

# The Zhou's Method for Solving the Euler Equidimensional Equation

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## Abstract

In this work, we apply the Zhou's method [1] or differential transformation method (DTM) for solving the Euler equidimensional equation. The Zhou's method may be considered as alternative and efficient for finding the approximate solutions of initial values problems. We prove superiority of this method by applying them on the some Euler type equation, in this case of order 2 and 3 [2]. The power series solution of the reduced equation transforms into an approximate implicit solution of the original equations. The results agreed with the exact solution obtained via transformation to a constant coefficient equation.

## Keywords

Zhou's Method, Equidimensional Equation, Euler Equation, DTM

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## 1. Introduction

We know that when the coefficients  $p(x)$  and  $q(x)$  are analytic functions on a given domain, then the equation  $y'' + p(x)y' + q(x)y = 0$  has analytic fundamental solution. We want to study equations with coefficients  $p$  and  $q$  having singularities, for this reason we study in this paper with one of the simplest cases, *Euler's equidimensional equation*. This is an important problem because many differential equations in physical sciences have coefficients with singularities [3]. One of the special features of the equidimensional equation is that order of each derivative is equal to the power of the independent variable. This means that this type of equations can be reduced to linear equation with constant coefficient by using a change of the form  $x = e^t$ .

Many numerical methods were developed for this type of equations, specifically on Euler's equations such that Laplace transform method and Adomian method [4]. The

method proposed in this paper was first established by Zhou to solve problems in electric circuits analysis. In this work, the differential transformation method is applied to solve the Euler equidimensional equations and to illustrate this method, several equations of this type are solved [5] [6].

## 2. The Euler Equidimensional Equation

A Euler equidimensional equation is a differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = g(x) \tag{1}$$

where  $a_n, \dots, a_0$  are constants and  $\frac{d^n y}{dx^n}$  is an  $n$ -th derivative of the function  $y(x)$  and  $g(x)$  is a continuous function.

Now, we consider a second order differential equation (homogeneous Euler equidimensional) of the form

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = 0, x > 0 \tag{2}$$

The solution can be obtained by using the change of variables

$$x = e^t \tag{3}$$

where  $\frac{dt}{dx} = \frac{1}{x}$ . In fact, for  $x > 0$ , we introduce  $x = e^t$ , therefore  $t = \ln(x)$ . Then, the first and second derivatives of  $y(x)$  are related by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt} \quad \text{and} \quad \frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{1}{x} \frac{dy}{dt} \right) = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2 y}{dt^2} \tag{4}$$

Now, substituting (4) in (2) yields a second order differential equation with constant coefficients, *i.e.*,

$$\begin{aligned} ax^2 \frac{1}{x^2} \left( \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) + bx \frac{1}{x} \frac{dy}{dt} + cy &= 0 \\ a \frac{d^2 y}{dt^2} - a \frac{dy}{dt} + b \frac{dy}{dt} + cy &= 0 \\ a \frac{d^2 y}{dt^2} + (b-a) \frac{dy}{dt} + cy &= 0 \end{aligned} \tag{5}$$

Equation (5) can be solved using the characteristic polynomial

$$am^2 + (b-a)m + c = 0 \tag{6}$$

where roots are  $m_1$  and  $m_2$  which give the general solution but depending on the type of roots it has, *i.e.*,

a) If  $m_1 \neq m_2$ , real or complex, then the general solution of the Equation (2) is given by

$$y(x) = \alpha_1 x^{m_1} + \alpha_2 x^{m_2}, \quad \alpha_1, \alpha_2 \in \mathbb{R}, x > 0$$

b) If  $m_1 = m_2$ , then the general solution of the Equation (2) is given by

$$y(x) = \alpha_1 x^{m_1} + \alpha_2 x^{m_1} \ln(x), \quad \alpha_1, \alpha_2 \in \mathbb{R}, x > 0$$

### 3. The Zhou’s Method or DTM

Differential transformation method (DTM) of the function  $y(x)$  is defined as

$$Y(k) = \frac{1}{k!} \left[ \frac{d^k y}{dx^k} \right]_{x=x_0} \tag{7}$$

In (7), we have that  $y(x)$  is the original function and  $Y(k)$  is the transformed function. The inverse differential transformation is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k) x^k, \tag{8}$$

but in real applications, function  $y(x)$  is expressed by a finite series and Equation (8) can be written as

$$y(x) = \sum_{k=0}^n Y(k) x^k, \tag{9}$$

which implies that

$$\sum_{k=n+1}^{\infty} Y(k) x^k$$

is negligibly small where  $n$  is decided by the convergence of natural frequency in this study.

