Applied Mathematics, 2017, 8, 663-670
http://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

On the Increments of Stable Subordinators

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How to cite this paper: Bahram, A. and Almohaimeed, B. (2017) On the Increments of Stable Subordinators. *Applied Mathematics*, **8**, 663-670.
https://doi.org/10.4236/am.2017.85053

Received: April 16, 2017
Accepted: May 20, 2017
Published: May 23, 2017

Abstract

Let $\{X(t), t \geq 0\}$ be a stable subordinator defined on a probability space $(\Omega, \mathcal{F}, \mathcal{A})$ and let a_t for $t > 0$ be a non-negative valued function. In this paper, it is shown that under varying conditions on a_t, there exists a function $\lambda_\beta(t)$ such that

$$\liminf_{t\to\infty} \frac{(X(t+a_T)-X(t))}{\lambda_\beta(t)} = 1 \quad a.s.,$$

where $\lambda_\beta(t) = \theta_\alpha a_t^{\frac{1}{\alpha}} \left(\log\frac{t}{a_t} + \beta \log\log t + (1-\beta)\log\log a_t\right)^{\frac{\alpha-1}{\alpha}}$, $0 \leq \beta \leq 1$,

$\theta_\alpha = (B(\alpha))^{\frac{1-\alpha}{\alpha}}$ and $B(\alpha) = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\left(\cos\left(\frac{\pi\alpha}{2}\right)\right)^{\frac{1}{\alpha-1}}$.

Keywords

Increments, Stable Subordinators, Iterated Logarithm Laws

1. Introduction

Let $\{X(t), t \geq 0\}$ be a stable ordinator with exponent α with $0 < \alpha < 1$, defined on a probability space $(\Omega, \mathcal{F}, \mathcal{A})$. Let a_t for $t > 0$ be a non-negative valued function and $Y(t) = X(t+a_t) - X(t)$, $t > 0$. Define

$$\lambda_\beta(t) = \theta_\alpha a_t^{\frac{1}{\alpha}} \left(\log\frac{t}{a_t} + \beta \log\log t + (1-\beta)\log\log a_t\right)^{\frac{\alpha-1}{\alpha}},$$

where $0 \leq \beta \leq 1$,

$$\theta_\alpha = (B(\alpha))^{\frac{1-\alpha}{\alpha}} \quad \text{and} \quad B(\alpha) = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\left(\cos\left(\frac{\pi\alpha}{2}\right)\right)^{\frac{1}{\alpha-1}}.$$

For any value of t, the characteristic function of $X(t)$ is of the form

$$E\left(e^{iuX(t)}\right)=\exp\left(-t|u|^{\alpha}\left(1-\frac{ui}{|u|}\tan\left(\frac{\pi\alpha}{2}\right)\right)\right),\quad 0<\alpha<1.$$

Limit theorems on the increments of stable subordinators have been investigated in various directions by many authors [1]-[6]. Among the above many results, we are interested in Fristedt [4] and Vasudeva and Divanji [3] whose results are the following limit theorems on the increments of stable subordinators.

Theorem 1 ([4])

$$\liminf_{t\to\infty}\theta_{\alpha}t^{-\frac{1}{\alpha}}(\log\log t)^{\frac{1-\alpha}{\alpha}}X(t)=1\quad \text{almost surely}\quad (a.s).$$

Theorem 2 ([3]) *Let* $0<a_t$ *for* $t>0$, *be a non-decreasing function of* t *such that*

(i) $0<a_t\le t$ for $t>0$,

(ii) $a_t\to\infty$ as $t\to\infty$, and

(iii) a_t/t is non-increasing. Then

$$\liminf_{t\to\infty}\frac{\left(X(t+a_t)-X(t)\right)}{\xi(t)}=1\quad a.s.,\tag{1}$$

where $\xi(t)=\theta_{\alpha}a_t^{\frac{1}{\alpha}}\left(\log\frac{t}{a_t}+\log\log t\right)^{\frac{\alpha-1}{\alpha}}$.

