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Variational Homotopy Perturbation Method for Solving Riccati Type Differential Problems

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Abstract

In this paper, a Variational homotopy perturbation method is proposed to solve nonlinear Riccati differential equation. By combining the Variational Iteration Method and the Homotopy Perturbation Method, this technique possesses a fast convergence rate with high accuracy. The results reveal that the proposed method is very effective and simple.

Keywords

Riccati Equation, Homotopy Perturbation Method, Variational Iteration Method, Variational Homotopy Perturbation Method

1. Introduction

The Riccati equation plays a great role in blueprint and analysis the linear and nonlinear optimal control problems. Numerical Solution of this equation has been acquired by applying Adomian's decomposition method [1], homotopy Analysis method HAM [2], variational iteration method VIM [3] and homotopy perturbation method HPM [4]. HPM introduced by He [5], it can solve a large class of nonlinear problems activity, accurately and easily.

The application of HPM on nonlinear problems has been implemented by scientists and engineers, because this method is to continuously deform a difficult problem under study into a simple problem easy to solve. VIM proved by Ji-Huan He [5]. It is simple and powerful method for solving a broad type of nonlinear Problem. It was shown that this method is operative and reliable analytic and numerical purposes. The method gives rapidly convergent successive approximation of the exact solution if such solution existed.

The nonlinear Riccati differential equation [6] has following form

$$\begin{aligned} &L(x) = A(x)u^2(x) + B(x)u(x) + C(x),\ \ 0 \le x \le X \\ &u(0) = \alpha \end{aligned} \tag{1}$$

where $L = \frac{\mathrm{d}}{\mathrm{d}x}$ or $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $A(x), B(x)$ and $C(x)$ are continuous functions and α is an arbitrary constant.

We organize the following paper as follows. In Section 2, we present the VIM, while in Section 3, we present the HPM. In Section 4, we apply the VHPM to solve quadratic Riccati equation. Moreover, we find solutions of some examples by VHPM in Section 5.

The results reveal that the proposed method is very effective and simple. We end this paper by conclusion that reveal that these methods are very effective and convenient for solving nonlinear Riccati equations.

2. Variational Iteration Method

To illustrate the basic concepts of the VIM we consider the following differential equation [7] [8]

$$L(u) + N(u) = f(x) \tag{2}$$

where L is a linear operator, N is a nonlinear operator, and $f(x)$ is an inhomogeneous term. Then, we can construct the correct functional as follows

$$u_{n+1} = u_n + \int_0^x \lambda(\xi)\left[L(u_n) + N(\tilde{u}_n) - f(\xi)\right]\mathrm{d}\xi \tag{3}$$

where λ is a general Lagrangian multiplier defined as [9]

$$\lambda(x,t) = \frac{(-1)^m}{(m-1)!}(x-t)^{m-1}, \ m \geq 1 \tag{4}$$

And $\tilde{u}_n$ are restricted variation which means $\delta\tilde{u}_n = 0$. Consequently, the solution $u = \lim_{n\to\infty} u_n$.

3. Homotopy Perturbation Method

To explain this method, we construct the following function [10]

$$A(u) - f(x) = 0, \ x \in \Omega \tag{5}$$

With boundary condition

$$B(u, \partial u/\partial n) = 0, \ x \in \Gamma \tag{6}$$

where A is a general differential operator, B is a boundary operator, $f(x)$ is a known analytical function. The operator A can be decomposed into two operators L and N, where L is a linear operator and N is a nonlinear operator.

By using the homotopy technique, we construct a homotopy $u(x,p): \Omega \times [0,1] \to \mathbb{R}$ which are satisfies

$$H(u,p) = (1-p)\left[L(u) - L(u_0)\right] + p\left[L(u) + N(u) - f(x)\right] = 0 \tag{7}$$

or

$$H(u,p) = L(u) - L(u_0) + p\left[L(u_0) + N(u) - f(x)\right] = 0 \tag{8}$$

where $p \in [0,1]$ is an embedding parameter. u_0 is an initial approximation of solution of equation.

$$L(u)+N(u)-f(x)=0 \tag{9}$$

We have

$$H(u,0)=L(u)+L(u_0)=0,\ H(u,1)=A(u)-f=0 \tag{10}$$

The solution can be written as a power series in p, $u(x)=\lim_{p\to1}u=u_0+pu_1+p^2u_2+\cdots$.