The following theorems that can be deduced from Equations (7) and (9) and the proofs are available in [4] [5] [6].

**Theorem 1** If  $y(x) = f(x) \pm g(x)$ , then  $Y(k) = F(k) \pm G(k)$ .

**Theorem 2** If  $y(x) = \alpha_1 f(x)$ , then  $Y(k) = \alpha_1 F(k)$  with  $\alpha_1$  constant.

**Theorem 3** If  $y(x) = \frac{d^n f}{dx^n}$ , then  $Y(k) = \frac{(k+n)!}{k!} F(k+n)$ .

**Theorem 4** If  $y(x) = f(x) g(x)$ , then  $Y(k) = \sum_{k_1=0}^k F(k_1) G(k-k_1)$ .

**Theorem 5** If  $y(x) = x^n$ , then  $Y(k) = \delta(k-n)$ , where

$$\delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

**Theorem 6** (Cárdenas, P). If  $y(x) = x^n f(x)$ , then

$$Y(k) = \begin{cases} 0, & k < n \\ F(k-n), & k \geq n \end{cases}$$

with  $n \in \mathbb{N}$ .

### 4. Numerical Results

To illustrate the ability of the Zhou’s method [2] [7] for the Euler equidimensional equation, the next problem is provided and the results reveal that this method is very effective.

**Example 1** (Homogeneous case). To begin, we consider the initial value problem

$$\begin{cases} x^2 y'' - 4xy' + 4y = 0 \\ y(1) = -2, y'(1) = -11 \end{cases} \tag{10}$$

Using the substitution (3) and (4), the IVP (10) is transformed to a second order differential equation with constant coefficients, i.e.,

$$\begin{aligned} x^2 \left( \frac{1}{x^2} (y''(t) - y'(t)) \right) - 4x \left( \frac{1}{x} y'(t) \right) + 4y(t) &= 0 \\ y''(t) - y'(t) - 4y'(t) + 4y(t) &= 0 \\ y''(t) - 5y'(t) + 4y(t) &= 0 \end{aligned} \tag{11}$$

Now, of the initial conditions we have that as  $x = 1$ , then  $t = 0$  and therefore  $y(0) = -2$  and  $y'(0) = -11$ . So, the new IVP is given by

$$\begin{cases} y'' - 5y' + 4y = 0 \\ y(0) = -2, y'(0) = -11 \end{cases} \tag{12}$$

The exact solution of the problem (12) is  $y(x) = x - 3x^4$ . Taking the differential transformation of this problem we obtain

$$\frac{(k+2)!}{k!} Y(k+2) - 5 \frac{(k+1)!}{k!} Y(k+1) + 4Y(k) = 0$$

or

$$Y(k+2) = \frac{1}{(k+2)(k+1)} [5(k+1)Y(k+1) - 4Y(k)] \tag{13}$$

where  $Y(0) = -2$  and  $Y(1) = -11$ . Therefore, the recurrence Equation (13) gives:

- $k = 0$ ,

$$Y(2) = \frac{1}{2} (5Y(1) - 4Y(0)) = \frac{1}{2} (-55 + 8) = -\frac{47}{2}$$

- $k = 1$ ,

$$Y(3) = \frac{1}{6} (10Y(2) - 4Y(1)) = \frac{1}{6} (-235 + 44) = -\frac{191}{6}$$

- $k = 2$ ,

$$Y(4) = \frac{1}{12} (15Y(3) - 4Y(2)) = \frac{1}{12} \left( -\frac{955}{2} + 94 \right) = -\frac{767}{24}$$

