Applied Mathematics, 2016, 7, 1919-1926
http://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Existence and Uniqueness of Solution for Cahn-Hilliard Hyperbolic Phase-Field System with Dirichlet Boundary Condition and Regular Potentials

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How to cite this paper: De Dieu Mangoubi, J., Moukoko, D., Moukamba, F. and Langa, F.D.R. (2016) Existence and Uniqueness of Solution for Cahn-Hilliard Hyperbolic Phase-Field System with Dirichlet Boundary Condition and Regular Potentials. *Applied Mathematics*, **7**, 1919-1926.
http://dx.doi.org/10.4236/am.2016.716157

Received: August 20, 2016
Accepted: October 11, 2016
Published: October 14, 2016

Abstract

Our aim in this paper is to study the existence and the uniqueness of the solutions for hyperbolic Cahn-Hilliard phase-field system, with initial conditions, Dirichlet boundary condition and regular potentials.

Keywords

Cahn-Hilliard Hyperbolic Phase-Field System, Regular Potential, Dirichlet Boundary Conditions

1. Introduction

G. Caginalp introduced in [1] the following phase-field system

$$\frac{\partial u}{\partial t} - \Delta^2 u - \Delta f(u) = -\Delta\theta \tag{1}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t} \tag{2}$$

where u is the order parameter and θ is the (relative) temperature. These equations model phase transition processes such as melting-solidification processes and have been studied, see [2]-[6], for a similar phase-field model with a nonlinear term.

These Cahn-Hilliard phase-fiel system are known as the conserved phase-field system (see [7]-[9]) based on type III heat conduction and with two temperatures (see [10]). The authors have proved the existence and the uniqueness of the solutions, the existence of global attractor and of exponential attractors with singularly or regular potentials.

In [11], Ntsokongo and Batangouna have studied the following Cahn-Hilliard phase-field system

$$\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=-\Delta\left(\frac{\partial \alpha}{\partial t}-\beta\Delta\frac{\partial \alpha}{\partial t}\right) \tag{3}$$

$$\frac{\partial^2 \alpha}{\partial t^2}-\Delta\frac{\partial^2 \alpha}{\partial t^2}-\Delta\frac{\partial \alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t} \tag{4}$$

where $\beta=1$, u is the order parameter and α is the (relative) temperature, they have proved the existence and the uniqueness solution with Dirichlet boundary condition and regular potentials.

In this paper, we consider the following Cahn-Hilliard hyperbolic phase-fiel system

$$\epsilon(-\Delta)\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=-\Delta\frac{\partial \alpha}{\partial t}, \tag{5}$$

$$\frac{\partial^2 \alpha}{\partial t^2}-\Delta\frac{\partial^2 \alpha}{\partial t^2}-\Delta\frac{\partial \alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t}, \tag{6}$$

$$u\big|_{\partial\Omega}=\alpha\big|_{\partial\Omega}=\Delta u\big|_{\partial\Omega}=0, \tag{7}$$

$$u\big|_{t=0}=u_0,\ \frac{\partial u}{\partial t}\bigg|_{t=0}=u_1,\ \alpha\big|_{t=0}=\alpha_0,\ \frac{\partial \alpha}{\partial t}\bigg|_{t=0}=\alpha_1, \tag{8}$$

which is the perturbed phase-field system of Cahn-Hilliard phase-field system (3)-(4) with $\beta=0$. In the above hyperbolic system Ω is a bounded and regular domain of $\mathbb{R}^n$ with $n=2$ or 3 and *f* is the nonlinear regular potentials.

The hyperbolic system has been extensively studied for Dirichlet boundary conditions and regular or singular potentials (see [12]-[14]). Whose certain have to end at existence of global attractor or at the existence of exponential attractors (see [15]).

In this paper we prove the existence and the uniqueness of solutions of (5)-(8). We consider the regular potential $f(s)=s^3-s$ which satisfies the following properties:

$$f \text{ is of class } C^2;\ f(0)=0, \tag{9}$$

$$-c_0\le f'(s),\quad c_0\ge 0,\quad \forall s\in\mathbb{R}, \tag{10}$$

$$-c_1\le F(s)\le f(s)s+c_2,\quad c_1,c_2\ge 0,\quad \forall s\in\mathbb{R}\ \text{ where } F(s)=\int_0^s f(\tau)\,\mathrm{d}\tau. \tag{11}$$

2. Notations

We denote by $\|.\|$ the usual L^2-norm (with associated product scalar (.,.)) and set $\|.\|_{-1}=\left\|(-\Delta)^{\frac{-1}{2}}.\right\|$, where $-\Delta$ denotes the minus Laplace operator with Dirichlet boundary conditions. More generally, $\|.\|_X$ denote the norm of Banach space *X*.