4. Variational Homotopy Perturbation Method

In this section, we apply the VHPM to Riccati Equation (1), we start this method by applying HPM in Equation (8) on Equation (1), we get

$$Lu_n(x)-Lu_0(x)+p\left[Lu_0(x)-A(x)u_n^2-B(x)u_n-C(x)\right]=0 \tag{11}$$

Now we using correction functional in Equation (5) to get

$$u_{n+1}=u_n+\int_0^x\lambda(\xi)\left[Lu_n-Lu_0+p\left(Lu_0-A(\xi)u_n^2-B(\xi)u_n-C(\xi)\right)\right]\mathrm{d}\xi \tag{12}$$

We can obtain

$$u_{n+1}=u_0+p\int_0^x\lambda(\xi)\left[Lu_0-A(\xi)u_n^2-B(\xi)u_n-C(\xi)\right]\mathrm{d}\xi \tag{13}$$

Now we can rewrite Equation (13) in the form

$$\sum_{n=0}^{\infty}p^nu_n=u_0+p\int_0^x\left[\lambda(\xi)\left(Lu_0-N\sum_{n=0}^{\infty}p^n\tilde{u}_n-C(\xi)\right)\right]\mathrm{d}\xi \tag{14}$$

As we see, the procedure is formulated by the coupling of VIM and HPM [11] [12] [13]. A comparison of like powers of p give solutions of various orders.

5. Numerical Examples

5.1. Example

Consider the following classical Riccati differential equation

$$u'(x)=-u^2+2u+1 \tag{15}$$

With initial condition $u(0)=0$.

For the above differential equation, the exact solution [14] is previously known to be

$$u(x)=1+\sqrt{2}\tanh\left(\sqrt{2}x+\frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) \tag{16}$$

The Taylor expansion of $u(x)$ about $x=0$ gives

$$u(x)=x+x^2+\frac{1}{3}x^3-\frac{1}{3}x^4-\frac{7}{15}x^5-\frac{7}{45}x^6+\frac{53}{315}x^7+\cdots \tag{17}$$

Suppose that the initial approximation is $u_0=x$.

To solve Equation (15), by the VHPM we substitution it in Equation (14)

$$\sum_{n=0}^{\infty}p^nu_n=u_0-p\int_0^x\left[-2\sum_{n=0}^{\infty}p^nu_n+\left(\sum_{n=0}^{\infty}p^nu_n\right)^2\right]\mathrm{d}\xi \tag{18}$$

Here $\lambda = -1$.

By comparing the coefficient of like powers of p, we have

$$
\begin{aligned}
& p^{(0)} : u_0 = x \\
& p^{(1)} : u_1 = -\int_0^x \left(-2u_0 + u_0^2\right) \mathrm{d}\xi = x^2 - \frac{1}{3}x^3 \\
& p^{(2)} : u_2 = -\int_0^x \left(-2u_1 + 2u_0 u_1\right) \mathrm{d}\xi = \frac{2}{3}x^3 - \frac{2}{3}x^4 + \frac{2}{15}x^5 \\
& p^{(3)} : u_3 = -\int_0^x \left(-2u_2 + 2u_0 u_2 + u_1^2\right) \mathrm{d}\xi = \frac{1}{3}x^4 - \frac{11}{15}x^5 + \frac{17}{45}x^6 - \frac{17}{315}x^7 \\
& \qquad \vdots
\end{aligned}
\tag{19}
$$

The other components of the VHPM can be determined in similar way. Finally, the approximate solution of Equation (15) is $u = u_0 + u_1 + u_2 + u_3 + \cdots$. Which converge to the exact solution in Equation (16).

5.2. Example

Consider the following quadratic Riccati differential equation

$$u'(x) = -2\mathrm{e}^x u^2 + 2\mathrm{e}^{2x} u + \mathrm{e}^x - \mathrm{e}^{3x} \tag{20}$$

With initial condition $u(0) = 1$.

For the above differential equation, the exact solution [14] is previously known to be

$$u(x) = \mathrm{e}^x \tag{21}$$

The Taylor expansion of $u(x)$ about $x = 0$ gives

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \cdots \tag{22}$$

Suppose that the initial approximation is $u_0 = x + 1$.

