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On the Injective Equitable Domination of Graphs

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Abstract

A dominating set D in a graph G is called an injective equitable dominating set (Inj-equitable dominating set) if for every $v \in V - D$, there exists $u \in D$ such that u is adjacent to v and $\left|\deg_{in}(u) - \deg_{in}(v)\right| \leq 1$. The minimum cardinality of such a dominating set is denoted by $\gamma_{ine}(G)$ and is called the Inj-equitable domination number of *G*. In this paper, we introduce the injective equitable domination of a graph and study its relation with other domination parameters. The minimal injective equitable dominating set, the injective equitable independence number $\beta_{ine}(G)$, and the injective equitable domatic number $d_{ine}(G)$ are defined.

Keywords

Domination, Injective Equitable Domination, Injective Equitable Domination Number

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected graph with neither loops nor multiple edges. The order and the size of G are denoted by n and m respectively, the open neighborhood $N(v) = \{u \in V : uv \in E\}$ and the closed neighborhood $N[v] = N(v) \cup \{v\}$. The degree of a vertex v in G is $d(v) = |N(v)|$. Let G and H be any two graphs with vertex sets $V(G)$, $V(H)$ and edge sets $E(G)$, $E(H)$, respectively. Then, the union $G \cup H$ is the graph whose vertex set is $V(G) \cup V(H)$ and edge set is $E(G) \cup E(H)$. For graph theoretic terminology, we refer to [1] and [2].

A set D of vertices in a graph $G = (V, E)$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in *D*. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. An excellent treatment of the fundamentals of domination is given by Hayens *et al.* [3]. A survey of several advanced topics in domination is given in the book edited by Haynes *et al.* [4].

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The injective domination of graphs has been introduced by A.Alwardi *et al.* [5]. For a graph G, a subset D of $V(G)$ is called an injective dominating set (Inj-dominating set) if for every vertex $v \in V-D$ there exists a vertex $u \in D$ such that $|\Gamma(u,v)| \geq 1$, where $|\Gamma(u,v)|$ is the number of common neighborhood between the vertices u and v. The minimum cardinality of such dominating set is denoted by $\gamma_{in}(G)$ and is called the injective domination number(Inj-domination number) of G. The Inj-neighborhood of a vertex $u \in V(G)$ denoted by $N_{in}(u)$ is defined as $N_{in}(u) = \{v \in V(G) : |\Gamma(u,v)| \geq 1\}$. The cardinality of $N_{in}(u)$ is called the injective degree of the vertex u and is denoted by $\deg_{in}(u)$ in G and $N_{in}[u] = N_{in}(u) \cup \{u\}$.

A subset D of V is called equitable dominating set of G if every vertex $u \in V-D$ adjacent to at least one vertex $v \in D$ and $|d(u)-d(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by $\gamma_e(G)$ and is called equitable domination number of G [6]. Equitable domination has interesting applications in the context of social networks. In a network, nodes with nearly equal capacity may interact with each other in a better way.

The importance of injective and equitable domination of graphs motivated us to introduce the injective equitable domination of graphs which mixes the two concepts.

As there are a lot of applications of domination, in particular the injective and equitable domination, we are expecting that our new concept has some applications.

2. The Injective Equitable Dominating Set

Definition 1 *A subset D of* $V(G)$ *is called injective equitable dominating set* (*Inj-equitable dominating set*) *if for every vertex* $v \in V-D$ *there exists a vertex* $u \in D$ *such that u is adjacent to v and* $|\deg_{in}(u)-\deg_{in}(v)| \leq 1$. *The minimum cardinality of such a dominating set is denoted by* γ_{ine} *and is called the Inj-equitable domination number of G. A* γ_{ine}*-set of G is the minimum dominating set of G.*

It is easy to see that any Inj-equitable dominating set in a graph G is also a dominating set, and then $\gamma(G) \leq \gamma_{ine}(G)$ and $\gamma_{ine}(G)=1$ if and only if $\gamma(G)=1$.