In this paper, our aim is to investigate Liminf behaviors of the increments of *Y*. We establish that, under certain conditions on a_t,

$$\liminf_{t\to\infty}\frac{Y(t)}{\lambda_{\beta}(t)}=1\quad a.s., \\ \text{where } Y(t)=X(t+a_t)-X(t).\tag{2}$$

Throughout the paper c and k (integer), with or without suffix, stand for positive constants. i.o. means infinitely often. We shall define for each $u\ge 0$ the functions $\log u=\log(\max(u,1))$ and $\log\log u=\log\log(\max(u,3))$.

2. Main Result

In this section, we reformulate the result obtained in Theorem 2 and establish our main result using $\lambda_{\beta}(t)$ with $0\le\beta\le 1$ instead of $\xi(t)$.

Theorem 3 *Let* a_t, $t>0$, *be a non-decreasing function of* t *such that*

(i) $0<a_t\le t$ for $t>0$,

(ii) $a_t\to\infty$ as $t\to\infty$, and

(iii) a_t/t is non-increasing. Then

$$\liminf_{t\to\infty}\frac{Y(t)}{\lambda_{\beta}(t)}=1\quad a.s.$$

Remark 1 *Let us mention some particular cases*

1. For $a_t=t$ we obtain Fristedt's iterated logarithm laws (see Thorem 1).

2. If $\beta = 1$ we have Vasudeva and Divanji theorem (see Theorem 2).

3. If $\beta = 0$ under assumptions (i), (ii) and (iii) of Theorem 3 we also have

$$\liminf_{t\to\infty} \frac{Y(t)}{\lambda_0(t)} = 1 \quad a.s.$$

In order to prove Theorem 3, we need the following Lemma

Lemma 1 (*see* [3] *or* [7]) *Let* X_1 *be a positive stable random variable with characteristic function*

$$E\left(\exp\{iuX_1\}\right) = \exp\left\{-|u|^{\alpha}\left(1 - \frac{iu}{|u|}\tan\left(\frac{\pi\alpha}{2}\right)\right)\right\}, 0 < \alpha < 1.$$

Then, as $x \to 0$,

$$P(X_1 \le x) \simeq \frac{x^{\frac{\alpha}{2(1-\alpha)}}}{\sqrt{2\pi\alpha B(\alpha)}} \exp\left\{-B(\alpha)\, x^{\frac{\alpha}{\alpha-1}}\right\}$$

where

$$B(\alpha) = (1-\alpha)\alpha^{\frac{\alpha-1}{\alpha}}\left(\cos\left(\frac{\pi\alpha}{2}\right)\right)^{\frac{1}{\alpha-1}}.$$

Proof of Theorem 3. Firstly, we show that for any given $\varepsilon > 0$, as $t \to \infty$,

$$P\left(Y(t) \le (1+\varepsilon)\lambda_\beta(t)\ \ i.o\right) = 1. \tag{3}$$

Let u_1 be a number such that $a_{u_1} > 1$. Define a sequence (u_k) through $u_{k+1} = u_k + a_{u_k}$, for $k = 1, 2, \cdots$. Now we show that

$$P\left(Y(u_k) \le (1+\varepsilon)\lambda_\beta(u_k)\ \ i.o\right) = 1.$$

From Mijhneer [8], we have

$$P\left(Y(u_k) \le (1+\varepsilon)\lambda_\beta(u_k)\right) = P\left(X(1) \le \frac{(1+\varepsilon)\lambda_\beta(u_k)}{a_{u_k}^{\frac{1}{\alpha}}}\right). \tag{4}$$

But

$$\frac{\lambda_\beta(u_k)}{a_{u_k}^{\frac{1}{\alpha}}} = \theta_\alpha\left(\log\frac{u_k}{a_{u_k}} + \beta\log\log u_k + (1-\beta)\log\log a_{u_k}\right)^{\frac{\alpha-1}{\alpha}}.$$