Therefore, using (9), the closed form of the solution can be easily written as

$$\begin{aligned} y(t) &= \sum_{k=0}^n Y(k) t^k = Y(0) + Y(1)t + Y(2)t^2 + Y(3)t^3 + \dots \\ &= -2 - 11t - \frac{47}{2}t^2 - \frac{191}{6}t^3 - \frac{767}{24}t^4 - \dots \end{aligned} \tag{14}$$

but since  $t = \ln(x)$ , then we obtain (see **Figure 1**)

$$y(x) \approx -2 - 11 \ln(x) - \frac{47}{2} (\ln(x))^2 - \frac{191}{6} (\ln(x))^3 - \frac{767}{24} (\ln(x))^4 - \dots$$

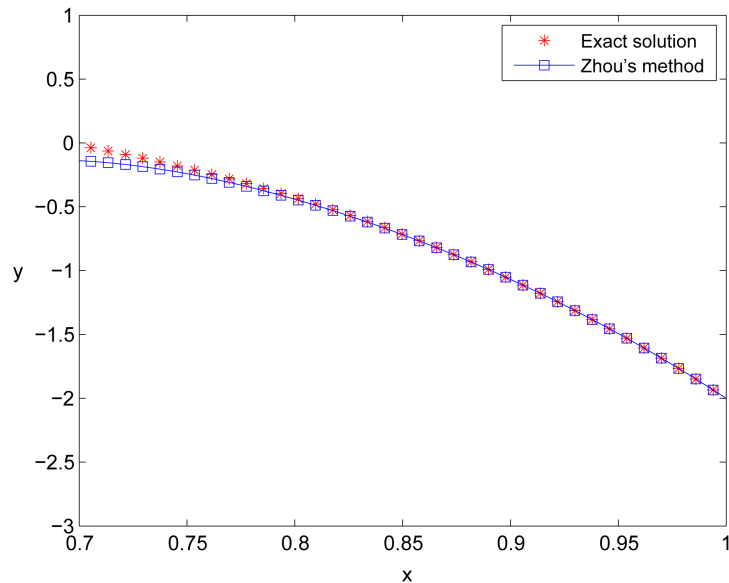


Figure 1. The Zhou's method vs. exact solution.

**Example 2 (Non-homogeneous case).** We consider the following IVP

$$\begin{cases} x^2 y'' + 4xy' + 2y = \ln(x) \\ y(1) = 2, y'(1) = 0 \end{cases} \tag{15}$$

Then, problem (15) is transformed to a second order differential equation with constant coefficient by using (3) and (4), i.e.,

$$\begin{aligned} x^2 \left( \frac{1}{x^2} (y''(t) - y'(t)) \right) + 4x \left( \frac{1}{x} y'(t) \right) + 2y(t) &= t \\ y''(t) - y'(t) + 4y'(t) + 2y(t) &= t \\ y''(t) + 3y'(t) + 2y(t) &= t \end{aligned} \tag{16}$$

We know that of the initial conditions  $x=1$  and therefore  $t=0$ , so we obtain  $y(0)=2$  and  $y'(0)=0$ . Then, the IVP is given by

$$\begin{cases} y'' + 3y' + 2y = t \\ y(0) = 2, y'(0) = 0 \end{cases} \tag{17}$$

The exact solution of the problem (15) is  $y(x) = 5x^{-1} - \frac{9}{4}x^{-2} + \frac{1}{2}\ln(x) - \frac{3}{4}$ . Now, the DTM of (17) is

$$\frac{(k+2)!}{k!} Y(k+2) + 3 \frac{(k+1)!}{k!} Y(k+1) + 2Y(k) = \delta(k-1)$$

or

$$Y(k+2) = \frac{1}{(k+2)(k+1)} [-3(k+1)Y(k+1) - 2Y(k) + \delta(k-1)] \tag{18}$$

with  $Y(0) = 2$  and  $Y(1) = 0$ . So, the recurrence Equation (18) gives:

- $k = 0,$

$$Y(2) = \frac{1}{2}(-3Y(1) - 2Y(0) + \delta(-1)) = \frac{1}{2}(-4) = -2$$

- $k = 1,$

$$Y(3) = \frac{1}{6}(-3(2)Y(2) - 2Y(1) + \delta(0)) = \frac{1}{6}(12 + 1) = \frac{13}{6}$$

- $k = 2,$

$$Y(4) = \frac{1}{12}(-3(3)Y(3) - 2Y(2) + \delta(1)) = \frac{1}{12}\left(-\frac{39}{2} + 4\right) = -\frac{31}{24}$$

