat{X}_i}\right)\right\|_{L^p(\mathbb{R}_j^{s-1})}$$

$$\le C\sum_{j=1}^{s} \left\| W_j\left(\prod_{1\le k\le s, k\ne j} T_k^{1/4}\right) \bar{\omega}_{r,p}\left(f_{\hat{X}_j}, T_j^{1/4} w_j, \frac{a_n}{n}\right)\right\|_{L^p(\mathbb{R}_j^{s-1})}$$

$$\sim \sum_{j=1}^{s} \left\| W_j\left(\prod_{1\le k\le s, k\ne j} T_k^{1/4}\right) \bar{K}_{r,p}\left(f_{\hat{X}_j}, T_j^{1/j} w_j, \left(\frac{a_n}{n}\right)^r\right)\right\|_{L^p(\mathbb{R}_j^{s-1})}$$

$$\le C\Omega_{r,p}\left(f, T^{\langle s\rangle}W, t\right) \sim \mathcal{K}_{r,p}\left(f, T^{\langle s\rangle}W, t^r\right).$$

6. Approximation for Functions with Bounded Variations

We define the modulus of continuity. For the Freud-type weight W (all of weights w_i are Freud-type), we define

$$\omega_{p,j}^{*}\left(f_{\hat{X}_j}, w_j, t\right) := \sup_{0<h\le t}\left\|w_j(x)\left(\Delta_h f_{\hat{X}_j}(x)\right)\right\|_{L^p(|x|\le\sigma_j(2t))} + \left\|\left(f_{\hat{X}_j}(x) - f_{\hat{X}_j}(0)\right)w_j(x)\right\|_{L^p(|x|\le\sigma_j(4t))}.$$

If W is Erdös-type (some weights w_i are Erdös-type), then we define

$$\omega_{p,j}^{*}\left(f_{\hat{X}_j}, w_j, t\right) := \sup_{0<h\le t}\left\|w_j(x)\left(\Delta_{h\Phi_{t,j}(x)} f_{\hat{X}_j}(x)\right)\right\|_{L^p(|x|\le\sigma_j(2t))} + \left\|\left(f_{\hat{X}_j}(x) - f_{\hat{X}_j}(0)\right)w_j(x)\right\|_{L^p(|x|\le\sigma_j(4t))}.$$

It is sufficient to consider only the modulus for the Erdös-type.

Assumption 6.1. Let $w_i \in \mathcal{F}(C^2+)(i=1,2,\cdots,s)$, and let $a_n^{(i)} \sim a_n, i=1,2,\cdots,s$. Suppose that f is continuous on $\mathbb{R}^s$, and $f_{\hat{X}_i}(x)$ has a bounded variation on any compact interval in $\mathbb{R}^s$ with

$$\int_{\mathbb{R}_i^{s-1}} W_i \int_{\mathbb{R}} w_i(x)\left|\mathrm{d}f_{\hat{X}_i}(x)\right| \mathrm{d}x_1\cdots\mathrm{d}_{x_{i-1}}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_s < \infty,\quad i=1,2,\cdots,s, \tag{6.1}$$

and if $p=\infty$, then we further suppose

$$\left|W(X) f_{\hat{X}_i}(x_i)\right| \to 0,\ \text{as}\ |x_i| \to \infty, i=1,2,\cdots,s. \tag{6.2}$$

In (6.7) and (6.8) below, we put $t = a_n/n$, where $a_n \sim a_n^{(i)}, i=1,2,\cdots,s$. Especially, in (6.8) we set

$$\left\|W_i \omega_{\infty,i}^{*}\left(f_{\hat{X}_i}, w_i; \frac{a_n^{(i)}}{n}\right)\right\|_{L^\infty\left(\mathbb{R}_i^{s-1}\right)} = o_n(1). \tag{6.3}$$

Theorem 6.2. We suppose $w_j = \exp(-Q_j) \in \mathcal{F}(C^2+)$, $j=1,2,\cdots,s$, and let (3.5) hold. Let $n \ge 1$. Then we have

$$\left\|\frac{W\left(f - v_n^{[s]}(f)\right)}{\prod_{k=1}^{s} T_k^{1/4}}\right\|_{L^1(\mathbb{R}^s)} \le \sum_{j=1}^{s} C_j \left\|W_j \omega_{p,j}^{*}\left(f_{\hat{X}_j}, w_j; c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}. \tag{6.4}$$

Now Assumption 6.1 holds. Then we have

$$\sum_{j=1}^{s} C_j \left\|W_j \omega_{p,j}^{*}\left(f_{\hat{X}_j}, w_j; c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)} \le C o_n(1)^{1-1/p}\left(\frac{a_n}{n}\right)^{\frac{1}{p}}. \tag{6.5}$$

Proof of (6.4). By Proposition 3.3 (1) and Proposition 3.4, we get

$$\left\|\frac{w_1(x)\left(f_{\hat{X}_1} - v_{n,1}\left(f_{\hat{X}_1}\right)\right)}{T_1^{1/4}(x)}\right\|_{L^p(\mathbb{R})} \le CE_{1,n}\left(f_{\hat{X}_1}, w_1\right) \le C_1 \omega_{p,1}^{*}\left(f_{\hat{X}_1}, w_1; c_1\frac{a_n^{(1)}}{n}\right),$$

where the constant C_1 and c_1 is independent of $\hat{X}_1$. Similarly, for $j=1,2,\cdots,s$,

$$\left\|\frac{w_j(x)\left(f_{\hat{X}_j}-v_{n,j}\left(f_{\hat{X}_j}\right)\right)}{T_j^{1/4}(x)}\right\|_{L^p(\mathbb{R})}\le C_j\omega_{p,j}^*\left(f_{\hat{X}_j},w_j;c_j\frac{a_n^{(j)}}{n}\right). \tag{6.6}$$

Using $f-v_n^{[s]}=\left(f-v_n^{[1]}(f)\right)+\left(v_n^{[1]}(f)-v_n^{[2]}(f)\right)+\cdots+\left(v_n^{[s-1]}(f)-v_n^{[s]}(f)\right)$, we get from Lemma 3.7 (1) and (6.6),