Applying Lemma 1 with

$$x = (1+\varepsilon)\theta_\alpha\left(\log\frac{u_k}{a_{u_k}} + \beta\log\log u_k + (1-\beta)\log\log a_{u_k}\right)^{\frac{\alpha-1}{\alpha}},$$

one can find a k_0 such that, for all $k \ge k_0$,

$$P\left(X(1) \le \frac{(1+\varepsilon)\lambda_\beta(u_k)}{a_{u_k}^{\frac{1}{\alpha}}}\right)$$

$$\ge \frac{c_0}{2\left(\log\left(\frac{u_k(\log u_k)^\beta(\log a_{u_k})^{1-\beta}}{a_{u_k}}\right)\right)^{1/2}}$$

$$\times \exp\left\{-(1+\varepsilon)^{\alpha/(\alpha-1)}\log\left(\frac{u_k(\log u_k)^\beta(\log a_{u_k})^{1-\beta}}{a_{u_k}}\right)\right\},$$

where c_0 is some positive constant. Notice that

$$(1+\varepsilon)^{\frac{\alpha}{\alpha-1}} = (1-\varepsilon_1) < 1 \text{ for some } \varepsilon_1 > 0.$$

Hence

$$P\left(X(1) \le \frac{(1+\varepsilon)\lambda_\beta(u_k)}{a_{u_k}^{\frac{1}{\alpha}}}\right)$$

$$\ge \frac{c_0}{2\left(\log\left(\frac{u_k(\log u_k)^\beta(\log a_{u_k})^{1-\beta}}{a_{u_k}}\right)\right)^{1/2}}\left(\frac{a_{u_k}}{u_k}\right)$$

$$\times\left(\frac{u_k}{a_{u_k}}\right)^{\varepsilon_1}\frac{1}{\left((\log u_k)^\beta(\log a_{u_k})^{1-\beta}\right)^{(1-\varepsilon_1)}}$$

$$= \frac{c_0}{2\left(\log\left(\frac{u_k(\log u_k)^\beta(\log a_{u_k})^{1-\beta}}{a_{u_k}}\right)\right)^{1/2}}\left(\frac{u_{k+1}-u_k}{u_k}\right)$$

$$\times\left(\frac{u_k}{a_{u_k}}\right)^{\varepsilon_1}\frac{1}{\left((\log u_k)^\beta(\log a_{u_k})^{1-\beta}\right)^{(1-\varepsilon_1)}}.$$

Let $1_k = u_k/a_{u_k}$ and $m_k = (\log u_k)^\beta(\log a_{u_k})^{1-\beta}$. Note that 1_k is non-decreasing and $m_k \to \infty$ as $k \to \infty$. In turn one finds a $k_1 \ge k_0$, such that

$$\frac{1_k^{\varepsilon_1} m_k^{\varepsilon_1}}{(\log 1_k m_k)^{1/2}} \ge 1, \quad \text{whenever } k \ge k_1.$$

Therefore, for all $k \ge k_1$, we have

$$P\left(X(1)\le\frac{(1+\varepsilon)\lambda_\beta(u_k)}{a_{u_k}^{\frac{1}{\alpha}}}\right)$$

$$\ge c_0\frac{(u_{k+1}-u_k)}{2u_k(\log u_k)^\beta(\log a_{u_k})^{1-\beta}}=c_0\frac{(u_{k+1}-u_k)}{2u_k}\left(\frac{\log a_{u_k}}{\log u_k}\right)^\beta\frac{1}{\log a_{u_k}} \quad (5)$$

$$\ge c_0\frac{(u_{k+1}-u_k)}{2u_k}\left(\frac{\log a_{u_k}}{\log u_k}\right)\frac{1}{\log a_{u_k}}=c_0\frac{(u_{k+1}-u_k)}{2u_k\log u_k}.$$

Observe that

$$\int_{k_1}^{\infty}\frac{\mathrm{d}t}{t\log t}\le\sum_{k=k_1}^{\infty}\frac{(u_{k+1}-u_k)}{u_k\log u_k}. \quad (6)$$

From the fact that $\int_{k_1}^{\infty}\frac{\mathrm{d}t}{t\log t}=\infty$ and from (4), (5), and (6) one gets

$$\sum_{k=1}^{\infty}P\left(Y(u_k)\le(1+\varepsilon)\lambda_\beta(u_k)\right)=\infty.$$

Observe that $(Y(u_k))$ is a sequence of mutually independent random variables (for, $u_{k+1}=u_k+a_{u_k}$) and by applying Borel-Cantelli lemma, we get

$$P\left(Y(u_k)\le(1+\varepsilon)\lambda_\beta(u_k)\ \ i.o\right)=1$$

which establishes (3).