Throughout this paper, the same letters c_1, c_2 and c_3 denote (generally positive) constants which may change from line to line, or even a same line.

3. A Priori Estimates

We multiply (5) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and (6) by $\frac{\partial \alpha}{\partial t}$, integrate over Ω and add the two resulting differential equalities. We find

$$\frac{\mathrm{d}E_1}{\mathrm{d}t}+2\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2=0,$$

where

$$E_1=\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2+\|\nabla u\|^2+2\int_\Omega F(u)\,\mathrm{d}x+\left\|\frac{\partial \alpha}{\partial t}\right\|^2+\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2+\|\nabla\alpha\|^2,$$

satisfies

$$E_1\ge C\left(\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2+\|u\|_{H^1}^2+\left\|\frac{\partial \alpha}{\partial t}\right\|^2+\left\|\frac{\partial \alpha}{\partial t}\right\|_{H^1}^2+\|\alpha\|_{H^1}^2\right)+C',\quad C>0.$$

Finaly, we conclude that $u,\alpha\in L^\infty\left(\mathbf{R}^+,H_0^1(\Omega)\right)$,

$$\frac{\partial u}{\partial t}\in L^\infty\left(\mathbf{R}^+,L^2(\Omega)\right)\cap L^2\left(0,T;H^{-1}(\Omega)\right)$$

and

$$\frac{\partial \alpha}{\partial t}\in L^\infty\left(\mathbf{R}^+,H_0^1(\Omega)\right)\cap L^2\left(0,T;H_0^1(\Omega)\right)$$

for all $T>0$.

Multiply (6) by $\frac{\partial^2\alpha}{\partial t^2}$ and integrate over Ω. We get.

$$\begin{aligned}2\left\|\frac{\partial^2\alpha}{\partial t^2}\right\|^2+2\left\|\nabla\frac{\partial^2\alpha}{\partial t^2}\right\|^2+\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2&=-2\left(\frac{\partial u}{\partial t},\frac{\partial^2\alpha}{\partial t^2}\right)-2\left(\nabla\alpha,\nabla\frac{\partial \alpha}{\partial t}\right)\\&\le 2\left\|\frac{\partial^2\alpha}{\partial t^2}\right\|\left\|\frac{\partial u}{\partial t}\right\|+2\|\nabla\alpha\|\left\|\nabla\frac{\partial^2\alpha}{\partial t^2}\right\|\end{aligned}$$

$$\left\|\frac{\partial^2\alpha}{\partial t^2}\right\|^2+\left\|\nabla\frac{\partial^2\alpha}{\partial t^2}\right\|^2+\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2\le\left\|\frac{\partial u}{\partial t}\right\|^2+\|\nabla\alpha\|^2.$$

Then $\frac{\partial^2\alpha}{\partial t^2}\in L^2\left(0,T;H_0^1(\Omega)\right)$.

In this study, we have three main results; existence theorem, uniqueness theorem and existence theorem with more regularity.

4. Existence and Uniqueness of Solutions

Theorem 4.1. (*Existence*) *We assume* $(u_0,u_1,\alpha_0,\alpha_1)\in H_0^1(\Omega)\times L^2(\Omega)\times\left(H_0^1(\Omega)\right)^2$ *then the system* (5) - (8) *possesses at least one solution* (u,α) *such that*

$$u,\alpha\in L^\infty\left(\mathbf{R}^+,H_0^1(\Omega)\right),$$

$$\frac{\partial u}{\partial t}\in L^\infty\left(\mathbf{R}^+,L^2(\Omega)\right)\cap L^2\left(0,T;H^{-1}(\Omega)\right),$$

$$\frac{\partial \alpha}{\partial t} \in L^{\infty}\left(\mathbf{R}^{+}, H_0^1(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right)$$

and $\frac{\partial^2 \alpha}{\partial t^2} \in L^2\left(0, T; H_0^1(\Omega)\right)$, for all $T > 0$.

