Scientific Research Publishing

Applied Mathematics, 2016, 7, 2418-2423
http://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

On a Boundary Value Problem for a Polynomial Pencil of the Sturm-Liouville Equation with Spectral Parameter in Boundary Conditions

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How to cite this paper: Adiloglu Nabiev, A. (2016) On a Boundary Value Problem for a Polynomial Pencil of the Sturm-Liouville Equation with Spectral Parameter in Boundary Conditions. *Applied Mathematics*, **7**, 2418-2423.
http://dx.doi.org/10.4236/am.2016.718190

Received: October 25, 2016
Accepted: December 25, 2016
Published: December 28, 2016

Abstract

The boundary value problem with a spectral parameter in the boundary conditions for a polynomial pencil of the Sturm-Liouville operator is investigated. Using the properties of the transformation operators for such operators, the asymptotic formulas for eigenvalues of the boundary value problem are obtained.

Keywords

Sturm-Liouville Equation, Boundary Value Problem, Transformation Operator, Spectral Theory of Differential Operators, Asymptotic Formulas, Fractional Derivative, Eigenvalue, Eigenfunction, Polynomial Pencil

1. Introduction

In this paper the boundary value problem, generated on the finite interval $0 \le x \le \pi$ by equation

$$-y'' + \left(q_0(x) + \lambda q_1(x) + \cdots + \lambda^{n-1} q_{n-1}(x)\right) y = \lambda^{2n} y \tag{1}$$

and the boundary conditions

$$P_{n1}(\lambda) y(0) - y'(0) = P_{n2}(\lambda) y(\pi) + y'(\pi) = 0 \tag{2}$$

is considered. Here we assume that $n > 1$, $q_0(x) \in C[0,\pi]$, $q_n(x) \in C^1[0,\pi]$ $\left(k = \overline{1, n-1}\right)$ are complex valued functions; λ is a complex parameter and

$$P_{nj}(\lambda) = i\lambda^n + \sum_{k=0}^{n-1} \beta_{kj} \lambda^k, \; j = 1; 2$$

with the given constants β_{kj}.

It is known that the Sturm-Liouville problems play an important role in solving many problems in mathematical physics. There has been a growing interest in Sturm-Liouville problems with spectral parameter in boundary conditions in recent years and there are a lot of articles on this subject in the literature. For more detailed analysis we refer to the papers [1]-[9] and the references therein. In the case $n>1$ the simple boundary value problem for the Equation (1) with conditions $y(0)=y(\pi)=0$ is investigated in [10] (also see [11]).

Note that many of these investigations are based on some integral representations for the fundamental solutions of the Sturm-Liouville equation called transformation operators. The transformation operators for Sturm-Liouville equation and quadratic pencil of the Sturm-Liouville equation are constructed and studied in [12] [13] and [14] [15] respectively, while the corresponding operators for the pencil (1) are investigated in [10] [16].

In this paper using the properties of transformation operators, the considering boundary value problem is investigated and asymptotic formula for the eigenvalues is obtained.

We studied in [10], the solutions $y_j(x,\lambda)$ $(j=1,2)$ of the Equation (1) satisfying the initial conditions

$$y_j(0,\lambda)=1, y_j'(0,\lambda)=(-1)^{j+1} i\lambda^n$$

and it is proved that in the sectors of complex plane

$$S_m=\left\{\lambda:\frac{m\pi}{n}\le\arg\lambda\le\frac{(m+1)\pi}{n}\right\}, m=\overline{0,2n-1}$$

the solutions $y_j(x,\lambda)$ have the following integral representations:

$$y_j(x,\lambda)=\mathrm{e}^{(-1)^{j+1}i\lambda^n x}\left[1+\int_{\frac{(-1)^{j+m}-1}{2}x}^{+\infty}K_{v,m}(x,t)\mathrm{e}^{(-1)^m 2i\lambda^n t}\mathrm{d}t\right] \tag{3}$$

where $v=j+\frac{1}{2}\left[(-1)^{j+m}-(-1)^j\right]$, $K_{1,m}(x,.), D_xK_{1,m}(x,.)$ and $K_{2,m}(x,.)$, $D_xK_{2,m}(x,.)$ belong to $L_1(-x;+\infty)$ and $L_1(0;+\infty)$ respectively. Moreover, if $D_{a,t}^{\alpha}\varphi(x,t)$ denotes Riemann-Liouville fractional derivative of order $\alpha(0<\alpha<1)$ (see [17]) with respect to *t*, *i.e.*

$$D_{a,t}^{\alpha}\varphi(x,t)\overset{\det}{=}\frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t(t-s)^{\alpha-1}\varphi(x,s)\mathrm{d}s$$