In the following propostion the Inj-equitable domination number of some standard graphs are determined.

Proposition 1

1) For any complete graph K_n, $\gamma_{ine}(K_n)=1$.

2) For any path P_n, with n vertices, $\gamma_{ine}(P_n)=\left\lceil \frac{n}{3} \right\rceil$.

3) For any cycle C_n on n vertices, $\gamma_{ine}(C_n)=\left\lceil \frac{n}{3} \right\rceil$.

4) For any complete bipartite graph $K_{r,s}$, where $r+s \geq 4$,

$$\gamma_{ine}(k_{r,s}) = \begin{cases} 2 & \text{if } |r-s| \leq 1; \\ r+s & \text{if } |r-s| \geq 2. \end{cases}$$

5) For any wheel graph $\gamma_{ine}(W_n)=1$.

Definition 1 motivated us to define the inherent Inj-equitable graph of any graph G

as follows:

Definition 2 *Let* $G=(V,E)$ *be a graph. The inherent Inj-equitable graph of G, denoted by* $IIE(G)$, *is defined as the graph with vertex set* $V(G)$ *and two vertices u and v are adjacent in the* $IIE(G)$ *if and only if u and v are adjacent in G and* $|\deg_{in}(u)-\deg_{in}(v)|\le 1$.

Theorem 2: *For any graph* G, $\gamma_{ine}(G)=\gamma(IIE(G))$.

Proof. Since any Inj-equitable dominating set of G is a dominating set of $IIE(G)$, then $\gamma(IIE(G))\le\gamma_{ine}(G)$. Now, let D be any γ-dominating set of $IIE(G)$. Then for any $u\in V(IIE(G))-D$, there exists $v\in D$ such that u and v are adjacent in $IIE(G)$. So, $|\deg_{in}(u)-\deg_{in}(v)|\le 1$. Therefore, *D* is Inj-equitable dominating set of *G*. Then, $\gamma_{ine}(G)\le\gamma(IIE(G))$. Hence, $\gamma_{ine}(G)=\gamma(IIE(G))$.

Definition 3 *The Inj-equitable neighborhood of* $u\in V$, $N_{ine}(u)$, *is defined as*

$$N_{ine}(u)=\{v\in V: v\in N(u)\text{ and }|\deg_{in}(u)-\deg_{in}(v)|\le 1\}.$$

The cardinality of $N_{ine}(u)$ is called the Inj-equitable degree of *u* and is denoted by $\deg_{ine}(u)$. The maximum and minimum Inj-equitable degree of a vertex in *G* are denoted respectively by $\Delta_{ine}(G)$ and $\delta_{ine}(G)$. That is,

$$\Delta_{ine}(G)=\max_{u\in V(G)}|N_{ine}(u)|$$

$$\delta_{ine}(G)=\min_{u\in V(G)}|N_{ine}(u)|.$$

Definition 4 *For any graph G, an edge* $e=uv$ *is called Inj-equitable edge if* $|\deg_{in}(u)-\deg_{in}(v)|\le 1$ *and we say u is Inj-equitable adjacent to v or u is Inj-equitable dominate v.*

Proposition 3 *For any graph* $G=(V,E)$, $\sum_{u\in V(G)}\deg_{ine}(u)=2q_{ine}$, *where* q_{ine} *is the number of Inj-equitable edges in G.*

Proof. Let *G* be a graph and let *H* be the Inj-equitable graph of *G*. Then $\sum_{u\in V(H)}\deg(u)=2q$, where q is the number of edges in *H*. Since the number of edges in *H* is the number of Inj-equitable edges in *G*, then q equals q_{ine}. Also, $\deg_{ine}(u)$ in *G* is equal to $\deg(u)$ in *H*. Hence, $\sum_{u\in V(G)}\deg_{ine}(u)=2q_{ine}$.