$$\begin{aligned}
&\left\|\frac{W\left(f-v_n^{[s]}(f)\right)}{T^{\langle s\rangle}}\right\|_{L^p(\mathbb{R}^s)}\\
&\le\left\|\frac{W\left(f-v_n^{[1]}(f)\right)}{T^{\langle 1\rangle}}\right\|_{L_p(\mathbb{R}^s)}+\sum_{j=2}^{s}\left\|\frac{W\left(v_n^{[j-1]}(f)-v_n^{[j]}(f)\right)}{T^{\langle j\rangle}}\right\|_{L^p(\mathbb{R}^s)}\\
&\le C\left[\left\|\frac{W\left(f-v_{n,1}(f)\right)}{T_1^{1/4}}\right\|_{L^p(\mathbb{R}^s)}+\sum_{j=2}^{s}\left\|\frac{Wv_n^{[j-1]}\left(\left\{f-v_{n,j}(f)\right\}\big/T_j^{1/4}\right)}{T^{\langle j-1\rangle}}\right\|_{L^p(\mathbb{R}^s)}\right]\\
&\le C\left[\left\|\frac{W\left(f-v_{n,1}(f)\right)}{T_1^{1/4}}\right\|_{L^p(\mathbb{R}^s)}+\sum_{j=2}^{s}\left\|\frac{W\left(f-v_{n,j}(f)\right)}{T_j^{1/4}}\right\|_{L^p(\mathbb{R}^s)}\right]\\
&\le C_1\left\|W_1\omega_{p,1}^*\left(f_{\hat{X}_1},w_1;c_1\frac{a_n^{(1)}}{n}\right)\right\|_{L^p\left(\mathbb{R}_1^{s-1}\right)}+\sum_{j=2}^{s}C_j\left\|W_j\omega_{p,j}^*\left(f_{\hat{X}_j},w_j;c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}\\
&\le\sum_{j=1}^{s}C_j\left\|\left(\prod\nolimits_{1\le i\le s,\,i\ne j}w_i\right)\omega_{p,j}^*\left(f_{\hat{X}_j},w_j;c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}.
\end{aligned}$$

Hence, we have (6.4). #

To prove Theorem 6.2 (6.5) we need the following two theorems.

Theorem 6.3 (cf. [12], Proposition 3.2). Let $W=\prod_{i=1}^{s}w_i$, $w_i\in\mathcal{F}\left(C^2+\right)$, $i=1,2,\cdots,s$. Let f hold Assumption 6.1, especially (6.1) and (6.2) hold. Then there exists a constant $C>0$ such that for every $t>0$ and $i=1,2,\cdots,s$,

$$\begin{aligned}
&\left\|W_i\omega_{1,i}^*\left(f_{\hat{X}_i},w_i;t\right)\right\|_{L^1\left(\mathbb{R}_i^{s-1}\right)}\\
&\le Ct\int_{\mathbb{R}_i^{s-1}}W_i\int_{\mathbb{R}}w_i(x)\left|\mathrm{d}f_{\hat{X}_i}(x)\right|\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_s
\end{aligned} \tag{6.7}$$

and

$$\lim_{t\to 0}\left\|W_i\omega_{\infty,i}^*\left(f_{\hat{X}_i},w_i;t\right)\right\|_{L^\infty\left(\mathbb{R}_i^{s-1}\right)}=0. \tag{6.8}$$

Proof. Let $g_{\hat{X}_i}(x):=f_{\hat{X}_i}(x)-f_{\hat{X}_i}(0)$. For $t>0$, we write $\sigma_i(t)=a_u^{(i)}$, and let $0<h\le t$ and $|x=x_i|\le\sigma_i(2t)<a_u^{(i)}$. Since $\Phi_t(x)\le 2$ for $|x|\le\sigma_i(2t)$, if we take t small enough, then

$$\begin{aligned}
&\int_{|x|\le\sigma_i(2t)}\left|\left\{g_{\hat{X}_i}\left(x+\frac{h}{2}\Phi_t(x)\right)-g_{\hat{X}_i}\left(x-\frac{h}{2}\Phi_t(x)\right)\right\}w(x)\right|\mathrm{d}x\\
&\le\int_{|x|\le\sigma_i(2t)}\int_{x-\frac{h}{2}\Phi_t(x)}^{x+\frac{h}{2}\Phi_t(x)}w_i(y)\left|\mathrm{d}g_{\hat{X}_i}(y)\right|\mathrm{d}x\\
&\le\int_{\mathbb{R}}\int_{x-h}^{x+h}w_i(y)\left|\mathrm{d}g_{\hat{X}_i}(y)\right|\mathrm{d}x\\
&\le\int_{\mathbb{R}}w_i(y)\int_{y-h}^{y+h}\mathrm{d}x\left|\mathrm{d}g_{\hat{X}_i}(y)\right|\le h\int_{\mathbb{R}}w_i(y)\left|\mathrm{d}f_{\hat{X}_i}(y)\right|.
\end{aligned}\tag{6.9}$$

On the other hand, by Proposition 4.2 with $p=1$,

$$\begin{aligned}
&\int_{\sigma_i(4t)}^{\infty}\left|\left\{f_{\hat{X}_i}(x)-f_{\hat{X}_i}(0)\right\}w_i(x)\right|\mathrm{d}x\\
&\le\frac{1}{Q_i'(\sigma_i(4t))}\int_{\sigma_i(4t)}^{\infty}Q_i'(x)\left|g_{\hat{X}_i}(x)\right|w_i(x)\mathrm{d}x\\
&\le Ct\int_0^{\infty}Q_i'(x_i)w_i(x_i)\int_0^{x}\left|\mathrm{d}f_{\hat{X}_i}(y)\right|\mathrm{d}x_i\\
&\le Ct\int_0^{\infty}\left(\int_y^{\infty}Q_i'(x_i)w_i(x_i)\mathrm{d}x\right)\left|\mathrm{d}f_{\hat{X}_i}(y)\right|\\
&=Ct\int_0^{\infty}w_i(y)\left|\mathrm{d}f_{\hat{X}_i}(y)\right|.
\end{aligned}$$

Similarly,

$$\int_{-\infty}^{-\sigma_i(4t)}\left|\left\{f_{\hat{X}_i}(x)-f_{\hat{X}_i}(0)\right\}w_i(x)\right|\mathrm{d}x\le Ct\int_{-\infty}^{0}w_i(y)\left|\mathrm{d}f_{\hat{X}_i}(y)\right|.$$

Therefore, we see

$$\int_{-\infty}^{\infty}\left|\left\{f_{\hat{X}_i}(x)-f_{\hat{X}_i}(0)\right\}w_i(x)\right|\mathrm{d}x\le Ct\int_{-\infty}^{\infty}w_i(y)\left|\mathrm{d}f_{\hat{X}_i}(y)\right|.\tag{6.10}$$

Consequently, from (6.9) and (6.10) we have

$$\begin{aligned}
&\left\|W_i\omega_{1,i}^{*}\left(f_{\hat{X}_i},w_i;t\right)\right\|_{L^1\left(\mathbb{R}_i^{s-1}\right)}\\
&\le Ct\int_{\mathbb{R}_i^{s-1}}W_i\int_{\mathbb{R}}w_i(x)\left|\mathrm{d}f_{\hat{X}_i}(x)\right|\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_s,
\end{aligned}$$

that is, we have (6.7). We show (6.8). Let $\varepsilon>0,|x|\ge L$ for large $L>0$. Then

$$\begin{aligned}
&\left|W_iw_i(x)\left(f_{\hat{X}_i}\left(x+\frac{h}{2}\Phi_t(x)\right)-f_{\hat{X}_i}\left(x-\frac{h}{2}\Phi_t(x)\right)\right)\right|\\
&\le 2C\left|W_iw_i(x)f_{\hat{X}_i}(x)\right|\le 2C\varepsilon.
\end{aligned}$$

