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Criteria for Maximizing Jobs in Imperfect Production Centers

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Abstract

The imperfect production center complexity to do with job maximization strategies is shown to have some criteria under which an optimal solution exists.
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Keywords

M/M/1 Queue, Governing Equations, Modified Equations, Group Structure

1. Introduction

The economies of oil producing nations depend heavily on oil price dynamics. These dynamics are the determinants of budgetary sizes and capital project allocations in nations with oil. As a result, it has gained attention even among mathematicians: Cai and Newt [1], Krugman [2] especially with the downturn dynamics of 2015; Lee and Huh [3]. Because to mathematicians, if nation A derives proceeds in a space X when the dynamics are positively increasing and sufficient for instance, the dynamics can be represented.[1] Practically, if T is an arbitrary operator and X is a proceed space for A, one can write that

$$T : X \to Y \tag{1}$$

Now, given additional information on (1) above, vital considerations can be made. Suppose a price shock occurs when X is complete. Then (1) is a transformation from Banach space to any space. Consequently, understanding the nature of inverse maps that reverse Y to X could be the solution of certain interesting problems. The direct interpretation is that to do with the needed policy maps that can take (1) to completeness once again.

[1]The converse is also true.

In the queuing literature, it is well known for N-homogenous jobs that the stationary probability of maintaining these jobs in a uni-server production center is given by

$$P_N = (1-\rho)\rho^N \tag{2}$$

The parameter ρ is the occupation rate of the center, Medhi [4]. Unfortunately, homogeneity of jobs is unrealistic, Krishnamoorthy [5]. Suppose X is isomorphic to a job space Z. Suddenly, an oil price shock[2] occurs in the neighborhood of X. Trivially, the completeness property of Z will alter similar to that of X a.s. Consequently, the extended job space Z_{ext} is nowhere dense in Z. Thus, working under homogenous assumptions in Z_{ext} is simplistic a.s. A re-consideration of the job size bracket $\{N\} \in Z_{ext}$ is necessary for a complete discussion and analysis in Z_{ext}. Moreover, the understanding of needed maps that takes Z_{ext} to Z is equivalent to that which takes X to Y given that the later space is a normed space.

In the past, a lot of studies considered the stationary behavior of jobs in Z_{ext} and Z identical. Nowadays, there are re-considerations proving otherwise. For instance, Krishnamoorthi and Sreenivan [6], Kumar and Sharma [7] and more recently, Som and Kumar [8]. On managing queuing systems in Z_{ext}, Ke and Pearn [9] studied an M/M/1 queuing system with server breakdowns and vacations where the arrival rate varies according to the server status and the vacation norm determined by the number of arrivals during the vacation period. Jayachitra and Albert [10] studied an Erlangian model under server breakdowns and multiple vacations and provided a cost model to determine the optimal operating policy at minimum cost. What is inherent in most of these management models is that the optimal criterion is proved from the service process. Essentially, optimal criterion from the number of jobs in the system is scarce in the literature.

Our aim is to present a methodology that studies the problem of strategies in imperfect production centers from the stationary number of jobs in the system. The most important gain is the generalization of known basic results. This extends the capacity of known results to centers with jobs of distinct characteristics. For instance, in service centers with normal jobs and constraint jobs, less time spending on jobs and delaying jobs, difficult to process jobs and easy to process jobs, etc. In this respect, our work is purely for operational research purpose geared towards best practices in centers with distinct job characteristics. We wish to provide the understanding of principal components of imperfect centers and develop optimal criteria under which production is maximized.

It turns out that[3] the problem herein is that of how best the coupling of system policies (T), occupation rates (ρ) and available constraints c can be tackled in Z_{ext}.

[2]Similar to that of 2015 that takes oil price from 100+ USD to the neighborhood of 20+ USD.

[3]From principal component analysis of several factors affecting a steady state system from the exterior, policies, occupation rates and constraint size have the largest Eigen values.

2. Preliminary Results

Lemma 1 *An operationally useful policy map* T *is necessarily compact and infinite dimensional in* Z_{ext}.

Proof.

It is enough to show that an infinite dimensional compact operator cannot have a close range in a complete normed space.

Let Z and Z_{ext} be arbitrary normed spaces where Z_{ext} is complete. Suppose that

$$T : Z \to Z_{ext}$$

is infinite dimensional and compact.

Suppose to the contrary that TZ is closed. Then TZ is complete.

Thus,

$$Z = \bigcup_{n=1}^{\infty} nB_Z \tag{3}$$

where B_Z is the unit ball of jobs in Z.

It follows directly from (3) that

$$TZ = \bigcup_{n=1}^{\infty} nTB_Z. \tag{4}$$

Since T is compact and nTB_Z is bounded, then nTB_Z is compact.