Definition 5 *Let* $G=(V,E)$ *be a graph. A vertex* $v\in V$ *is called Inj-equitable isolated vertex if* $N_{ine}(v)=\phi$. *The set of all Inj-equitable isolated vertices is denoted by* I_{ine}. *Hence* $I\subset I_{ine}\subset D$ *for every Inj-equitable dominating set D, where I is the set of isolated vertices.*

Definition 6 *A graph G is called Inj-equitable totally disconnected graph if it has no Inj-equitable edge.*

Theorem 4 *For any graph G with n vertices,* $1\le\gamma_{ine}(G)\le n$. *Further,* $\gamma_{ine}(G)=1$ *if and only if there exists at least one vertex v in G such that* $\deg_{ine}(v)=n-1$. $\gamma_{ine}(G)=n$ *if and only if G is Inj-equitable totally disconnected graph.*

Proof. It is obviously that $1\le\gamma_{ine}(G)$. Also, for any graph $G=(V,E)$, $V(G)$ is an injective equitable dominating set. Therefore, $\gamma_{ine}(G) \le n$. Hence, $1 \le \gamma_{ine}(G) \le n$.

Now, we want to prove that $\gamma_{ine}(G) = 1$ if and only if there exists at least one vertex v in G such that $\deg_{ine}(v) = n-1$. Suppose that $\gamma_{ine}(G) = 1$ and $D = \{v\}$ is a γ_{ine}-set. So, for all $u \in V - D$, $uv \in E(G)$ and $|\deg_{in}(u) - \deg_{in}(v)| \le 1$. Hence, $\deg_{ine}(v) = n-1$.

conversely, suppose that there exists at least one vertex v in G such that $\deg_{ine}(v) = n-1$. Then, $D = \{v\}$ is an Inj-equitable dominating set. Hence, $\gamma_{ine}(G) = 1$.

To prove that $\gamma_{ine}(G) = n$ if and only if G is Inj-equitable totally disconnected graph, suppose that G is Inj-equitable totally disconnected graph. So, all the vertices are Inj-equitable isolated. Hence, $\gamma_{ine}(G) = n$.

Conversely, suppose that G has at least one Inj-equitable edge, say $e = uv$. So, $|\deg_{in}(u) - \deg_{in}(v)| \le 1$. Therefore, $V - \{u\}$ is an Inj-equitable dominating set, and so, $\gamma_{ine}(G) \le n-1$ contradicts that $\gamma_{ine}(G) = n$. Hence, G is Inj-equitable totally disconnected graph.

Proposition 5 *If a graph G has no Inj-equitable isolated vertices, then*

$$\gamma_{ine}(G) \le \frac{n}{2}.$$

In the following theorem, we present the graph for which $\gamma_{ine}(G)$ and $\gamma(G)$ are equal.

Theorem 6 *Let G be a graph such that any two adjacent vertices contained in a triangle or G is regular triangle-free graph. Then,* $\gamma_{ine}(G) = \gamma(G)$.

Proof. Suppose that G is a regular triangle-free graph and D is a γ-set of G. Then $\gamma(G) = |D|$. Let u and v be any two adjacent vertices in G. Then $N(u) \cap N_{in}(u) = \phi$. Therefore, $\deg_{in}(u) = \deg(v)$. Since G is regular, $\deg(v) = \deg(u)$. So, $|\deg_{in}(u) - \deg_{in}(v)| = |\deg(u) - \deg(v)| = 0 \le 1$. Therefore, D is an Inj-equitable dominating set. So that, $\gamma_{ine}(G) \le |D| = \gamma(G)$. But $\gamma(G) \le \gamma_{ine}(G)$. Hence, $\gamma_{ine}(G) = \gamma(G)$.

Let G be a graph such that any two adjacent vertices contains in a triangle. It is clear that for any $u \in V(G)$, $\deg_{in}(u) = \deg(u)$. So, $|\deg_{in}(u) - \deg_{in}(v)| = |\deg(u) - \deg(v)| = 0 \le 1$. By the same way of the proof of regular triangle-free graph we can prove that $\gamma_{ine}(G) = \gamma(G)$.