The proof is based on a priori estimates obtained in the previous section and on a standard Galerkin scheme.

Theorem 4.2. (*Uniqueness*) *Let the assumpptions of Theorem* 4.1 *hold. Then, the system* (5) - (8) *possesses a unique solution* (u, α) *such that*

$$u, \alpha \in L^{\infty}\left(\mathbf{R}^{+}, H_0^1(\Omega)\right),$$

$$\frac{\partial u}{\partial t} \in L^{\infty}\left(\mathbf{R}^{+}, L^2(\Omega)\right) \cap L^2\left(0, T; H^{-1}(\Omega)\right),$$

$$\frac{\partial \alpha}{\partial t} \in L^{\infty}\left(\mathbf{R}^{+}; H_0^1(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right)$$

and $\frac{\partial^2 \alpha}{\partial t^2} \in L^2\left(0, T; H_0^1(\Omega)\right)$ for all $T > 0$.

Proof. Let $\left(u^{(1)}, \alpha^{(1)}\right)$ and $\left(u^{(2)}, \alpha^{(2)}\right)$ be two solutions of the system (5)-(8) with initial data $\left(u_0^{(1)}, u_1^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}\right)$ and $\left(u_0^{(2)}, u_1^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}\right) \in H_0^1(\Omega) \times L^2(\Omega) \times \left(H_0^1(\Omega)\right)^2$, respectively. We set $u = u^{(1)} - u^{(2)}$ and $\alpha = \alpha^{(1)} - \alpha^{(2)}$, then (u, α) is solution of the following system

$$\epsilon(-\Delta)\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \Delta^2 u - \Delta\left(f\left(u^{(1)}\right) - f\left(u^{(2)}\right)\right) = -\Delta\frac{\partial \alpha}{\partial t}, \tag{12}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t}, \tag{13}$$

$$u\big|_{\partial\Omega} = \Delta u\big|_{\partial\Omega} = \alpha\big|_{\partial\Omega} = 0,$$

$$u\big|_{t=0} = u_0 = u_0^{(1)} - u_0^{(2)}, \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = u_1 = u_1^{(1)} - u_1^{(2)}$$

$$\alpha\big|_{t=0} = \alpha_0 = \alpha_0^{(1)} - \alpha_0^{(2)}, \quad \frac{\partial \alpha}{\partial t}\bigg|_{t=0} = \alpha_1 = \alpha_1^{(1)} - \alpha_1^{(2)}.$$

We multiply (12) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and integrate over Ω. We find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2 + \|\nabla u\|^2\right) + 2\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + 2\left(f\left(u^{(1)}\right) - f\left(u^{(2)}\right), \frac{\partial u}{\partial t}\right) = 2\left(\frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t}\right). \tag{14}$$

Multiplying (13) by $\frac{\partial \alpha}{\partial t}$ and integrating over Ω, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \|\nabla\alpha\|^2\right) + 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 = -2\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t}\right). \tag{15}$$

Now summing (14) and (15) we obtain

$$\frac{\mathrm{d}E_2}{\mathrm{d}t}+2\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2=-2\left(f\left(u^{(1)}\right)-f\left(u^{(2)}\right),\frac{\partial u}{\partial t}\right)$$
$$\le\left\|f\left(u^{(1)}\right)-f\left(u^{(2)}\right)\right\|^2+\left\|\frac{\partial u}{\partial t}\right\|^2, \qquad (16)$$

where

$$E_2=\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2+\|\nabla u\|^2+\left\|\frac{\partial \alpha}{\partial t}\right\|^2+\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2+\|\nabla\alpha\|^2.$$

Lagrange theorem gives a estimates

$$f\left(u^{(1)}\right)-f\left(u^{(2)}\right)=\int_0^1 f'\left(u^{(2)}+s\left(u^{(1)}-u^{(2)}\right)\right)\mathrm{d}su$$
$$=\int_0^1\left(3\left(su^{(1)}+(1-s)u^{(2)}\right)^2-1\right)\mathrm{d}s|u|,$$

which implies

$$\left\|f\left(u^{(1)}\right)-f\left(u^{(2)}\right)\right\|^2\le 36\int_\Omega\left(\left(u^{(2)}\right)^2+\left(u^{(1)}\right)^2+1\right)^2|u|^2\,\mathrm{d}x$$
$$\le 36\left(\left\|u^{(2)}\right\|_{L^6}^4+\left\|u^{(1)}\right\|_{L^6}^4+1\right)\|u\|_{L^6}^2$$
$$\le C\left(\left\|\nabla u^{(2)}\right\|^4+\left\|\nabla u^{(1)}\right\|^4+1\right)\|\nabla u\|^2.$$