then for all $x\in[0,\pi]$ the functions $\left(D_{-x,t}^{\frac{1}{n}}\right)^p K_{1,n}(x,t)$ and $\left(D_{0,t}^{\frac{1}{n}}\right)^p K_{2,m}(x,t)\left(p=\overline{1,n}\right)$ belong to $L_1(-x,+\infty)$ and $L_2(0,+\infty)$ respectively. Furthermore, the following equalities are valid:

$$\int_{-x}^{+\infty} K_{1,m}(x,t)\mathrm{e}^{(-1)^m 2i\lambda^n t}\mathrm{d}t = -\sum_{k=0}^{n-1}\gamma_{k+1}\lambda^{-k-1}\alpha_k^{(1)}(x) + (-1)^m (2i\lambda^n)^{-1}\int_{-x}^{+\infty}\mathrm{e}^{(-1)^m 2i\lambda^n t}\left(D_{-x,t}^{\frac{1}{n}}\right)^n K_{1,m}(x,t)\mathrm{d}t, \tag{4}$$

$$\int_{0}^{+\infty} K_{2,m}(x,t)\mathrm{e}^{(-1)^m 2i\lambda^n t}\mathrm{d}t = \sum_{k=0}^{n-1}\gamma_{k+1}\lambda^{-k-1}\alpha_k^{(2)}(x) + (-1)^m (2i\lambda^n)^{-1}\int_{0}^{+\infty}\mathrm{e}^{(-1)^m 2i\lambda^n t}\left(D_{0,t}^{\frac{1}{n}}\right)^n K_{2,m}(x,t)\mathrm{d}t, \tag{5}$$

where

$$\gamma_k = 2^{-\frac{k}{n}}\mathrm{e}^{\frac{i\pi k}{2n}}, k = \overline{1, n-1},$$

$$\alpha_0^{(1)}(x) = \alpha_0^{(2)}(x) = \gamma_{n-1}\int_0^x q_{n-1}(s)\mathrm{d}s,$$

$$\alpha_k^{(j)}(x) = \gamma_{n-k-1}\int_0^x q_{n-k-1}(s)\mathrm{d}s + (-1)^j\sum_{p=1}^{k}\gamma_{n-p}\int_0^x q_{n-p}(s)\alpha_{k-p}^{(j)}(s)\mathrm{d}s,\quad j = 1,2; k = \overline{1, n-1}. \tag{6}$$

2. Asymptotic Formulas for the Solutions and Eigenvalues

By $s(x,\lambda)$ and $c(x,\lambda)$ we denote the solutions of the Equation (1) with initial conditions

$$s(0,\lambda) = c'(0,\lambda) = 0, s'(0,\lambda) = c(0,\lambda) = 1. \tag{7}$$

Using integral representations (3) and formulae (4), (5), it is easy to show that for each $\lambda \in S_m$

$$s(x,\lambda) = \frac{\sin\lambda^n x}{\lambda^n} + (-1)^m (2i\lambda^n)^{-1}\mathrm{e}^{(-1)^m i\lambda^n x}\int_{-x}^{\infty} K_{1,m}(x,t)\mathrm{e}^{(-1)^m 2i\lambda^n t}\mathrm{d}t + (-1)^{m+1}(2i\lambda^n)^{-1}\mathrm{e}^{(-1)^{m+1} i\lambda^n x}\int_0^{\infty} K_{2,m}(x,t)\mathrm{e}^{(-1)^m 2i\lambda^n t}\mathrm{d}t, \tag{8}$$

$$\begin{aligned} s'(x,\lambda) = {} & \cos\lambda^n x - \frac{1}{2i}\sum_{k=0}^{n-1}\gamma_{k+1}\lambda^{-k-1}\alpha_k^{(1)}(x)\mathrm{e}^{i\lambda^n x} + \frac{1}{2i}\sum_{k=0}^{n-1}\gamma_{k+1}\lambda^{-k-1}\alpha_k^{(2)}(x)\mathrm{e}^{-i\lambda^n x} \\ & + (-1)^m (2i\lambda^n)^{-1}\mathrm{e}^{(-1)^m i\lambda^n x}\int_{-x}^{\infty}\left[D_x K_{1,m}(x,t) - \frac{1}{2}\left(D_{-x,t}^{\frac{1}{n}}\right)^n K_{1,m}(x,t)\right]\mathrm{e}^{(-1)^m 2i\lambda^n t}\mathrm{d}t \\ & + (-1)^{m+1}(2i\lambda^n)^{-1}\mathrm{e}^{(-1)^{m+1} i\lambda^n x}\int_0^{\infty}\left[D_x K_{2,m}(x,t) + \frac{1}{2}\left(D_{0,t}^{\frac{1}{n}}\right)^n K_{2,m}(x,t)\right]\mathrm{e}^{(-1)^m 2i\lambda^n t}\mathrm{d}t, \end{aligned} \tag{9}$$