Next, for $|x|\le L$ we see

$$\left|W_iw_i(x)\left(f_{\hat{X}_i}\left(x+\frac{h}{2}\Phi_t(x)\right)-f_{\hat{X}_i}\left(x-\frac{h}{2}\Phi_t(x)\right)\right)\right|\to 0\ \text{ as } t\to 0,$$

hence for $0<t\le t_0$ small enough,

$$\left|W_iw_i(x)\left(f_{\hat{X}_i}\left(x+\frac{h}{2}\Phi_t(x)\right)-f_{\hat{X}_i}\left(x-\frac{h}{2}\Phi_t(x)\right)\right)\right|\le\varepsilon.$$

On the other hand,

$$\left\|W_i w_i(x)\left(f_{\hat{X}_i}(x)-f_{\hat{X}_i}(0)\right)\right\|_{L^\infty(|x|\ge\sigma_i(4t))}$$
$$\le\left\|W_i w_i(x) f_{\hat{X}_i}(x)\right\|_{L^\infty(|x|\ge\sigma_i(4t))}+\left|f_{\hat{X}_i}(0)\right|\left\|W_i w_i(x)\right\|_{L^\infty(|x|\ge\sigma_i(4t))}\to 0 \text{ as } t\to 0.$$

Hence for $0<t\le t_0$ small enough,

$$\left\|W(X)\left(f_{\hat{X}_i}(x)-f_{\hat{X}_i}(0)\right)\right\|_{L^\infty(|x|\ge\sigma_i(4t))}\le\varepsilon.$$

Consequently, we have

$$\lim_{t\to 0}\left\|W_i\omega^*_{\infty,i}\left(f_{\hat{X}_i},w_i;t\right)\right\|_{L^\infty\left(\mathbb{R}_i^{s-1}\right)}=0,$$

that is, (6.8). #

Theorem 6.4. Under Assumption 6.1, we have

$$\int_{\mathbb{R}_i^{s-1}}\left|W_i\omega^*_{p,i}\left(f_{\hat{X}_i},w_i:t\right)\right|^p DX_i\le Co_t(1)^{p-1}t,$$

where

$$DX_i:=\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_s,\ i=1,2,\cdots,s$$

and

$$o_t(1):=\left\|W_i\omega^*_{\infty,i}\left(f_{\hat{X}_i},w_i;t\right)\right\|_{L^\infty\left(\mathbb{R}_i^{s-1}\right)}.$$

Proof. Let $g_{\hat{X}_i}(x):=f_{\hat{X}_i}(x)-f_{\hat{X}_i}(0)$.

$$\int_{|x_i|\le\sigma_i(2t)}\left|\left\{g_{\hat{X}_i}\left(x_i+\frac{h}{2}\Phi_t(x_i)\right)-g_{\hat{X}_i}\left(x_i-\frac{h}{2}\Phi_t(x_i)\right)\right\}w_i(x_i)\right|^p \mathrm{d}x_i$$
$$\le\left[\sup_{|x_i|\le\sigma_i(2t)}\left|\left\{g_{\hat{X}_i}\left(x_i+\frac{h}{2}\Phi_t(x_i)\right)-g_{\hat{X}_i}\left(x_i-\frac{h}{2}\Phi_t(x_i)\right)\right\}w_i(x_i)\right|^{p-1}\right.$$
$$\left.+\sup_{|x_i|\ge\sigma_i(4t)}\left|\left\{f_{\hat{X}_i}(x_i)-f_{\hat{X}_i}(0)\right\}w_i(x_i)\right|^{p-1}\right]$$
$$\times\left[\int_{|x_i|\le\sigma_i(2t)}\left|\left\{g_{\hat{X}_i}\left(x_i+\frac{h}{2}\Phi_t(x_i)\right)-g_{\hat{X}_i}\left(x_i-\frac{h}{2}\Phi_t(x_i)\right)\right\}w_i(x_i)\right|\mathrm{d}x_i\right.$$
$$\left.+\int_{|x_i|\ge\sigma_i(4t)}\left|\left\{f_{\hat{X}_i}(x_i)-f_{\hat{X}_i}(0)\right\}w_i(x_i)\right|\mathrm{d}x_i\right]$$
$$\le\left[\sup_{|x_i|\le\sigma_i(2t)}\left|\left\{g_{\hat{X}_i}\left(x_i+\frac{h}{2}\Phi_t(x_i)\right)-g_{\hat{X}_i}\left(x_i-\frac{h}{2}\Phi_t(x_i)\right)\right\}w_i(x_i)\right|\right.$$
$$\left.+\sup_{|x_i|\ge\sigma_i(4t)}\left|\left\{f_{\hat{X}_i}(x_i)-f_{\hat{X}_i}(0)\right\}w_i(x_i)\right|\right]^{p-1}$$
$$\times\left[\int_{|x_i|\le\sigma_i(2t)}\left|\left\{g_{\hat{X}_i}\left(x_i+\frac{h}{2}\Phi_t(x_i)\right)-g_{\hat{X}_i}\left(x_i-\frac{h}{2}\Phi_t(x_i)\right)\right\}w_i(x_i)\right|\mathrm{d}x_i\right.$$
$$\left.+\int_{|x_i|\ge\sigma_i(4t)}\left|\left\{f_{\hat{X}_i}(x_i)-f_{\hat{X}_i}(0)\right\}w_i(x_i)\right|\mathrm{d}x_i\right]$$
$$=\omega^*_{\infty,i}\left(f_{\hat{X}_i},w_i;t\right)^{p-1}\times\omega^*_{1,i}\left(f_{\hat{X}_i},w_i;t\right).$$

Similarly,

$$\begin{aligned}
&\int_{|x_i|\ge\sigma_i(4t)}\left|\left\{f_{\hat{X}_i}(x_i)-f_{\hat{X}_i}(0)\right\}w_i(x_i)\right|^p \mathrm{d}x_i \\
&\le\Bigg[\sup_{|x_i|\ge\sigma_i(4t)}\left|\left\{f_{\hat{X}_i}(x_i)-f_{\hat{X}_i}(0)\right\}w_i(x_i)\right|^{p-1} \\
&\quad+\sup_{|x_i|\le\sigma_i(2t)}\left|\left\{g_{\hat{X}_i}\left(x_i+\frac{h}{2}\Phi_t(x_i)\right)-g_{\hat{X}_i}\left(x_i-\frac{h}{2}\Phi_t(x_i)\right)\right\}w_i(x_i)\right|^{p-1}\Bigg] \\
&\quad\times\Bigg[\sup_{|x_i|\ge\sigma_i(4t)}\left|\left\{f_{\hat{X}_i}(x_i)-f_{\hat{X}_i}(0)\right\}w_i(x_i)\right| \\
&\quad+\int_{|x_i|\le\sigma_i(2t)}\left|\left\{g_{\hat{X}_i}\left(x_i+\frac{h}{2}\Phi_t(x_i)\right)-g_{\hat{X}_i}\left(x_i-\frac{h}{2}\Phi_t(x_i)\right)\right\}w_i(x_i)\right|\mathrm{d}x_i\Bigg] \\
&\le\Bigg[\sup_{|x_i|\ge\sigma_i(4t)}\left|\left\{f_{\hat{X}_i}(x_i)-f_{\hat{X}_i}(0)\right\}w_i(x_i)\right| \\
&\quad+\sup_{|x_i|\le\sigma_i(2t)}\left|\left\{g_{\hat{X}_i}\left(x_i+\frac{h}{2}\Phi_t(x_i)\right)-g_{\hat{X}_i}\left(x_i-\frac{h}{2}\Phi_t(x_i)\right)\right\}w_i(x_i)\right|\Bigg]^{p-1} \\
&\quad\times\Bigg[\int_{|x_i|\ge\sigma_i(4t)}\left|\left\{f_{\hat{X}_i}(x_i)-f_{\hat{X}_i}(0)\right\}w_i(x_i)\right|\mathrm{d}x_i \\
&\quad+\int_{|x_i|\le\sigma_i(2t)}\left|\left\{g_{\hat{X}_i}\left(x_i+\frac{h}{2}\Phi_t(x_i)\right)-g_{\hat{X}_i}\left(x_i-\frac{h}{2}\Phi_t(x_i)\right)\right\}w_i(x_i)\right|\mathrm{d}x_i\Bigg] \\
&=\omega^*_{\infty,i}\left(f_{\hat{X}_i},w_i;t\right)^{p-1}\times\omega^*_{1,i}\left(f_{\hat{X}_i},w_i;t\right).
\end{aligned}$$