To solve Equation (20), by the VHPM we substitution it in Equation (14), then we get

$$\sum_{n=0}^{\infty} p^n u_n = u_0 - p\int_0^x \left[1 + \sum_{n=0}^{\infty} \frac{(2\xi)^n}{n!}\left(\sum_{n=0}^{\infty} p^n u_n\right) - 2\sum_{n=0}^{\infty} \frac{(2\xi)^n}{n!}\left(\sum_{n=0}^{\infty} p^n u_n\right)^2 + \sum_{n=0}^{\infty} \frac{(3\xi)^n}{n!} - \sum_{n=0}^{\infty} \frac{\xi^n}{n!}\right] \mathrm{d}\xi \tag{23}$$

Here $\lambda = -1$.

By comparing the coefficient of like powers of p, we have

$$
\begin{aligned}
& p^{(0)} : u_0 = x + 1 \\
& p^{(1)} : u_1 = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{24}x^5 - \frac{49}{270}x^6 - \frac{37}{270}x^7 - \frac{1091}{40320}x^8 \\
& \qquad\qquad + \frac{247}{45360}x^9 - \frac{1}{50400}x^{10} \\
& p^{(2)} : u_2 = \frac{1}{10}x^5 + \frac{5}{36}x^6 + \frac{13}{126}x^7 + \frac{23}{480}x^8 + \frac{19}{2592}x^9 - \frac{3389}{302400}x^{10} - \cdots \\
& p^{(3)} : u_3 = -\frac{1}{20}x^5 - \frac{5}{72}x^6 - \frac{13}{252}x^7 - \frac{1}{120}x^8 + \frac{59}{1728}x^9 + \frac{32119}{604800}x^{10} + \cdots \\
& \qquad \vdots
\end{aligned}
\tag{24}
$$

The other components of the VHPM can be determined in similar way. Finally, the approximate solution of Equation (20) is $u = u_0 + u_1 + u_2 + u_3 + \cdots$. Which converge to the exact solution in Equation (21).

5.3. Example

Consider the Riccati Type Painleve's First Transcendent equation [15]

$$u''(x) = 6u^2 + \mu x, \quad \mu = 1 \tag{25}$$

With initial conditions $u(0) = 1, u'(0) = 0$.

The above differential equation without known exactly solutions and we suppose that the initial approximation is $u_0 = 1 + x^2$.

To solve Equation (25) by the VHPM, substitution it in Equation (14), then we get

$$\sum_{n=0}^{\infty} p^n u_n = u_0 + p\int_0^x \lambda(\xi)\left[2 - 6\left(\sum_{n=0}^{\infty} p^n u_n\right)^2 - \xi\right]\mathrm{d}\xi \tag{26}$$

In this example $\lambda = (\xi - x)$.

By comparing the coefficient of like powers of p, we have

$$\begin{aligned}
&p^{(0)} : u_0 = 1 + x^2 \\
&p^{(1)} : u_1 = 2x^2 + \frac{1}{6}x^3 + x^4 + \frac{1}{5}x^6 \\
&p^{(2)} : u_2 = 2x^4 + \frac{1}{10}x^5 + \frac{6}{5}x^6 + \frac{1}{21}x^7 + \frac{9}{35}x^8 + \frac{2}{75}x^{10} \\
&p^{(3)} : u_3 = \frac{8}{5}x^6 + \frac{13}{105}x^7 + \frac{1877}{1680}x^8 + \frac{11}{210}x^9 + \frac{11}{35}x^{10} + \frac{17}{1925}x^{11} \\
&\qquad + \frac{254}{5775}x^{12} + \frac{x^{14}}{325} \\
&\quad \vdots
\end{aligned} \tag{27}$$

The other components of the VHPM can be determined in similar way. Finally, the approximate solution of Equation (25) is $u \approx u_0 + u_1 + u_2 + u_3 + \cdots$. In Table 1 we present the comparison between the approximate solution founded by VHPM with Truncated Taylor series(TTS) [16], and Rational approximation (RA) [17].