$$c(x,\lambda)=\cos\lambda^n x-\frac{1}{2}\mathrm{e}^{i\lambda^n x}\sum_{k=0}^{n-1}\frac{\gamma_{k+1}}{\lambda^{k+1}}\alpha_k^{(1)}(x)+\frac{1}{2}\mathrm{e}^{-i\lambda^n x}\sum_{k=0}^{n-1}\frac{\gamma_{k+1}}{\lambda^{k+1}}\alpha_k^{(2)}(x)$$
$$+(-1)^{m+1}\left(4i\lambda^n\right)^{-1}\mathrm{e}^{(-1)^m i\lambda^n x}\int_{-x}^{\infty}\mathrm{e}^{(-1)^m 2i\lambda^n t}\left(D_{-x,t}^{\frac{1}{n}}\right)^n K_{1,m}(x,t)\mathrm{d}t \tag{10}$$
$$+(-)^{m+1}\left(4i\lambda^n\right)^{-1}\mathrm{e}^{(-1)^{m+1} i\lambda^n x}\int_{0}^{\infty}\mathrm{e}^{(-1)^m 2i\lambda^n t}\left(D_{0,t}^{\frac{1}{n}}\right)^n K_{2,m}(x,t)\mathrm{d}t,$$

$$c'(x,\lambda)=-\lambda^n\sin\lambda^n x+\frac{1}{2i}\mathrm{e}^{i\lambda^n x}\sum_{k=0}^{n-1}\gamma_{k+1}\lambda^{n-k-1}\alpha_k^{(1)}(x)+\frac{1}{2i}\mathrm{e}^{-i\lambda^n x}\sum_{k=0}^{n-1}\gamma_{k+1}\lambda^{n-k-1}\alpha_k^{(2)}(x)$$
$$+\frac{1}{2}\mathrm{e}^{(-1)^{m+1} i\lambda^n x}\int_{0}^{\infty}\left[D_x K_{2,m}(x,t)+\frac{1}{2}\left(D_{0,t}^{\frac{1}{n}}\right)^n K_{2,m}(x,t)\right]\mathrm{e}^{(-1)^m 2i\lambda^n t}\mathrm{d}t \tag{11}$$
$$+\frac{1}{2}\mathrm{e}^{(-1)^{m} i\lambda^n x}\int_{-x}^{\infty}\left[D_x K_{1,m}(x,t)-\frac{1}{2}\left(D_{-x,t}^{\frac{1}{n}}\right)^n K_{1,m}(x,t)\right]\mathrm{e}^{(-1)^m 2i\lambda^n t}\mathrm{d}t.$$

Let us consider the boundary problem (1), (2). Denote by $\Delta(\lambda)$ the characteristic function of this problem. Then

$$\Delta(\lambda)=\begin{vmatrix}1 & -P_{n1}(\lambda)\\ P_{n2}(\lambda)s(\pi,\lambda)+s'(\pi,\lambda) & P_{n2}(\lambda)c(\pi,\lambda)+c'(\pi,\lambda)\end{vmatrix} \tag{12}$$

Zeros of the function $\Delta(\lambda)$ we'll call eigenvalues of the problem (1), (2). Let $w(\lambda,x;h)$ be the solution of the Equation (1) with initial conditions

$$w(\lambda,0;h)=1,\ w'(\lambda,0;h)=P_{n1}(\lambda) \tag{13}$$

It is clear that

$$\begin{aligned}w(\lambda,x;h)&=P_{n1}(\lambda)s(x,\lambda)+c(x,\lambda)\\ w'(\lambda,x;h)&=P_{n1}(\lambda)s'(x,\lambda)+c'(x,\lambda)\end{aligned} \tag{14}$$

and

$$\Delta(\lambda)=P_{n2}(\lambda)w(\lambda,\pi;h)+w'(\lambda,\pi;h) \tag{15}$$

From formulae (8)-(11) we find that

$$\begin{aligned}w(\lambda,\pi;h)=&P_{n1}(\lambda)\frac{\sin\lambda^n\pi}{\lambda^n}+\cos\lambda^n\pi-\frac{P_{n1}(\lambda)+i\lambda^n}{2i\lambda^n}\mathrm{e}^{i\lambda^n\pi}\sum_{k=0}^{n-1}\lambda^{-k-1}\gamma_{k+1}\alpha_k^{(1)}(\pi)\\ &-\frac{P_{n1}(\lambda)-i\lambda^n}{2i\lambda^n}\mathrm{e}^{-i\lambda^n\pi}\sum_{k=0}^{n-1}\lambda^{-k-1}\gamma_{k+1}\alpha_k^{(2)}(\pi)+o\left(\lambda^{-n}\right)\mathrm{e}^{\left|\operatorname{Im}\lambda^n\pi\right|},|\lambda|\to+\infty\end{aligned} \tag{16}$$