Lemma 1 *For any two graphs* G_1 *and* G_2, $\gamma_{ine}(G_1 \cup G_2) = \gamma_{ine}(G_1) + \gamma_{ine}(G_2)$.

Proof. Let $G \cong G_1 \cup G_2$ and let S_1 and S_2 be the minimum Inj-equitable dominating set of G_1 and G_2, respectively, such that $|S_1| = \gamma_{ine}(G_1)$ and $|S_2| = \gamma_{ine}(G_2)$. Now, it is obviously that $S_1 \cup S_2$ is an Inj-equitable dominating set of $G \cong G_1 \cup G_2$. Therefore,

$$\gamma_{ine}(G) \le |S_1 \cup S_2| = \gamma_{ine}(G_1) + \gamma_{ine}(G_2).$$

That is,

$$\gamma_{ine}(G) \le \gamma_{ine}(G_1) + \gamma_{ine}(G_2). \tag{1}$$

To prove $\gamma_{ine}(G) \ge \gamma_{ine}(G_1) + \gamma_{ine}(G_2)$ by contradiction. Let S' be the minimum Inj-equitable dominating set of G such that $|S'| = \gamma_{ine}(G)$. Let

$\gamma_{ine}(G) < \gamma_{ine}(G_1) + \gamma_{ine}(G_2)$. Then there exist S_1' and S_2', where S_1' is the minimum Inj-equitable dominating set of G_1 and S_2' is the minimum Inj-equitable dominating set of G_2 and either $|S_1'| < |S_1|$ or $|S_2'| < |S_2|$ which is a contradiction. Hence

$$\gamma_{ine}(G) \geq \gamma_{ine}(G_1) + \gamma_{ine}(G_2). \quad (2)$$

From 1 and 2, we get

$$\gamma_{ine}(G) = \gamma_{ine}(G_1) + \gamma_{ine}(G_2).$$

By mathematical induction, we can generalize Lemma 1 as follows:

Proposition 7 *Let* $G = \bigcup_{j=1}^{m} G_j$ *be a graph. Then* $\gamma_{ine}(G) = \sum_{j=1}^{m} \gamma_{ine}(G_j)$.

Theorem 8 *Let G be a graph with* $n \geq 2$ *vertices. Then* $\gamma_{ine}(G) = n-1$ *if and only if* $G \cong H \cup K_2$, *where H is Inj-equitable totally disconnected graph*.

Proof. Let G be a graph with $n \geq 2$ vertices and let $\gamma_{ine}(G) = n-1$. By Theorem 2, $\gamma(IIE(G)) = n-1$ which implies that $IIE(G)$ will be of the form $H \cup K_2$. By the Definition 2, all the edges of G are not Inj-equitable edge except one edge. Therefore, $G \simeq H \cup K_2$.

Conversely, let $G \simeq H \cup K_2$ where H is an Inj-equitable totally disconnected graph. By Lemma 1, $\gamma_{ine}(G) = \gamma_{ine}(H) + \gamma_{ine}(K_2) = n-2+1 = n-1$.

Definition 7 *An Inj-equitable dominating set D is said to be a minimal Inj-equitable dominating set if no proper subset of D is an Inj-equitable dominating set. A minimal Inj-equitable dominating set D of maximum cardinality is called* Γ_{ine} *-set and its cardinality, denoted by* $\Gamma_{ine}(G)$, *is called upper Inj-equitable domination number*.

The following theorem gives the characterization of the minimal Inj-equitable dominating set .

Theorem 9 *An Inj-equitable dominating set D is minimal if and only if for every vertex* $u \in D$ *one of the following holds*:

1) u is not Inj-equitable adjacent to any vertex in D.

2) There exists a vertex $v \in V - D$ such that $N_{ine}(v) \cap D = \{u\}$.