Given that $dimT = \infty$; nTB_Z is nowhere dense in TZ. Hence, TZ is of the first Baire category.

This contradiction completes the prove. □

Lemma 2 *Let* $\{z_n\}_{n\ge 0}$ *be a job sequence in* Z_{ext}. *If* $\{z_n\}_{n\ge 0}$ *has a convergent point* z *in* Z, *then the measurable policy map* T *such that* $Tz_n \to Tz$ *is a strong job policy.*

Proof. It suffices to show that if Z and Z_{ext} are normed job spaces, then for any $\{z_n\}_{n\ge 0} \in Z_{ext}$ and a point $z \in Z$ given that $T \in K(Z, Z_{ext})$ and $z_n \xrightarrow{w} z$, then $Tz_n \to Tz$ strongly.

Let $y_n = Tz_n$ and $y = Tz$. We show that $y_n \xrightarrow{w} y$. Suppose that $y^* \in Z^*$ and define z^* by

$$z^*(z) = y^*(Tz). \tag{5}$$

Clearly z^* is linear. Similarly, since T is compact z^* is compact. Thus, z^* is bounded.

Given that $z_n \xrightarrow{w} z$, we have $z^*(z_n) \to z^*(a)$. Hence, $y^*(Tz_n) \to y^*(Tz)$. Now to show the last component of the lemma, suppose that Tz_n does not converge to Tz. Then (Tz_n) has a sub sequence ($Tz_{n'}$) such that $\|Ta_{n'} - Ta\| > \alpha$ for some $\alpha > 0$. Since $z_{n'} \xrightarrow{w} z$, then ($z_{n'}$) is bounded. Given that T is compact, then ($Tz_{n'}$) has a Cauchy sub-sequence.

Consequently,

$$\|Tz_{n''} - Tz\| \to 0. \tag{6}$$

This contradiction completes the proof. □

Lemma 3 *A job policy* T^* *on* Z_{ext} *is compact if a corresponding job policy* $T \in Z$ *is compact.*

Proof. Suppose that Z is a normed space and Z_{ext} is a Banach space.

Furthermore, suppose that T is compact. Let (y_n^*) be a sequence in $B\left(Z_{ext}^*, Z^*\right)$. If ψ is a functional such that for any $n > 0$, we have

$$\psi_n\left(y\right) = \left(y, y^*\right) \tag{7}$$

Then

$$\left|\psi_n\left(y\right)\right| = \left|\left(y, y^*\right)\right| \le \|y\|\|y_n^*\| \le \|y\| \tag{8}$$

Thus, ψ_n's are uniformly bounded. In addition,

$$\left|\psi_n\left(y\right) - \psi_n\left(y'\right)\right| = \left|\left(y - y', y_n^*\right)\right| \le \|y - y'\|. \tag{9}$$

Thus, ψ_n's are equicontinuous and so $\left\{\psi_n\left(y\right): y \in TB_Z\right\}$ is compact. This implies that there exist a sub sequence

$$\left\{\psi_{n'}\left(Tz\right)\right\} = \left\{\left(z, T^* y_n^*\right)\right\} \tag{10}$$

which converges uniformly on $B\left(Z, Z_{ext}\right)$. Hence T^* is compact. □

Lemma 4 *Given* $\{N\} = \{c\} \oplus \{n\}; \{c\} \cap \{n\} = \{\ \}$; *then*

$$E\left[N\right] = \frac{\rho^c\left[\rho + c\left(1-\rho\right)\right]}{1-\rho}. \tag{11}$$

Proof. Denote by $V\left(z\right)$ the probability generating function (PGF) for the new jobs $n \in \{N\}$ when there are $c \ge 0$ fixed constraint jobs in a system such that

$$V\left(z\right) = \sum_{c\oplus n=0}^{\infty} P_{c\oplus n} z^{c\oplus n};\ n = 0,1,2,3,\cdots \tag{12}$$

That means

$$V\left(z\right) = P_c + P_{c+1}z + P_{c+2}z^2 + P_{c+3}z^3 + \cdots; n = 0,1,2,3,\cdots \tag{13}$$

And in view of (2), we have

$$V\left(z\right) = \left(1-\rho\right)\rho^c + \left(1-\rho\right)\rho^{c+1}z + \left(1-\rho\right)\rho^{c+2}z^2 + \cdots \tag{14}$$

So that

$$V\left(z\right) = \frac{\left(1-\rho\right)\rho^c z^c}{1-\rho z} \tag{15}$$

The lemma holds upon differentiating (15) at $z = 1$. □

It is interesting to note that if $c = 0$ in (11) above, then $\{c\} \oplus \{n\} = \{\ \} \oplus \{n\} = \{n\}$. Consequently, there is only one job group $\{n\}$ in the system (homogenous).[4] In this case, the result goes to that of a classical production center with homogenous jobs as expected.