Now we complete the proof by showing that, for any $\varepsilon\in(0,1)$,

$$P\left(Y(t)\le(1-\varepsilon)\lambda_\beta(t_k)\quad i.o\right)=0. \quad (7)$$

Define a subsequence (t_k), such that

$$a_{t_k}=(t_{k+1}-t_k)\big/(\log t_k)^{(1-\beta)(1+\varepsilon)},\ k=1,2,\cdots \quad (8)$$

and the events A_t and B_k as

$$A_t=\{Y(t)\le(1-\varepsilon)\lambda_\beta(t)\}$$

and

$$B_k=\left\{\inf_{t_k\le t\le t_{k+1}}Y(t)\le(1-\varepsilon)\lambda_\beta(t_{k+1})\right\},\quad k=1,2,\cdots.$$

Note that

$$(A_t\quad i.o.,t\to\infty)\subset(B_k\quad i.o.,k\to\infty).$$

Further, define

$$C_k=\left\{X(t_k+a_{t_k})-X(t_{k+1})\le(1-\varepsilon)\lambda_\beta(t_{k+1})\right\}$$

and observe that

$$(B_k\quad i.o.,k\to\infty)\subset(C_k\quad i.o.,k\to\infty).$$

Hence in order to prove (7) it is enough to show that

$$P(C_k\quad i.o.)=0. \quad (9)$$

We have

$$P\left(X\left(t_k+a_{t_k}\right)-X\left(t_{k+1}\right)\le(1-\varepsilon)\lambda_\beta\left(t_{k+1}\right)\right)=P\left(X(1)\le\frac{(1-\varepsilon)\lambda_\beta\left(t_{k+1}\right)}{\left(a_{t_k}+t_k-t_{k+1}\right)^{1/\alpha}}\right)$$

and

$$\frac{(1-\varepsilon)\lambda_\beta\left(t_{k+1}\right)}{\left(a_{t_k}+t_k-t_{k+1}\right)^{1/\alpha}}$$
$$\simeq(1-\varepsilon)\theta_\alpha\left(\frac{a_{t_{k+1}}}{a_{t_k}}\right)^{1/\alpha}\left(\log\left(\frac{t_{k+1}\left(\log t_{k+1}\right)^\beta\left(\log a_{t_k}\right)^{1-\beta}}{a_{t_k}}\right)\right)^{(\alpha-1)/\alpha}.$$

The fact that a_t/t is non-increasing as $t\to\infty$ implies that

$$\frac{a_{t_{k+1}}}{t_{k+1}}\le\frac{a_{t_k}}{t_k}\quad\text{or}\quad\frac{a_{t_{k+1}}}{a_{t_k}}\le\frac{t_{k+1}}{t_k}.$$

Hence for a given $\varepsilon_1>0$ satisfying $(1-\varepsilon)(1+\varepsilon_1)^{1/\alpha}<1$, there exists a k_2 such that

$$a_{t_{k+1}}/a_{t_k}\le(1+\varepsilon_1),\ \text{for all } k\ge k_2.$$

Let $(1-\varepsilon))(1+\varepsilon_1)^{1/\alpha}=(1-\varepsilon_2)$. Then, for $k\ge k_2$,

$$P(C_k)\le P\left(X(1)\le(1-\varepsilon_2)\theta_\alpha\left(\log\frac{t_{k+1}}{a_{t_{k+1}}}\left(\log t_{k+1}\right)^\beta\left(\log a_{t_{k+1}}\right)^{1-\beta}\right)^{(\alpha-1)/\alpha}\right).$$