Hence,

$$\omega^*_{p,i}\left(f_{\hat{X}_i},w_i,t\right)\le C\left(\omega^*_{\infty,i}\left(f_{\hat{X}_i},w_i,t\right)^{p-1}\times\omega^*_{1,i}\left(f_{\hat{X}_i},w_i,t\right)\right),$$

Consequently, we have

$$\begin{aligned}
&\int_{\mathbb{R}_i^{s-1}}\left|W_i\omega^*_{p,i}\left(f_{\hat{X}_i},w_i;t\right)\right|^p DX_i \\
&\le C\int_{\mathbb{R}_i^{s-1}}\left\{W_i\omega^*_{\infty,i}\left(f_{\hat{X}_i},w_i;t\right)\right\}^{p-1}\left\{W_i\omega^*_{1,i}\left(f_{\hat{X}_i},w_i;t\right)\right\}DX_i \\
&\le C\left\|W_i\omega^*_{\infty,i}\left(f_{\hat{X}_i},w_i;t\right)\right\|^{p-1}_{L^\infty\left(\mathbb{R}_i^{s-1}\right)}\int_{\mathbb{R}_i^{s-1}}W_i\omega^*_{1,i}\left(f_{\hat{X}_i},w_i;t\right)DX_i \\
&\le Co_t(1)^{p-1}t,
\end{aligned}$$

where

$$o_t(1):=\left\|W_i\omega^*_{\infty,i}\left(f_{\hat{X}_i},w_i;t\right)\right\|_{L^\infty\left(\mathbb{R}_i^{s-1}\right)}. \qquad \#$$

Proof of Theorem 6.2 (6.5). Using Theorems 6.3 and 6.4 with $t=a_n/n$, we easily obtain (6.5). #

We will give an analogy of Theorem 6.2. To do so we use the following weights . They are guaranteed by Proposition 2.3.

$$w_i(x_i)T_i^{1/4}(x_i) \sim w_i^{\$} \in \mathcal{F}(C^2+), i=1,2,\cdots,s,$$
$$W^{\$} := \prod_{i=1}^{s} w_i^{\$},\ W_i^{\$} := \prod_{1\le j\le s, j\ne i} w_j^{\$}. \tag{6.11}$$

Assumption 6.5. Let $w_i \in \mathcal{F}(C^3+), 0<\lambda<3/2\,(i=1,\cdots,s)$. Suppose that f is continuous on $\mathbb{R}^s$, and $f_{\hat{X}_i}(x)$ has a bounded variation on any compact interval in $\mathbb{R}^s$ with

$$\int_{\mathbb{R}_i^{s-1}} W_i^{\$} \int_{\mathbb{R}} w_i^{\$}(x)\left|\mathrm{d}f_{\hat{X}_i}(x)\right| \mathrm{d}x_1\cdots\mathrm{d}_{x_{i-1}}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_s < \infty, \quad i=1,2,\cdots,s, \tag{6.12}$$

and if $p=\infty$, then we further suppose

$$\left|W^{\$}(X)\, f_{\hat{X}_i}(x_i)\right| \to 0,\ \text{as} \left|x_i\right| \to \infty, i=1,2,\cdots,s, \tag{6.13}$$

where the weights $w_i^{\$}, W^{\$}$ and $W_i^{\$}$ are defined by (6.11).

Theorem 6.6. We suppose $w_j = \exp(-Q_j) \in \mathcal{F}_\lambda(C^3+)(0<\lambda<3/2),\ j=1,2,\cdots,s$, and let

$$T_j\left(a_n^{(j)}\right) \le c\left(\frac{n}{a_n^{(j)}}\right)^{\frac{2}{3}},\ j=1,2,\cdots,s. \tag{6.14}$$

Let $n \ge 1, 1 \le p \le \infty$. Then we have

$$\left\|W\left(f - v_n^{(s)}(f)\right)\right\|_{L^p(\mathbb{R}^s)} \le \sum_{j=1}^{s} C_j \left\| W_j\left(\prod_{k=1}^{j-1} T_k^{1/4}\right)\omega_{p,j}^*\left(f_{\hat{X}_j}, T_j^{1/4}w_j; c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}. \tag{6.15}$$

Now Assumption 6.5 holds. Then we have

$$\sum_{j=1}^{s} C_j \left\| W_j\left(\prod_{k=1}^{j-1} T_k^{1/4}\right)\omega_{p,j}^*\left(f_{\hat{X}_j}, T_j^{1/4}w_j; c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)} \le C o_n(1)^{1-1/p}\left(\frac{a_n}{n}\right)^{\frac{1}{p}}. \tag{6.16}$$

Proof of (6.15). By Proposition 3.3 (2) and Proposition 3.4, we get

$$\left\|w_1(x)\left(f_{\hat{X}_1} - v_{n,1}\left(f_{\hat{X}_1}\right)\right)\right\|_{L^p(\mathbb{R})} \le CE_{p,n}\left(f_{\hat{X}_1}; T_1^{1/4}w_1\right) \le C_1\omega_{p,1}^*\left(f_{\hat{X}_1}, T_1^{1/4}w_1; c_1\frac{a_n^{(1)}}{n}\right),$$

where the constant C_1 and c_1 are independent of $\hat{X}_1$. Similarly, for $j=1,2,\cdots,s$,

$$\left\|w_j(x)\left(f_{\hat{X}_j} - v_{n,j}\left(f_{\hat{X}_j}\right)\right)\right\|_{L^p(\mathbb{R})} \le C_j\omega_{p,j}^*\left(f_{\hat{X}_j}, T_j^{1/4}w_j; c_j\frac{a_n^{(j)}}{n}\right). \tag{6.17}$$