5.4. Example

Consider the Riccati Type Painleve's Second Transcendent equation [15]

$$u''(x) = 2u^3 + xu + \mu, \ \mu = 1 \tag{28}$$

With initial conditions $u(0) = 1, u'(0) = 0$.

The above differential equation without known exactly solutions and we suppose that the initial approximation is $u_0 = 1 + x^2$.

To solve Equation (28) by the VHPM, substitution it in Equation (14), then we get

$$\sum_{n=0}^{\infty} p^n u_n = u_0 + p\int_0^x \lambda(\xi)\left[1 - 2\left(\sum_{n=0}^{\infty} p^n u_n\right)^3 - \xi\sum_{n=0}^{\infty} p^n u_n\right]\mathrm{d}\xi \tag{29}$$

Table 1. Comparison between the approximate solution $u=\sum_{i=0}^{20} u_i$ with TTS and RA.

x	VHPM	TTS	RA
0.0	1.000000	1.0000	1.0000
0.1	1.030471	1.030471	1.0305
0.2	1.126366	1.126366	1.1264
0.3	1.301454	1.301453	1.3015
0.4	1.583055	1.583054	1.5831
0.5	2.022763	2.022771	2.0228
0.6	2.721246	2.721242	2.7212
0.7	3.890893	3.890886	3.8909
0.8	6.038351	6.038340	6.0383
0.9	10.622497	10.622610	10.6223
1.0	23.363804	23.393600	23.3860

And we have $\lambda=(\xi-x)$.

By comparing the coefficient of like powers of p, we have

$$\begin{aligned}
&p^{(0)}: u_0=1+x^2\\
&p^{(1)}: u_1=\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{2}x^4+\frac{1}{20}x^5+\frac{1}{5}x^6+\frac{1}{28}x^8\\
&p^{(2)}: u_2=\frac{1}{4}x^4+\frac{3}{40}x^5+\frac{11}{36}x^6+\frac{1}{51}x^7+\frac{41}{224}x^8+\frac{1}{224}x^9\\
&\qquad+\frac{131}{2100}x^{10}+\frac{47}{15400}x^{11}+\frac{19}{1540}x^{12}+\frac{3}{2548}x^{14}\\
&p^{(3)}: u_3=\frac{1}{10}x^6+\frac{17}{420}x^7+\frac{383}{2240}x^8+\frac{703}{12960}x^9+\frac{6887}{50400}x^{10}\\
&\qquad+\frac{4013}{123200}x^{11}+\frac{1039}{15840}x^{12}+\frac{2759}{257400}x^{13}+\frac{117269}{5605600}x^{14}\\
&\qquad+\frac{151}{77000}x^{15}+\frac{74933}{16816800}x^{16}+\frac{28671}{190590400}x^{17}+\frac{5653}{9529520}x^{18}\\
&\qquad+\frac{15}{387296}x^{20}\\
&\quad\vdots
\end{aligned}\tag{30}$$

The other components of the VHPM can be determined in similar way. Finally, the approximate solution of Equation (28) is $u\approx u_0+u_1+u_2+u_3+\cdots$. In Table 2 we present the comparison between the approximate solution founded by VHPM with TTS [15], and RA [17].

6. Conclusion

In this paper, we studied the solution of nonlinear Riccati differential equation. We have applied a recently introduced technique called the VHPM to solve this nonlinear differential equation. This method is more efficient and simpler. The results of the method exhibit excellent agreement with the exact solution. The

Table 2. Comparison between the approximate solution $u = \sum_{i=0}^{20} u_i$ with TTS and RA.

x	VHPM	TTS	RA
0.0	1.000000	1.0000	1.0000
0.1	1.015244	1.01520	1.0152
0.2	1.062614	1.06260	1.0626
0.3	1.146376	1.14640	1.1464
0.4	1.274152	1.27420	1.2742
0.5	1.459213	1.45920	1.4592
0.6	1.725376	1.72540	1.7254
0.7	2.118444	2.11840	2.1184
0.8	2.736936	2.73690	2.7369
0.9	3.834399	3.83440	3.8343
1.0	6.309968	6.31100	6.3104

comparison between the numerical results with TTS and RA in Problems 3 and 4 of validates the accuracy of the VHPM method for problems without known exactly solutions that have been advanced for solving Riccati equation shows that the new technique is reliable and powerful.

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