$$\begin{aligned}w'(\lambda,\pi;h)=&P_{n1}(\lambda)\cos\lambda^n\pi-\lambda^n\sin\lambda^n\pi-\frac{P_{n1}(\lambda)-\lambda^n}{2i\lambda^n}\mathrm{e}^{i\lambda^n\pi}\sum_{k=0}^{n-1}\gamma_{k+1}\lambda^{n-k-1}\alpha_k^{(1)}(\pi)\\ &+\frac{P_{n1}(\lambda)+\lambda^n}{2i\lambda^n}\mathrm{e}^{-i\lambda^n\pi}\sum_{k=0}^{n-1}\gamma_{k+1}\lambda^{n-k-1}\alpha_k^{(2)}(\pi)+o(1)\mathrm{e}^{\left|\operatorname{Im}\lambda^n\pi\right|},|\lambda|\to+\infty\end{aligned} \tag{17}$$

Then for $\Delta(\lambda)$ we can write the asymptotic formula

$$\begin{aligned}\Delta(\lambda)=&-\lambda^n\sin\lambda^n\pi+\mathrm{e}^{i\lambda^n\pi}\left(i\lambda^n+\sum_{k=0}^{n-1}\theta_k\lambda^{n-k-1}\right)\\ &+\mathrm{e}^{-i\lambda^n\pi}\left(i\lambda^n+\sum_{k=0}^{n-1}h_k\lambda^{n-k-1}\right)+\mathrm{e}^{\left|\operatorname{Im}\lambda^n\pi\right|}o(1),|\lambda|\to+\infty\end{aligned} \tag{18}$$

where θ_k and h_k are constants. From this we conclude that there exists the constant $L > 0$ such that

$$\left|\Delta(\lambda) + \lambda^n \Delta_0(\lambda)\right| \le Le^{\left|\mathcal{J}_m \lambda^n \pi\right|} \tag{19}$$

for all λ, where

$$\Delta_0(\lambda) = \sin \lambda^n \pi - \mathrm{e}^{i\lambda^n \pi}\left(i + \sum_{k=0}^{n-2} \lambda^{-(k+1)} \theta_k\right) - \mathrm{e}^{-i\lambda^n \pi}\left(i + \sum_{k=0}^{n-2} \lambda^{-(k+1)} h_k\right). \tag{20}$$

From (20) we have that for sufficiently large positive integer k there are a finite number of zeros of $\Delta_0(\lambda)$ in the circle $O_k\left(|\lambda| = \sqrt[n]{k + \frac{1}{2}}\right)$. In other words, the total number of zeros of $\Delta_0(\lambda)$ in O_k is equal to the total number of zeros of the function $\sin \lambda^n \pi$. Moreover, there exists a positive number N such that on the circle $O_k\left(|\lambda| = \sqrt[n]{k + \frac{1}{2}}\right)$ the estimation

$$\left|\lambda^n \Delta_0(\lambda)\right| > N|\lambda|^n \mathrm{e}^{\left|\mathcal{J}_m \lambda^n \pi\right|} \tag{21}$$

satisfies. Hence, from (28), (30) and the equality

$$\Delta(\lambda) = -\lambda^n \Delta_0(\lambda) + \left(\Delta(\lambda) + \lambda^n \Delta_0(\lambda)\right) \tag{22}$$

according to Rouche's theorem we conclude that $\Delta(\lambda)$ and $\lambda^n \Delta_0(\lambda)$ have the same number of zeros in the circle O_k for sufficiently large k. Using a simple asymptotic estimations (see [2]), we obtain that zeros having sufficiently large module of the function $\Delta(\lambda)$ lie near rays $\arg \lambda = \frac{m\pi}{n}$, and so the eigenvalues of the problem (1), (2) consist of $2n$ series. Solving the equation $\Delta(\lambda) = 0$ asymptotically we find the following asymptotic formula for $m^{\underline{th}}$ series of eigenvalues of the problem (1), (2):

$$\lambda_{k,m} = \mathrm{e}^{\frac{2im\pi}{n}} \sqrt[n]{k} + \sum_{s=1}^{n} \frac{b_s^{(m)}}{k^{1+\frac{s-1}{n}}} + 0\left(\frac{1}{k^{1+\frac{n-1}{n}}}\right), k \to +\infty \tag{23}$$

where $\sup_s \left|b_s^{(m)}\right| < \infty$.

Theorem 2. Boundary value problem (1), (2) has a countable number of eigenvalues. The eigenvalues having sufficiently large module are placed near the rays $\arg \lambda = \frac{m\pi}{n}\left(m = \overline{0, m-1}\right)$, and $m^{\underline{th}}$ series of these satisfy the asymptotic formula (23).

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