Using $f - v_n^{[s]} = \left(f - v_n^{[1]}(f)\right) + \left(v_n^{[1]}(f) - v_n^{[2]}(f)\right) + \cdots + \left(v_n^{[s-1]}(f) - v_n^{[s]}(f)\right)$, we get from Lemma 3.7 (2) and (6.17),

$$\begin{aligned}
&\left\|W\left(f-v_n^{[s]}(f)\right)\right\|_{L^p(\mathbb{R}^s)} \\
&\le \left\|W\left(f-v_n^{[1]}(f)\right)\right\|_{L^p(\mathbb{R}^s)}+\sum_{j=2}^{s}\left\|W\left(v_n^{[j-1]}(f)-v_n^{[j]}(f)\right)\right\|_{L^p(\mathbb{R}^s)} \\
&\le C\left[\left\|W\left(f-v_{n,1}(f)\right)\right\|_{L^p(\mathbb{R}^s)}+\sum_{j=2}^{s}\left\|Wv_n^{[j-1]}\left(f-v_{n,j}(f)\right)\right\|_{L^p(\mathbb{R}^s)}\right] \\
&\le C\left[\left\|W\left(f-v_{n,1}(f)\right)\right\|_{L^p(\mathbb{R}^s)}+\sum_{j=2}^{s}\left\|W\left(\prod_{k=1}^{j-1}T_k^{1/4}\right)\left(f-v_{n,j}(f)\right)\right\|_{L^p(\mathbb{R}^s)}\right] \\
&\le C_1\left\|W_1\omega_{p,1}^*\left(f_{\hat{X}_1},T_1^{1/4}w_1;c_1\frac{a_n^{(1)}}{n}\right)\right\|_{L^p(\mathbb{R}_j^{s-1})} \\
&\quad+\sum_{j=2}^{s}C_j\left\|W_j\left(\prod_{k=1}^{j-1}T_k^{1/4}\right)\omega_{p,j}^*\left(f_{\hat{X}_j},T_j^{1/4}w_j;c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p(\mathbb{R}_j^{s-1})}
\end{aligned}$$

by (3.11)

$$\le\sum_{j=1}^{s}C_j\left\|W_j\left(\prod_{k=1}^{j-1}T_k^{1/4}\right)\omega_{p,j}^*\left(f_{\hat{X}_j},T_j^{1/4}w_j;c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p(\mathbb{R}_j^{s-1})},$$

where $\prod_{k=1}^{j-1}T_k^{1/4}=1$ for $j=1$, that is, we have (6.15).

(6.16) follows from the following theorems. #

In Theorems 6.3 and 6.4 we replace w_i, W with $w_i^{\$}, W^{\$}$ of (6.11), where $w_i\in\mathcal{F}_\lambda\left(C^3+\right), 0<\lambda<3/2\,(i=1,2,\cdots,s)$, then we easily have the following Theorem 6.7.

Theorem 6.7 (cf. [Theorem 6.3 in this paper]). Let $W^{\$}=\prod_{i=1}^{s}w_i^{\$}$, $w_i^{\$}\in\mathcal{F}\left(C^2+\right), i=1,2,\cdots,s$. Let f hold Assumption 6.5, especially (6.18) and (6.19) hold. Then there exists a constant $C>0$ such that for every $t>0$ and $i=1,2,\cdots,s$,

$$\begin{aligned}
&\left\|W_i^{\$}\omega_{1,i}^*\left(f_{\hat{X}_i},w_i^{\$};t\right)\right\|_{L^1(\mathbb{R}_i^{s-1})} \\
&\le Ct\int_{\mathbb{R}_i^{s-1}}W_i^{\$}\int_{\mathbb{R}}w_i^{\$}(x)\left|\mathrm{d}f_{\hat{X}_i}(x)\right|\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_s,
\end{aligned}\tag{6.18}$$

and

$$\lim_{t\to 0}\left\|W_i^{\$}\omega_{\infty,i}^*\left(f_{\hat{X}_i},w_i^{\$};t\right)\right\|_{L^\infty(\mathbb{R}_i^{s-1})}=0.\tag{6.19}$$

Theorem 6.8. Under Assumption 6.5, we have

$$\int_{\mathbb{R}_i^{s-1}}\left|W_i^{\$}\omega_{p,i}^*\left(f_{\hat{X}_i},w_i^{\$};t\right)\right|^p DX_i\le Co_t(1)^{p-1}t,$$

where

$$DX_i:=\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_s,\ i=1,2,\cdots,s$$

and

$$o_t(1) := \left\| W_i^{\$} \omega_{\infty,i}^{*}\left(f_{\hat{X}_i}, w_i^{\$}; t \right) \right\|_{L^\infty\left(\mathbb{R}_i^{s-1}\right)}.$$

7. Approximation for Functions of the Lipschitz Class

Through this section we consider the weight $w = \exp(-Q) \in \mathcal{F}_\lambda\left(C^3+\right)$.

Theorem 7.1. (1) We suppose $w_j = \exp\left(-Q_j\right) \in \mathcal{F}_\lambda\left(C^3+\right)\left(0<\lambda<3/2\right)$, $j=1,2,\cdots,s$, and let (3.6) hold. Let $n \geq 1, 1 \leq p \leq \infty$. Then we have

$$\begin{aligned} &\left\| W\left(f - v_n^{[s]}(f) \right) \right\|_{L^p\left(\mathbb{R}^s\right)} \\ &\leq \sum_{j=1}^{s} C_j \left\| W_j \left(\prod_{k=1}^{j-1} T_k^{1/4} \right) \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, T_j^{1/4} w_j; c_j \frac{a_n^{(j)}}{n} \right) \right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}, \end{aligned} \tag{7.1}$$

where $\mathbb{R}_j^{s-1}$ is defined in (3.8). Now, we suppose that f is continuous on $\mathbb{R}^s$. Let $\left\| W^\delta f \right\|_{L^\infty(\mathbb{R})} < \infty$ for some $0<\delta<1$, and let $W^\delta f \in Lip(\alpha)$ for some $0<\alpha\leq 1$, that is,

$$\left| W^\delta f\left(X_1\right) - W^\delta f\left(X_2\right) \right| \leq \left| X_1 - X_2 \right|^\alpha. \tag{7.2}$$

Then we have

$$\begin{aligned} &\sum_{j=1}^{s} C_j \left\| W_j \left(\prod_{k=1}^{j-1} T_k^{1/4} \right) \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, T_j^{1/4} w_j; c_j \frac{a_n^{(j)}}{n} \right) \right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)} \\ &\leq C \sum_{i=1}^{s} \left(\frac{a_n^{(i)}}{n} \right)^\alpha. \end{aligned} \tag{7.3}$$

(2) We suppose $w_j = \exp\left(-Q_j\right) \in \mathcal{F}\left(C^2+\right)$, $j=1,2,\cdots,s$, and hold (3.6). Let $n \geq 1, 1 \leq p \leq \infty$. Then we have

$$\left\| \frac{W\left(f - v_n^{[s]}(f) \right)}{\prod_{k=1}^{s} T_k^{1/4}} \right\|_{L^p\left(\mathbb{R}^s\right)} \leq \sum_{j=1}^{s} C_j \left\| W_j \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, w_j; c_j \frac{a_n^{(j)}}{n} \right) \right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}. \tag{7.4}$$

Now, we suppose that f has the conditions as (1). Then we have

$$\sum_{j=1}^{s} C_j \left\| W_j \bar{\omega}_{p,j}\left(f_{\hat{X}_j}, w_j; c_j \frac{a_n^{(j)}}{n} \right) \right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)} \leq C \sum_{i=1}^{s} \left(\frac{a_n^{(i)}}{n} \right)^\alpha. \tag{7.5}$$

Proof. We suppose $0<\delta<1$.