- $k = 3,$

$$Y(5) = \frac{1}{20}(-12Y(4) - 2Y(3) + \delta(2)) = \frac{1}{20}\left(\frac{31}{2} - \frac{13}{3}\right) = \frac{67}{120}$$

Therefore, using (9), the closed form of the solution can be easily written as

$$\begin{aligned} y(t) &= \sum_{k=0}^n Y(k)t^k = Y(0) + Y(1)t + Y(2)t^2 + Y(3)t^3 + \dots \\ &= 2 + 0t - 2t^2 + \frac{13}{6}t^3 - \frac{31}{24}t^4 + \frac{67}{120}t^5 + \dots \end{aligned} \tag{19}$$

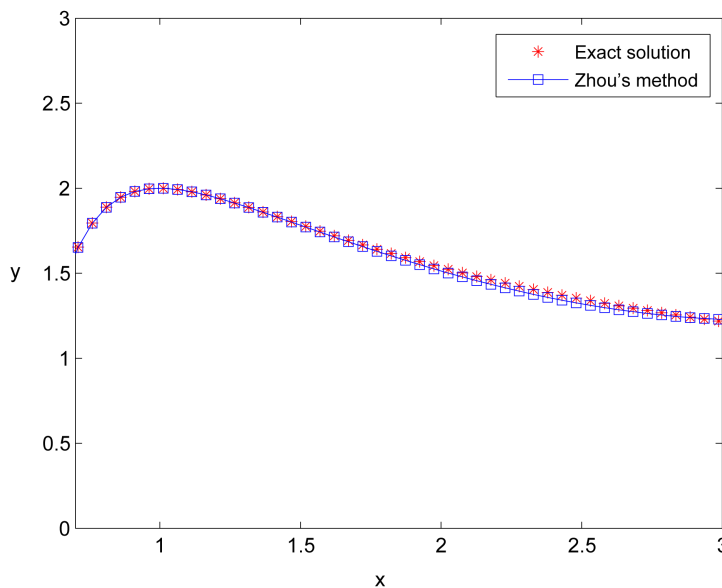
But since  $t = \ln(x)$ , then we obtain (see **Figure 2**)

$$y(x) \approx 2 - 2(\ln(x))^2 + \frac{13}{6}(\ln(x))^3 - \frac{35}{24}(\ln(x))^4 + \frac{67}{120}(\ln(x))^5 + \dots$$

**Example 3** (Third order Euler’s equation). Consider the following IVP

$$\begin{cases} x^3 y''' + 10x^2 y'' - 20xy' + 20y = 0 \\ y(1) = 0, y'(1) = -1, y''(1) = 1 \end{cases} \tag{20}$$

Now, to find  $y'''(x)$  we use the chain rule. In fact we obtain



**Figure 2.** The Zhou’s method vs. exact solution.

$$\begin{aligned} \frac{d^3 y}{dx^3} &= \frac{d}{dx} \left[ \frac{1}{x^2} \left( \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) \right] = -\frac{2}{x^3} \left[ \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right] + \frac{1}{x^2} \left[ \frac{d^3 y}{dt^3} - \frac{d^2 y}{dt^2} \right] \\ &= \frac{1}{x^3} \frac{d^3 y}{dt^3} - \frac{3}{x^3} \frac{d^2 y}{dt^2} + \frac{2}{x^3} \frac{dy}{dt} \end{aligned} \tag{21}$$

Therefore, using (3), (4) and (21) we have

$$\begin{aligned} x^3 \left[ \frac{1}{x^3} (y''' - 3y'' + 2y') \right] + 10x^2 \left[ \frac{1}{x^2} (y'' - y') \right] - 20x \left[ \frac{1}{x} y' \right] + 20y &= 0 \\ y''' - 3y'' + 2y' + 10y'' - 10y' - 20y' + 20y &= 0 \\ y''' + 7y'' - 28y' + 20y &= 0 \end{aligned} \tag{22}$$

Now, as in the previous example  $x=1$  and then  $t=0$ . So, the new initial conditions are given by  $y(0)=0, y'(0)=-1$  and  $y''(0)=1$ . Using (7) we find that  $Y(0)=0, Y(1)=-1$  and  $Y(2)=\frac{1}{2!}$ . Therefore, we obtain the IVP

$$\begin{cases} y''' + 7y'' - 28y' + 20y = 0 \\ y(0) = 0, y'(0) = -1, y''(0) = 1 \end{cases} \tag{23}$$