Inserting the above estimate into (16), we have

$$\frac{\mathrm{d}E_2}{\mathrm{d}t}+2\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2\le K\left(\|\nabla u\|^2+\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2\right), K>0.$$

Applying Gronwall's lemma, we obtain for all $t\in(0,T)$

$$E_2(t)+2\int_0^t\left(\left\|\frac{\partial u}{\partial t}(\tau)\right\|_{-1}^2+\left\|\nabla\frac{\partial \alpha}{\partial t}(\tau)\right\|^2\right)\mathrm{e}^{k(t-\tau)}\mathrm{d}\tau\le E_2(0)\mathrm{e}^{kT}.$$

We deduce the continuous dependence of the solution relative to the initial conditions, hence the uniqueness of the solution.

The existence and uniqueness of the solution of problem (5)-(8) being proven in a larger space, we will seek the solution with more regularity. □

Theorem 4.3. *Assume*

$$(u_0,u_1,\alpha_0,\alpha_1)\in\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times H_0^1(\Omega)\times\left(H^2(\Omega)\cap H_0^1(\Omega)\right)^2,$$

then the system (5)-(8) *possesses a unique solution* (u,α) *such that*

$$u,\alpha\in L^\infty\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right),$$

$$\frac{\partial u}{\partial t}\in L^\infty\left(0,T;H_0^1(\Omega)\right)\cap L^2\left(0,T;L^2(\Omega)\right),$$

$$\frac{\partial \alpha}{\partial t}\in L^\infty\left(\mathbf{R}^+,H^2(\Omega)\cap H_0^1(\Omega)\right)\cap L^2\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right),$$

$$\frac{\partial^2 \alpha}{\partial t^2} \in L^2\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right)$$

and $\frac{\partial^2 u}{\partial t^2} \in L^2\left(0,T;L^2(\Omega)\right)$, for all $T>0$.

Proof. Following theorems 4.1 and 4.2, the system (5)-(8) possesses the unique solution (u,α) such that

$$u,\alpha \in L^\infty\left(0,T;H_0^1(\Omega)\right),$$

$$\frac{\partial u}{\partial t} \in L^\infty\left(\mathbf{R}^+,L^2(\Omega)\right)\cap L^2\left(0,T;H^{-1}(\Omega)\right),$$

$$\frac{\partial \alpha}{\partial t} \in L^\infty\left(\mathbf{R}^+,H_0^1(\Omega)\right)\cap L^2\left(0,T;H_0^1(\Omega)\right)$$

and $\frac{\partial^2 \alpha}{\partial t^2} \in L^2\left(0,T;H_0^1(\Omega)\right)$, for all $T>0$.

Multiply (2.1) by $\frac{\partial u}{\partial t}$ and integrate over Ω. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+\|\Delta u\|^2\right)+2\left\|\frac{\partial u}{\partial t}\right\|^2=2\left(\nabla\frac{\partial\alpha}{\partial t},\nabla\frac{\partial u}{\partial t}\right)-2\left(\nabla f(u),\nabla\frac{\partial u}{\partial t}\right)$$

we deduce the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+\|\Delta u\|^2\right)+2\left\|\frac{\partial u}{\partial t}\right\|^2\le 2\left(\nabla\frac{\partial\alpha}{\partial t},\nabla\frac{\partial u}{\partial t}\right)+2\int_\Omega\left|f'(u)\right|\left|\nabla u\right|\left|\nabla\frac{\partial u}{\partial t}\right|\mathrm{d}x. \quad (17)$$

Thanks to use $f'(s)$, we find the following estimate

$$\begin{aligned}2\int_\Omega\left|f'(u)\right|\left|\nabla u\right|\left|\nabla\frac{\partial u}{\partial t}\right|\mathrm{d}x &\le \int_\Omega\left|3u^2-1\right|\left|\nabla u\right|\left|\nabla\frac{\partial u}{\partial t}\right|\mathrm{d}x\\ &\le \int_\Omega\left(3u^2+1\right)\left|\nabla u\right|\left|\nabla\frac{\partial u}{\partial t}\right|\mathrm{d}x\\ &\le C\left(\|u\|_{L^6}^4+1\right)\|\nabla u\|_{L^6}^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2.\end{aligned}$$