(1) (7.2) follows from (3.7). We will show (7.3). Now, we use $T^{\langle j-1 \rangle} := \prod_{i=1}^{j-1} T_i^{1/4}$, where $T^{\langle 0 \rangle} = 1$, and $W_j := \prod_{1 \leq i \leq s, i \neq j} w_i$. We will estimate

$$\left\| W_j T^{\langle j-1 \rangle} \left(\frac{1}{t} \int_0^t \left\| T_j^{1/4}(x) w_j(x) \left(\Delta_{h\Phi_{t,j}(x)} f_{\hat{X}_j}(x) \right) \right\|_{L^p\left(|x| \leq \sigma_j(2t)\right)}^p \mathrm{d}h \right)^{1/p} \right\|_{L^p\left(\mathbb{R}_j^{s-1}\right)}.$$

From $W^{\delta} f \in Lip(\alpha)$ and $w_j(x) \sim w\left(x+(h/2)\Phi_{t,j}(x)\right) \sim w\left(x-(h/2)\Phi_{t,j}(x)\right)$ (see[3], Lemma 7) we have

$$\left(\frac{1}{t}\int_0^t \left\|T_j^{1/4}(x) w_j(x) \Delta_{h\Phi_{t,j}(x)} f_{\hat{X}_j}(x)\right\|_{L^p(|x|\le\sigma_j(2t))}^p \mathrm{d}h\right)^{1/p}$$

$$= \left(\frac{1}{t}\int_0^t \left\|T_j^{1/4}(x) w_j(x)\left\{ f_{\hat{X}_j}\left(x+\frac{h}{2}\Phi_{t,j}(x)\right) - f_{\hat{X}_j}\left(x-\frac{h}{2}\Phi_{t,j}(x)\right)\right\}\right\|_{L^p(|x|\le\sigma_j(2t))}^p \mathrm{d}h\right)^{1/p}$$

$$\le \left(\frac{1}{t}\int_0^t \left\| \frac{T_j^{1/4}(x) w_j(x)}{w_j^{\delta}\left(x+\frac{h}{2}\Phi_{t,j}(x)\right)}\left[\left| w_j^{\delta}\left(x+\frac{h}{2}\Phi_{t,j}(x)\right) f_{\hat{X}}\left(x+\frac{h}{2}\Phi_{t,j}(x)\right) - w_j^{\delta}\left(x-\frac{h}{2}\Phi_{t,j}(x)\right) f_{\hat{X}}\left(x-\frac{h}{2}\Phi_{t,j}(x)\right)\right|\right.\right.\right.$$

$$\left.\left.\left. + \left|\left\{ w_j^{\delta}\left(x+\frac{h}{2}\Phi_{t,j}(x)\right) - w_j^{\delta}\left(x-\frac{h}{2}\Phi_{t,j}(x)\right)\right\} f_{\hat{X}_j}\left(x-\frac{h}{2}\Phi_{t,j}(x)\right)\right|\right]\right\|_{L^p(|x|\le\sigma_j(2t))}^p \mathrm{d}h\right)^{1/p}$$

$$\le C\left(\frac{1}{t}\int_0^t \left\| T_j^{1/4}(x) w_j^{(1-\delta)}(x)\left[\left| w_j^{\delta}\left(x+\frac{h}{2}\Phi_{t,j}(x)\right) f_{\hat{X}}\left(x+\frac{h}{2}\Phi_{t,j}(x)\right) - w_j^{\delta}\left(x-\frac{h}{2}\Phi_{t,j}(x)\right) f_{\hat{X}}\left(x-\frac{h}{2}\Phi_{t,j}(x)\right)\right|\right.\right.\right.$$

$$\left.\left.\left. + \left|\left\{ w_j^{\delta}\left(x+\frac{h}{2}\Phi_{t,j}(x)\right) - w_j^{\delta}\left(x-\frac{h}{2}\Phi_{t,j}(x)\right)\right\} f_{\hat{X}_j}\left(x-\frac{h}{2}\Phi_{t,j}(x)\right)\right|\right]\right\|_{L^p(|x|\le\sigma_j(2t))}^p \mathrm{d}h\right)^{1/p}$$

$$=: I_1 + I_2.$$

By (7.2) we see

$$\left\|W_j T^{\langle j-1\rangle} I_1\right\|_{L^p(\mathbb{R}^{s-1})} = \left\| W_j^{1-\delta} T^{\langle j-1\rangle}\left(\frac{1}{t}\int_0^t \left\|\left(W_{\hat{X}_j}^{\delta} f_{\hat{X}_j}\right)\left(x+\frac{h}{2}\Phi_{t,j}(x)\right)\right.\right.\right.$$

$$\left.\left.\left. -\left(W_{\hat{X}_j}^{\delta} f_{\hat{X}_j}\right)\left(x-\frac{h}{2}\Phi_{t,j}(x)\right)\right\|_{L^p(|x|\le\sigma_j(2t))}^p \mathrm{d}h\right)^{1/p}\right\|_{L^p(\mathbb{R}^{s-1})}$$

$$\le C\left\| W_j^{1-\delta} T^{\langle j-1\rangle}\left(\frac{1}{t}\int_0^t h^{p\alpha}\mathrm{d}h\right)^{1/p}\right\|_{L^p(\mathbb{R}^{s-1})}$$

$$\le t^{\alpha}\left\| W_j^{1-\delta} T^{\langle j-1\rangle}\right\|_{L^p(\mathbb{R}^{s-1})} \le Ct^{\alpha}.$$

On the other hand, for I_2 we estimate

$$= C\left\| W_j T^{\langle j-1\rangle}\left(\frac{1}{t}\int_0^t \left\| T_j^{1/4}(x) w_j^{(1-\delta)/2}(x) h\Phi_{t,j}(x) w_j^{\delta}(\xi)\right.\right.\right.$$

$$\left.\left.\left. \times \delta Q_j'(\xi) f_{\hat{X}_j}\left(x-\frac{h}{2}\Phi_{t,j}(x)\right)\right\|_{L^p(|x|\le\sigma_j(2t))}^p \mathrm{d}h\right)^{1/p}\right\|_{L^p(\mathbb{R}^{s-1})},$$

$$x-\frac{h}{2}\Phi_{t,j}(x) < \xi < x+\frac{h}{2}\Phi_{t,j}(x),$$

$$\le C\left\| W_j^{1-\delta} T^{\langle j-1\rangle}\left(\frac{1}{t}\int_0^t h^p \left| w_j^{(1-\delta)/2}(x) Q_j'(\xi)\left(W_{\hat{X}_j}^{\delta} f_{\hat{X}_j}\right)\left(x-\frac{h}{2}\Phi_{t,j}(x)\right)\right| \mathrm{d}h\right)^{1/p}\right\|_{L^p(\mathbb{R}^s)},$$