[4]This is the only case the classical M/M/1 model depicts in its expectation.

Lemma 5 *For a finite capacity imperfect production center with two distinct job groups* $\{n\}$ *and* $\{c\}$*, we have*

$$E[N] = \frac{\rho^c\left[\left(1-\rho^K\right)\left(c(1-\rho)+\rho\right)-K(1-\rho)\rho^K\right]}{1-\rho} \tag{16}$$

Proof. In view of (2) and for $N = K < \infty$, it can be shown that the PGF $W(z)$ is

$$W(z) = \frac{(1-\rho)(\rho z)^c\left(1-(\rho z)^K\right)}{1-\rho z} \tag{17}$$

Differentiating (17) w.r.t z, we have

$$W'(z) = \frac{(1-\rho z)\left[c\rho(1-\rho)(\rho z)^{c-1}\left(1-(\rho z)^K\right)-K\rho(1-\rho)(\rho z)^{c+K-1}\right]+\phi(z)}{(1-\rho z)^2} \tag{18}$$

where

$$\phi(z) = \rho(1-\rho)(\rho z)^c\left(1-(\rho z)^K\right) \tag{19}$$

Which gives

$$E[\mathbf{N}] = \frac{\rho^c\left[(1-\rho)\left(1-\rho^K\right)\left[c(1-\rho)+\rho\right]-K(1-\rho)^2\rho^K\right]}{(1-\rho)^2} \tag{20}$$

upon substituting z = 1 in (18) above. Finally, the lemma holds if (20) is rearranged and simplified. □

Corollary 6 *From the numerical results* (*Tables* 1-6 *in the appendix*)*, it is clear that*

1) $\rho \to \rho(c)$.
2) $\{N(\rho)\} \to \{N(\rho,c)\}$.

Lemma 7 (First Criterion) *A maximizer of the group* $\{N\}$ *is the solution for the policy map-constraint-occupation rate problem* $G(.)$ *such that*

$$G\left(\partial_T N, \partial_\rho N, \partial_c N, N, T, c, \rho\right) = 0. \tag{21}$$

Proof. We seek a solution $N(c,\rho,T)$ for $G(.)$ such that

$$G(.) = \partial_c N + F\left(\partial_T N, \partial_\rho N, N, T, c, \rho\right). \tag{22}$$

where $F(.)$ is a real valued semi-linear continuous function with respect to all its arguments. (22) is equivalent to

$$G(.) = \partial_c N + \partial_T N + h(T,c,\rho,N)\partial_\rho N + \bar{h}(T,c,\rho,N). \tag{23}$$

By Lemma 1, *T* is necessarily compact. Let *T* be a fixed point of G(.). Then (23) reduces to

$$\partial_c N = -h(c,\rho,N)\partial_\rho N - \bar{h}(c,\rho,N). \tag{24}$$

Consider a differentiable arc $\mathbf{B}:\rho\to\rho(c)$ in the (ρ,c) plane such that points on $\mathbf{B}\to(\rho(c),c)$.

By multivariate chain rule on the left hand side of (24), we have

$$\frac{\mathrm{d}}{\mathrm{d}c}N(\rho(c),c)=\partial_c N(\rho(c),c)+\partial_\rho N(\rho(c),c)\frac{\mathrm{d}\rho(c)}{\mathrm{d}c}. \tag{25}$$

Assuming that $N(\rho,c)$ solves (21) and combining (24) and (25) and rearranging, we have

$$\frac{\mathrm{d}}{\mathrm{d}c}N(\rho(c),c)=\left(-h(..)+\frac{\mathrm{d}\rho(c)}{\mathrm{d}c}\right)\partial_\rho N(\rho(c),c)-\bar{h}(..) \tag{26}$$

So[5] that the couple differential equations

$$\frac{\mathrm{d}}{\mathrm{d}\rho}N(\rho(c),c)=h(\rho(c),c,N(\rho(c),c)) \tag{27}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}c}N(\rho(c),c)=\left(-\bar{h}(\rho(c),c,N(\rho(c),c))\right) \tag{28}$$

constitute in general a solution for $G(.)$ for $N(\rho(c),c)$ when T is a fixed point of $G(.)$. □

Lemma 8 (Second Criterion) *Any continuously differentiable solution* $N(\rho,c)$ *for* $G(.)$ *satisfying the first criterion above must coincide with the original solution* $\tilde{N}(\tilde{\rho}(c),c)$ *along the base curve* $\rho=\rho(c)$.