Applying DTM to (23) we obtain

$$\frac{(k+3)!}{k!} Y(k+3) + 7 \frac{(k+2)!}{k!} Y(k+2) - 28 \frac{(k+1)!}{k!} Y(k+1) + 20Y(k) = 0$$

or

$$\begin{aligned} Y(k+3) &= \frac{1}{(k+3)(k+2)(k+1)} \left[ -7(k+2)(k+1)Y(k+2) + 28(k+1)Y(k+1) - 20Y(k) \right] \end{aligned} \tag{24}$$

So, the recurrence equation (24) gives:

- $k=0,$

$$Y(3) = \frac{1}{6} (-7(2)Y(2) + 28Y(1) - 20Y(0)) = \frac{1}{6} (-7 - 28) = -\frac{35}{6}$$

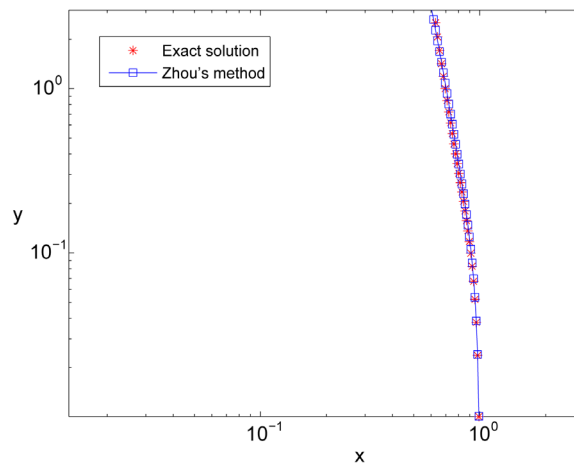
- $k=1,$

$$\begin{aligned} Y(4) &= \frac{1}{24} (-7(3)(2)Y(3) + 28(2)Y(2) - 20Y(1)) \\ &= \frac{1}{24} \left( -42 \left( \frac{-35}{6} \right) + 56 \left( \frac{1}{2} \right) - 20(-1) \right) = \frac{293}{24} \end{aligned}$$

- $k=2,$

$$\begin{aligned} Y(5) &= \frac{1}{60} (-7(4)(3)Y(4) + 28(3)Y(3) - 20Y(2)) \\ &= \frac{1}{60} \left( -84 \left( \frac{293}{24} \right) + 84 \left( \frac{-35}{6} \right) - 20 \left( \frac{1}{2} \right) \right) = \frac{-1017}{40} \end{aligned}$$

Therefore, using (9), the closed form of the solution can be easily written as



**Figure 3.** The Zhou's method vs. exact solution.

$$\begin{aligned}
 y(t) &= \sum_{k=0}^n Y(k)t^k = Y(0) + Y(1)t + Y(2)t^2 + Y(3)t^3 + \dots \\
 &= 0 + (-1)t + \frac{1}{2}t^2 - \frac{35}{6}t^3 + \frac{293}{24}t^4 - \frac{1017}{40}t^5 + \dots \\
 &= -t + \frac{1}{2}t^2 - \frac{35}{6}t^3 + \frac{293}{24}t^4 - \frac{1017}{40}t^5 + \dots
 \end{aligned}
 \tag{25}$$

But since  $t = \ln(x)$ , then we obtain (see **Figure 3**)

$$y(x) \approx -(\ln(x)) + \frac{1}{2}(\ln(x))^2 - \frac{35}{6}(\ln(x))^3 + \frac{293}{24}(\ln(x))^4 - \frac{1017}{40}(\ln(x))^5 + \dots$$

### 5. Conclusion

In this paper, we presented the definition and handling of one-dimensional differential transformation method or Zhou's method. Using the substitutions (3) and (4), Euler's equidimensional equations were transformed to a second and third order differential equations with constant coefficients, next using DTM these equations were transformed into algebraic equations (iterative equations). The new scheme obtained by using the Zhou's method yields an analytical solution in the form of a rapidly convergent series. This method makes the solution procedure much more attractive. The figures [4] [5] and [6] clearly show the high efficiency of DTM with the three examples proposed.

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