Proof. Suppose that D is minimal Inj-equitable dominating set and suppose that $u \in D$. Then, $D - \{u\}$ is not Inj-equitable dominating set. Therefore, there exists a vertex $v \in (V - D) \cup \{u\}$ which is not Inj-equitable adjacent to any vertex in $D - \{u\}$. Then, either $v = u$ or $v \neq u$. If $v = u$, then u is not Inj-equitable adjacent to any vertex in D. If $v \neq u$, then $v \in V - D$ and not Inj-equitable adjacent to any vertex in $D - \{u\}$. But V is Inj-equitable dominated by D. So, V is Inj-equitable adjacent only to vertex u in D. Hence, $N_{ine}(v) \cap D = \{u\}$.

Conversely, suppose that D is an Inj-equitable dominating set and for every vertex $u \in D$ one of the two conditions holds. We want to prove that D is minimal. Suppose D is not minimal. Then there exists a vertex $u \in D$ such that $D - \{u\}$ is an Inj-equitable dominating set. Therefore, there exists $v \in D - \{u\}$ such that v Inj-equitable adjacent to u. Therefore, u does not satisfy (i). Also, if $D - \{u\}$ is Inj-equitable dominating set, then every vertex $v \in V - D$ is Inj-equitable adjacent to at least one vertex in $D-\{u\}$. So, condition (ii) does not hold which is a contradiction. Hence, D is a minimal Inj-equitable dominating set.

Theorem 10 *A graph G has a unique minimal Inj-equitable dominating set if and only if the set of all Inj-equitable isolated vertices forms an Inj-equitable dominating set.*

Proof. Let G has a unique minimal Inj-equitable dominating set D and let $v \in D - I_{ine}$. Since v is not an Inj-equitable isolated, $V-\{v\}$ is an Inj-equitable dominating set. Therefore, there exists a minimal Inj-equitable dominating set $D_1 \subseteq V-\{v\}$ and $D_1 \neq D$, which contradicts that G has a unique minimal Inj-equitable dominating set. Hence, $D = I_{ine}$.

Conversely, let I_{ine} forms an Inj-equitable dominating set. Then it is clear that G has a unique minimal Inj-equitable dominating set.

Theorem 11 *If G is a graph has no Inj-equitable isolated vertices, then the complement* $V-S$ *of any minimal Inj-equitable dominating set S is also an Inj-equitable dominating set.*

Proof. Let S be any minimal Inj-equitable dominating set of G and $V-S$ is not Inj-equitable dominating set. So, there exist at least one vertex $u \in S$ which is not Inj-equitable dominated by any vertex in $V-S$. Since G has no Inj-equitable isolated vertices, the vertex u must be Inj-equitable dominated by at least one vertex in $S-\{u\}$. Thus, $S-\{u\}$ is an Inj-equitable dominating set of G, which contradicts the minimality of S. Hence, $V-S$ is an Inj-equitable dominating set.

Theorem 12 *For any graph with n vertices*

$$\frac{n}{1+\Delta_{ine}} \leq \gamma_{ine}(G).$$

Proof. Let S be a γ_{cne}-set of G. Then for all $u \in S$,

$$|N_{ine}(u)| \leq \Delta_{ine}(G)$$

Thus,

$$|N_{ine}(S)| \leq \gamma_{ine}(G)\Delta_{ine}(G).$$

Now,

$$n = |N_{ine}[S]| = |S \cup N_{ine}(S)|$$

Therefore,

$$n \leq \gamma_{ine}(G) + \gamma_{ine}(G)\Delta_{ine}(G).$$

Hence,

$$\frac{n}{1+\Delta_{ine}} \leq \gamma_{ine}(G).$$

Definition 8 *Let* $G=(V,E)$. *A subset S of* $V(G)$ *is called an Inj-equitable independent set if for any* $u \in S$, $v \notin N_{ine}(u)$ *for all* $v \in S-\{u\}$. *The maximum cardinality of an Inj-equitable independent set is denoted by* β_{ine}.