Proof. Since (27) and (28) are coupled systems, only in rare cases analytic solution exists in closed form. However, if we specify an initial value for c_0, ρ_0 and N_0, then the existence of a unique solution pair $\tilde{\rho}(c)$ and $\tilde{N}(\tilde{\rho}(c),c)$ is guaranteed. A solution $\hat{N}(\tilde{\rho}(c),c)$ that did not pass through the origin leading to $\tilde{N}(\tilde{\rho}(c),c)$ cannot be a solution for $G(.)$ since it is nowhere differentiable around $G(.)$. □

Lemma 9 (Optimality Criterion) *Suppose* $\|T\|\to\|T_{\max}\|$ *so that the constraint dependent occupation rate* $\rho_c\to\rho_{\max}(c)\in(0,1)$. *A solution* $N(\rho(c),c)$ *that passes through the origin* $\tilde{N}(\tilde{\rho}(c),c)$ *for* $G(.)$ *is optimal a.s.*

Proof. Given that $\|T\|\to\|T_{\max}\|$, we have $\rho_c\to\rho_{\max}(c)\in(0,1)$. By the numerical approximation (*Tables 1-6 in the appendix*) $N(\rho(c),c)\to N_{\max}(\rho(c),c)$. Given that $G(.)$ is $N(\rho(c),c)$ dependent, it is then trivial. □

3. Scope for Future Work

There is a scope in extending our results to some special cases of the problem solved in this work. For instance, when the function $h(..)$ is independent of N or when $h(..)$ and $\bar{h}(..)$ are linear or even non-linear combination of ρ and T and N. The author are grateful to all literature sources used.

[5] $(..)=(\rho(c),c,N(\rho(c),c))$.

Competing Interest

There is no competing interest of any kind within the authorship of this work.

Authors Contribution

SS: Drafted the entire manuscript, provided the introductory chapter (Section 1) and proved Lemmas 1, 2, 3, 5, 7, 8 and 9 together with Corollary 6.

HM: Participated in the sequence alignment of the manuscript and provided the numerical simulations.

MLM: Participated in the design of the manuscript and proved lemma 4 under my supervision.

All authors read and approved the final manuscript.

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Appendix

For a numerical approximation, we study the model in (11) under various sizes of constraint numbers c and varying occupation rate ρ for $E[\mathbf{N}]$. The following numerical results are obtained.

Table 1. $E[\mathbf{N}]$ when $c = 0$.

S/N	ρ	$E[\mathbf{N}]$
1	0.1	0.1111111
2	0.5	1.0000000
3	0.7	2.3333333
4	0.8	4.0000000
5	0.9	9.0000000
6	0.99	99.000000
7	0.999	999.00000
8	0.9999	9999.0000
9	0.99999	99999.000
10	0.999999	999999.00

Table 2. $E[\mathbf{N}]$ when $c = 10$.

S/N	ρ	$E[\mathbf{N}]$
1	0.5	0.0107422
2	0.65	0.1596297
3	0.75	0.7320757
4	0.8	1.5032386
5	0.85	3.0843657
6	0.9	6.6248904
7	0.99	98.577647
8	0.999	998.95528
9	0.9999	9998.9955
10	0.99999	99998.999

Table 3. $E[\mathbf{N}]$ when $c = 23$.

S/N	ρ	$E[\mathbf{N}]$
1	0.75	0.0347842
2	0.80	0.1593799
3	0.85	0.6823583
4	0.89	2.1310988
5	0.90	2.8361402
6	0.95	12.908988
7	0.97	27.462288
8	0.99	96.820943
9	0.999	998.75077
10	0.9999	9998.9747

Table 4. $E[\mathbf{N}]$ when $\rho = 0.5$.

S/N	c	$E[\mathbf{N}]$
1	0	1.000000
2	3	0.500000
3	5	0.187500
4	7	0.062500
5	9	0.019531
6	10	0.010742
7	11	0.005859
8	13	0.001709
9	14	0.000916
10	15	0.0004883

Table 5. $E[\mathbf{N}]$ when $\rho = 0.75$.

S/N	c	$E[\mathbf{N}]$
1	0	3.000000
2	5	1.898436
3	10	0.7320757
4	15	0.240542
5	20	0.072938
6	25	0.021071
7	30	0.005893
8	35	0.001610
9	40	0.000432
10	45	0.000115

Table 6. $E[\mathbf{N}]$ when $\rho = 0.9$.

S/N	c	$E[\mathbf{N}]$
1	0	9.000000
2	5	8.266860
3	10	6.624890
4	15	4.941387
5	20	3.525723
6	25	2.440853
7	30	1.653255
8	35	1.101388
9	40	0.724263
10	45	0.471310