by the boundedness of $T_j^{1/4}(x)w_j^{(1-\delta)/2}(x)$

$$\le Ct\left\|W^\delta f\right\|_{L^p(\mathbb{R}^s)},$$

by the boundedness of $w_j^{(1-\delta)/2}(x)\left|Q_j'(\xi)\right|$ (note $\left|Q'\right|\le Q^C$) and Lemma 3.10

$$\le Ct^\alpha \quad \left(\text{by } w^\delta f\in L^p(\mathbb{R})\right).$$

Hence we conclude

$$\left\|W_j T^{\langle j-1\rangle}\left(\frac{1}{t}\int_0^t\left\|T_j^{1/4}(x)w_j(x)\left(\Delta_{h\Phi_{t,j}(x)}f_{\hat{X}_j}(x)\right)\right\|_{L^p(|x|\le\sigma_j(2t))}^p \mathrm{d}h\right)^{1/p}\right\|_{L^p(\mathbb{R}_j^{s-1})}\le Ct^\alpha. \quad (7.6)$$

Now, we see

$$\left\|W_j T^{\langle j-1\rangle}\inf_{c_j(\text{constant})}\left\|\left(f_{\hat{X}_j}(x)-c_j\right)w_j(x)\right\|_{L^p(\mathbb{R}\setminus[-\sigma_j(4t),\sigma_j(4t)])}\right\|_{L^p(\mathbb{R}_j^{s-1})}$$

$$\le\left\|W_j T^{\langle j-1\rangle}\left\|f_{\hat{X}_j}(x)w_j(x)\right\|_{L^p(\mathbb{R}\setminus[-\sigma_j(4t),\sigma_j(4t)])}\right\|_{L^p(\mathbb{R}_j^{s-1})}$$

$$\le w_j^{1-\delta}\left(\sigma_j(4t)\right)\left\|W^\delta f\right\|_{L^p(\mathbb{R}^s)}\le Cw_j^{1-\delta}\left(\sigma_j(4t)\right).$$

Here, if we set $4t=a_u/u$, then we see

$$w_j^{1-\delta}\left(\sigma_j(4t)\right)=\exp\left(-(1-\delta)Q_j(a_u)\right)\sim\exp\left(-C\frac{u}{\sqrt{T(a_u)}}\right)\le \mathrm{e}^{-u^\eta}$$

for some $0<\eta<1$, that is,

$$w_j^{1-\delta}\left(\sigma_j(4t)\right)\le C\mathrm{e}^{-u^\delta}\le C\frac{a_u}{4u}=Ct.$$

Therefore, we have

$$\left\|W_j T^{\langle j-1\rangle}\inf_{c_j(\text{constant})}\left\|\left(f_{\hat{X}_j}(x)-c_j\right)w_j(x)\right\|_{L^p(\mathbb{R}\setminus[-\sigma_j(4t),\sigma_j(4t)])}\right\|_{L^p(\mathbb{R}_j^{s-1})}\le Ct. \quad (7.7)$$

Consequently, by (7.6) and (7.7) we have

$$\left\|W_j T^{\langle j-1\rangle}\bar{\omega}_{p,j}\left(f_{\hat{X}_j},w_j;t\right)\right\|_{L^p(\mathbb{R}_j^{s-1})}\le Ct^\alpha.$$

So we have (7.3), that is,

$$\sum_{j=1}^s C_j\left\|W_j T^{\langle j-1\rangle}\bar{\omega}_{p,j}\left(f_{\hat{X}_j},w_j;c_j\frac{a_n^{(j)}}{n}\right)\right\|_{L^p(\mathbb{R}_j^{s-1})}\le C\sum_{i=1}^s\left(\frac{a_n^{(i)}}{n}\right)^\alpha.$$

(2) (7.4) follows from (3.7). The estimate (7.5) follows as (1). We omit the proof. #

8. Approximation for Differentiable Functions

In this section, we treat the differentiable functions.

Let $s>1, r\ge 1$ be fixed integers, and let $w_j \in \mathcal{F}_\lambda\left(C^3+\right), j=1,\cdots,s$. We suppose that the multivariate function $f\left(x_1,\cdots,x_s\right)$ is r-times partial differentiable, and then with norm:

$$\|Wf\|_{\mathcal{W}_{r,p}} := \sum_{i=1}^{s}\left\|w_i T_i^{1/4} D_i^r f\right\|_{L^p\left(\mathbb{R}^s\right)} < \infty,$$

where

$$D_i^r f := \frac{\partial^r f}{\partial^r x_i},\ i=1,2,\cdots,s.$$

The class of all functions $f\left(x_1,\cdots,x_s\right)$ with $\|Wf\|_{\mathcal{W}_{r,p}}<\infty$ will be denoted by $\mathcal{W}_{r,p}$. In the sequel, if $1\le i\le s$ is an integer, then $\|f\|_{L_{p,i}(\mathbb{R})}$ will denote the L_p-norm of f taken with respect to the i-th variable.

Theorem 8.1. We suppose $w_j=\exp\left(-Q_j\right)\in \mathcal{F}_\lambda\left(C^3+\right)$ and $r\ge 1$ is an integer. Let $n\ge 1, 1\le p\le\infty$, and let $Wf\in\mathcal{W}_{r,p}$. Then we have

$$\left\|W\left(f-v_n^{[s]}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)} \le C\left(\frac{a_n}{n}\right)^r \|Wf\|_{\mathcal{W}_{r,p}}, \tag{8.1}$$

where $a_n=\max\left\{a_n^{(i)}, i=1,2,\cdots,s\right\}$.

Remark 8.2. Especially, for $r=1$ we have

$$\left\|W\left(f-v_n^{[s]}(f)\right)\right\|_{L^p\left(\mathbb{R}^s\right)} \le C\left(\frac{a_n}{n}\right)\sum_{i=1}^{s}\left\|w_i T_i^{1/4} D_i^1 f\right\|_{L^p\left(\mathbb{R}^s\right)}.$$

Theorem 8.3 ([7], Cororally 8). We suppose $w\in\mathcal{F}\left(C^2+\right)$. Let $1\le p\le\infty$ and $r(\ge 1)$ is an integer. If $g^{(r-1)}$ be absolutely continuous and $wg^{(r)}\in L^p(\mathbb{R})$, then we have

$$E_{p,n}(g)\le C\left(\frac{a_n}{n}\right)^r \left\|wg^{(r)}\right\|_{L^p(\mathbb{R})}.$$