Since $u\in L^\infty\left(0,T;H_0^1(\Omega)\right)$, then the estimate (17) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+\|\Delta u\|^2\right)+2\left\|\frac{\partial u}{\partial t}\right\|^2\le 2\left(\nabla\frac{\partial\alpha}{\partial t},\nabla\frac{\partial u}{\partial t}\right)+C\|\Delta u\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2. \quad (18)$$

Multiplying (6) by $-\Delta\frac{\partial\alpha}{\partial t}$ and integrating over Ω, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2+\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2+\|\Delta\alpha\|^2\right)+2\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2=-2\left(\nabla\frac{\partial\alpha}{\partial t},\nabla\frac{\partial u}{\partial t}\right). \quad (19)$$

Now summing (18) and (19), we obtain

$$\frac{\mathrm{d}E_3}{\mathrm{d}t}+2\left\|\frac{\partial u}{\partial t}\right\|^2+2\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2\le C\|\Delta u\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2$$

where

$$E_3 = \epsilon\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + \|\Delta u\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2 + \|\Delta\alpha\|^2.$$

Appling the Gronwall's lemma, we deduce that $u,\alpha \in L^\infty\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right)$,

$$\frac{\partial u}{\partial t} \in L^\infty\left(0,T;H_0^1(\Omega)\right)\cap L^2\left(0,T;L^2(\Omega)\right)$$

and

$$\frac{\partial \alpha}{\partial t} \in L^\infty\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right)\cap L^2\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right).$$

Multiplying (5) by $(-\Delta)^{-1}\frac{\partial^2 u}{\partial t^2}$ and integrating ovre Ω, we obtain

$$\begin{aligned}2\epsilon\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 &= 2\left(\frac{\partial \alpha}{\partial t},\frac{\partial^2 u}{\partial t^2}\right) + 2\left(\Delta u,\frac{\partial^2 u}{\partial t^2}\right) - 2\left(f(u),\frac{\partial^2 u}{\partial t^2}\right),\\ &\le 2\left\|\frac{\partial \alpha}{\partial t}\right\|\left\|\frac{\partial^2 u}{\partial t^2}\right\| + 2\|\Delta u\|\left\|\frac{\partial^2 u}{\partial t^2}\right\| + 2\int_\Omega |f(u)|\left|\frac{\partial^2 u}{\partial t^2}\right|\mathrm{d}x.\end{aligned} \tag{20}$$

Thanks to use $f(s)$ and the fact that $u \in L^\infty\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right)$, we get

$$\begin{aligned}\int_\Omega |f(u)|\left|\frac{\partial^2 u}{\partial t^2}\right|\mathrm{d}x &\le \|u^2\|_{L^\infty}\int_\Omega |u|\left|\frac{\partial^2 u}{\partial t^2}\right|\mathrm{d}x + \int_\Omega |u|\left|\frac{\partial^2 u}{\partial t^2}\right|\mathrm{d}x\\ &\le C\|\nabla u\|^2 + \frac{\epsilon}{3}\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2.\end{aligned}$$

Inserting the above estimate into (20), we obtain

$$\epsilon\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 \le C_1\left\|\frac{\partial \alpha}{\partial t}\right\|^2 + C_2\|\Delta u\|^2 + C_3\|\nabla u\|^2, C_1, C_2, C_3 > 0$$

which implies that $\frac{\partial^2 u}{\partial t^2} \in L^2\left(0,T;L^2(\Omega)\right)$.

Multiplying (6) by $-\Delta\frac{\partial^2 \alpha}{\partial t^2}$ and integrating over Ω, we find

$$2\left\|\nabla\frac{\partial^2 \alpha}{\partial t^2}\right\|^2 + \left\|\Delta\frac{\partial^2 \alpha}{\partial t^2}\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2 \le 2\left\|\frac{\partial u}{\partial t}\right\|^2 + 2\|\Delta\alpha\|^2,$$

that implies $\frac{\partial^2 \alpha}{\partial t^2} \in L^2\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right)$. □

5. Conclusion

We have just shown the theorems of existence and uniqueness of the solutions for perturbed Cahn-Hilliard hyperbolic phase-field system with regular potentials.

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