Definition 9 *An Inj-equitable independent set S is called maximal if any vertex set properly containing S is not Inj-equitable independent set. The lower Inj-equitable* *independent number* i_{ine} *is the minimum cardinality of the maximal Inj-equitable independent set.*

Theorem 13 *Let S be a maximal Inj-equitable independent set. Then S is a minimal Inj-equitable dominating set.*

Proof. Let S be a maximal Inj-equitable independent set. Let $v \in V - S$. If $v \notin N_{ine}(u)$ for every $u \in S$, then $S \cup \{v\}$ is an Inj-equitable independent set, a contradiction to the maximality of *S*. So, $v \in N_{ine}(u)$ for some $u \in S$. Hence, *S* is ann Inj-equitable dominating set. Since for any $v \in S$, $v \notin N_{ine}(u)$ for every $u \in S - \{v\}$, either $N(u) \cap S = \phi$ or $|\deg_{in}(u) - \deg_{in}(v)| \geq 2$ for all $v \in N(u) \cap S$. Therefore, S is minimal Inj-equitable dominating set.

Theorem 14 *For any graph G,*

$$\gamma_{ine} \leq i_{ine} \leq \beta_{ine} \leq \Gamma_{ine}.$$

3. Injective Equitable Domatic Number

The maximum order of a partition of a vertex set *V* of a graph *G* into dominating sets is called the domatic number of *G* and is denoted by $d(G)$ [7]. In this section we present a few basic results on the Inj-equitable domatic number of a graph.

Definition 10 *An Inj-equitable domatic partition of a graph G is a partition* $\{V_1, V_2, \cdots, V_k\}$ *of* $V(G)$ *in which each* V_i *is Inj-equitable dominating set of G. The Inj-equitable domatic number is the maximum order of an Inj-equitable domatic partition and is denoted by* $d_{ine}(G)$.

Example 1 *For the graph G given in* Figure 1, $\{\{v_1, v_2\}, \{v_3, v_4\}\}$ *is an Inj-equitable domatic partition of maximum order. Therefore, the Inj-equitable domatic number of G is* $d_{ine}(G) = 2$.

Proposition 15

1) For any path P_n with $n \geq 2$, $d_{ine}(P_n) = 2$.

2) For any cycle C_n with $n \geq 3$, $d_{ine}(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3; \\ 2 & \text{otherwise.} \end{cases}$

3) For any complete graph K_n, $d_{ine}(K_n) = n$.

4) For any complete bipartite graph $K_{r,s}$, where $r + s \geq 4$,

$$d_{ine} \leq d_{ine}(K_{r,s}) = \begin{cases} \min\{r, s\} & \text{if } |r - s| \leq 1; \\ 1 & \text{if } |r - s| \geq 2. \end{cases}$$

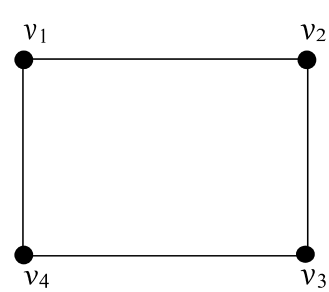

Figure 1. Circle with 4 vertices C_4.

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Proposition 16 *For any graph G,* $d_{ine} \leq d(G)$, *where* $d(G)$ *is the domatic number of G.*

Proof. Since any partition of V into Inj-equitable dominating set is also partition of V into dominating set, $d_{ine} \leq d(G)$.

4. Conclusions

In this paper, we introduced the Inj-equitable domination of graphs and some other related parameters like Inj-equitable independent number, uper Inj-equitable domination number and domatic Inj-equitable domination number.

There are many other related parameters for future studies like connected Inj-equitable domination, total Inj-equitable domination, independent Inj-equitable domination, split Inj-equitable domination and clique Inj-equitable domination.

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