From lemma 1, we can find a $k_3(\ge k_2)$ such that for all $k\ge k_3$,

$$P(C_k)\le c_1\left(\log\frac{t_{k+1}}{a_{t_{k+1}}}\left(\log t_{k+1}\right)^\beta\left(\log a_{t_{k+1}}\right)^{1-\beta}\right)^{-\frac{1}{2}}$$
$$\times\exp\left\{(1-\varepsilon_2)^{\alpha/(\alpha-1)}\left(\log\frac{t_{k+1}}{a_{t_k}}\left(\log t_{k+1}\right)^\beta\left(\log a_{t_{k+1}}\right)^{1-\beta}\right)\right\},$$

where c_1 is a positive constant.

Let $(1-\varepsilon_2)^{\alpha/(\alpha-1)}=(1+\varepsilon_3)$, $\varepsilon_3>0$. Then, for all $k\ge k_3$,

$$P(C_k)\le c_1\left(\log\frac{t_{k+1}}{a_{t_k}}\left(\log t_{k+1}\right)^\beta\left(\log a_{t_{k+1}}\right)^{1-\beta}\right)^{-1/2}\left(\frac{a_{t_{k+1}}}{t_k}\right)^{(1+\varepsilon_3)}$$
$$\left(\left(\log t_{k+1}\right)^\beta\left(\log a_{t_{k+1}}\right)^{1-\beta}\right)^{-(1+\varepsilon_3)}.$$

Since

$$\left(a_{t_{k+1}}/t_{k+1}\right)^{(1+\varepsilon_3)}\le\left(a_{t_k}/t_k\right)^{(1+\varepsilon_3)}\le a_{t_k}/t_k,$$

then from (8) and for all $k\ge k_3$, we have

$$P(C_k)\le c_1\left(\log\frac{t_k}{a_{t_k}}\left(\log t_k\right)^\beta\left(\log a_{t_k}\right)^{1-\beta}\right)^{-1/2}\left(\frac{a_{t_k}}{t_k}\right)\left(\left(\log t_k\right)^\beta\left(\log a_{t_k}\right)^{1-\beta}\right)^{-(1+\varepsilon_3)}.$$

$$P(C_k) \le c_1 \left(\log \frac{t_k}{a_{t_k}} (\log t_k)^{\beta} \left(\log a_{t_k}\right)^{1-\beta} \right)^{-1/2} \left(\frac{t_{k+1}-t_k}{t_k} \right)$$

$$\times \frac{1}{(\log t_k)^{1+\varepsilon_3}} \frac{1}{\left(\log a_{t_{k+1}}\right)^{(1-\beta)(1+\varepsilon_3)}}$$

$$\le c_1 \left(\frac{t_{k+1}-t_k}{t_k} \right) \frac{1}{(\log t_k)^{(1+\varepsilon_3)}}.$$

Observe that

$$\int_{k_4}^{\infty} \frac{\mathrm{d}t}{t(\log t)^{(1+\varepsilon_3)}} \ge \sum_{k=k_4}^{\infty} \frac{(t_{k+1}-t_k)}{t_{k+1}(\log t_{k+1})^{(1+\varepsilon_3)}},$$

and

$$\frac{(t_{k+1}-t_k)}{t_{k+1}(\log t_{k+1})^{(1+\varepsilon_3)}} \simeq \frac{(t_{k+1}-t_k)}{t_k(\log t_k)^{(1+\varepsilon_3)}}.$$

Hence

$$\sum_{k=k_4}^{\infty} \frac{(t_{k+1}-t_k)}{t_k(\log t_k)^{(1+\varepsilon_3)}} < \infty.$$

Now we get $\sum_{k=k_4}^{\infty} P(C_k) < \infty$, which in turn establishes (9) by applying to the Borel-Cantelli lemma. The proof of Theorem 3 is complete.

3. Conclusion

In this paper, we developed some limit theorems on increments of stable subordinators. We reformulated the result obtained by Vasudeva and Divanji [3], and established our result by using $\lambda_\beta(t)$.

Acknowledgments

Our thanks to the experts who have contributed towards development of our paper.

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