Equivalently,

$$E_{p,n}(g)\le C\left(\frac{a_n}{n}\right)^r E_{p,n-1}\left(g^{(r)}\right).$$

Proof of Theorem 8.1. We use Proposition 2.3, that is, $T^\alpha w\sim w_\alpha\in\mathcal{F}\left(C^2+\right)$. In view of Theorem 3.3 (1) and a repeated application of Theorem 8.3, we get

$$\begin{aligned}
&\left\|w_1(x)\left(f_{\hat{X}_1}-v_n^{[s]}\left(f_{\hat{X}_1}\right)\right)\right\|_{L^p(\mathbb{R})}\\
&\le E_{p,n}\left(T_1^{1/4}w_1, f_{\hat{X}_1}\right)\le C\left(\frac{a_n^{(1)}}{n}\right)^r\left\|T_1^{1/4}(x)w_1(x)f_{\hat{X}_1}^{(r)}\right\|_{L^p(\mathbb{R})}\\
&=C_1\left(\frac{a_n^{(1)}}{n}\right)^r\left\|w_1(x)T_1^{1/4}(x)D_1^r f\right\|_{L^p(\mathbb{R})},
\end{aligned}$$

where the constant C_1 is independent of $\hat{X}_1$, and D_1 denotes differentiation with respect to the first variable. Similarly, for $j = 1, 2, \cdots, s$,

$$\left\| w_j \left(f_{\hat{X}_j} - v_{n,j}\left(f_{\hat{X}_j} \right) \right) \right\|_{L^p(\mathbb{R})} \le C_j \left(\frac{a_n^{(j)}}{n} \right)^r \left\| w_j T_j^{1/4} D_j^r f \right\|_{L^p(\mathbb{R})}, \tag{8.2}$$

where D_j^r denotes the derivative with respect to the *j*-th variable.

Using Lemma 3.7 and (8.2), we obtain for integer $j \ge 2$,

$$\begin{aligned} &\left\| W\left(v_n^{[j-1]}(f) - v_n^{[j]}(f) \right) \right\|_{L^p(\mathbb{R}^s)} = \left\| W v_n^{[j-1]}\left(f - v_{n,j}(f) \right) \right\|_{L^p(\mathbb{R}^s)} \\ &\le C \left\| W \left(\prod_{i=1}^{j-1} T_i^{1/4} \right) \left(f - v_{n,j}(f) \right) \right\|_{L^p(\mathbb{R}^s)} \le \left(\frac{a_n^{(j)}}{n} \right)^r \left\| w_j T_j^{1/4} D_j^r f \right\|_{L^p(\mathbb{R}^s)}. \end{aligned} \tag{8.3}$$

From (8.2) and (8.3), we get

$$\begin{aligned} &\left\| W\left(f - v_n^{[s]}(f) \right) \right\|_{L^p(\mathbb{R}^s)} \\ &= \left\| W\left(f - v_n^{[1]}(f) \right) \right\|_{L^p(\mathbb{R}^s)} + \sum_{j=2}^{s} \left\| W\left(v_n^{[j-1]}(f) - v_n^{[j]}(f) \right) \right\|_{L^p(\mathbb{R}^s)} \\ &\le C \left(\frac{a_n}{n} \right)^r \sum_{j=1}^{s} \left\| W \left(\prod_{i=1}^{j} T_i^{1/4}(x_i) \right) D_j^r f \right\|_{L^p(\mathbb{R}^s)}. \end{aligned}$$

Therefore, we conclude

$$\left\| W\left(f - v_n^{[s]}(f) \right) \right\|_{L^p(\mathbb{R}^s)} \le C \left(\frac{a_n}{n} \right)^r \sum_{j=1}^{s} \left\| W T^{\langle s \rangle} D_j^{(r)} f \right\|_{L^p(\mathbb{R}^s)} = C \left(\frac{a_n}{n} \right)^r \left\| Wf \right\|_{W_{r,p}}. \quad \#$$

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Appendix

In this Appendix we state two inequalities which play important roles in the study of approximation theory. In fact, we use Theorem A_1 in the proof of Theorem 5.2. Let $a_n^{(i)}$ be the MRS number of $w_i = \exp(-Q_i)$.

Theorem A_1 (Markov-Bernstein inequalities). Let $0 < p \le \infty$, and let $r_1, \cdots, r_s \ge 0$ be integers. There exists $C \ne C(n, P) > 0$ such that for $n \ge 1$ and $P \in \mathcal{P}_{n;s}(\mathbb{R}^s)$.

(1) if $w_i \in \mathcal{F}(C^2+)$, then we have

$$\left\| P(X)^{(r_1,\cdots,r_j)} W(X) \right\|_{L^p(\mathbb{R}^s)} \le C \prod_{i=1}^s \left(\frac{n \left(T_i\left(a_n^{(i)}\right) \right)^{1/2}}{a_n^{(i)}} \right)^{r_i} \left\| P(X) W(X) \right\|_{L^p(\mathbb{R}^s)}.$$

(2) if $w_i \in \mathcal{F}_\lambda(C^3+)$, then we have

$$\left\| \frac{P(X)^{(r_1,\cdots,r_j)} W(X)}{\prod_{i=1}^s T_i^{r_i/2}} \right\|_{L^p(\mathbb{R}^s)} \le C \prod_{i=1}^s \left(\frac{n}{a_n^{(i)}} \right)^{r_i} \left\| P(X) W(X) \right\|_{L^p(\mathbb{R}^s)}.$$

The following, so called, Nikolskee-type inequality is useful.

Theorem A_2 (Nikolskii-type inequality). Let $w_i = \exp(-Q) \in \mathcal{F}_\lambda(C^3+)(0 < \lambda < 3/2)$, and let $P \in \mathcal{P}_{n;s}$. For $0 < p \le q \le \infty$, we have

$$\|WP\|_{L_p(\mathbb{R}^s)} \le C \prod_{i=1}^s \left(a_n^{(i)}\right)^{\frac{1}{p}-\frac{1}{q}} \|WP\|_{L_q(\mathbb{R}^s)},$$

and for $1 \le q < p \le \infty$, we have

$$\left\| \prod_{i=1}^s T_i^{\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} WP \right\|_{L_p(\mathbb{R}^s)} \le C \prod_{i=1}^s \left(\frac{n}{a_n^{(i)}} \right)^{\frac{1}{q}-\frac{1}{p}} \|WP\|_{L_q(\mathbb{R})}.$$

To prove Theorem A_1 we need the Proposition 5.3.

Proof of Theorem A_1. To prove (1), we use the second inequality in Proposition 5.3, repeatedly. Then we easily obtain the result.

(2) Using the first inequality in Proposition 5.3, repeatedly, we have the result. #

The proof of Theorem A_2 is obtained by repeatedly using the following proposition.

Proposition A_3 ([8], Theorem 18). Let $w = \exp(-Q) \in \mathcal{F}_\lambda(C^3+)(0 < \lambda < 3/2)$, and let $P \in \mathcal{P}_n$. For $0 < p \le q \le \infty$, we have

$$\|wP\|_{L_p(\mathbb{R})} \le C a_n^{\frac{1}{p}-\frac{1}{q}} \|wP\|_{L_q(\mathbb{R})},$$

and for $1 \le q < p \le \infty$, we have

$$\left\| T^{\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} wP \right\|_{L_p(\mathbb{R})} \le C \left(\frac{n}{a_n} \right)^{\frac{1}{q}-\frac{1}{p}} \|wP\|_{L_q(\mathbb{R})}.$$

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