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Oscillation Properties of Third Order Neutral Delay Differential Equations*

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Abstract

Oscillation criteria are established for third-order neutral delay differential equations with deviating arguments. These criteria extend and generalize those results in the literature. Moreover, some illustrating examples are also provided to show the importance of our results.

Keywords

Oscillation, Third Order, Neutral Delay, Differential Equations

1. Introduction

This article is concerned with the oscillation and the asymptotic behavior of solutions of the third-order neutral delay differential equations with deviating argument of the form

$$\left(r(t)\left[z''(t)\right]^{\alpha}\right)' + \int_c^d q(t,\xi)x^{\alpha}\left(g(t,\xi)\right)\mathrm{d}\xi = 0, t \ge t_0, \tag{E}$$

where $z(t) = x(t) + \int_a^b p(t,\eta)x\left(\tau(t,\eta)\right)\mathrm{d}\eta$. We assume that:

(H) $r \in C\left([t_0,\infty),(0,\infty)\right)$; $p,\tau \in C\left([t_0,\infty)\times[a,b],R\right)$; $q,g \in C\left([t_0,\infty)\times[c,d],R\right)$, α is a quotient of odd positive integers, $0 \le \int_a^b p(t,\eta)\mathrm{d}\eta \le p < 1$, $\tau(t,\eta) \le t$, $g(t,\xi) \le t$, $\lim_{t\to\infty}\tau(t,\eta) = \lim_{t\to\infty} g(t,\xi) = \infty$ and $q(t,\xi) > 0$.

A function $x(t) \in C\left([t_x,\infty)\right), t_x \ge t_0$ is called a solution of (E), if it has the properties $z(t) \in C^1\left([t_x,\infty)\right)$, $z'(t) \in C^1\left([t_x,\infty)\right)$, $r(t)\left[z''(t)\right]^{\alpha} \in C^1\left([t_x,\infty)\right)$ and satisfies (E) on $[t_x,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\left\{|x(t)| : t \ge T\right\} > 0$ for all $T \ge t_x$. We assume that (E) possesses such solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[t_x,\infty)$; otherwise,

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it is called nonoscillatory.

In the recent years, great attention in the oscillation theory has been devoted to the oscillatory and asymptotic properties of the third-order differential equations (see [1]-[14]). Baculikova *et al.* [2] [3], Dzurina *et al.* [4] and Mihalikova *et al.* [11] studied the oscillation of the third-order nonlinear differential equation

$$\left(r(t)\left[\left(x(t)+p(t)x(\tau(t))\right)''\right]^{\alpha}\right)'+q(t)f\left(x(g(t))\right)=0,\ t\geq t_0,$$

under the condition

$$\int_{t_0}^{\infty} r^{-1/\alpha}(s)=\infty.$$

Li *et al.* [10] considered the oscillation of

$$\left(r_2(t)\left[r_1(t)\left(x(t)+p(t)x(\tau(t))\right)'\right]'\right)'+q(t)x(g(t))=0,\ t\geq t_0,$$

under the assumption

$$\int_{t_0}^{\infty} r_1^{-1}(s)=\infty \quad \text{and} \quad \int_{t_0}^{\infty} r_2^{-1}(s)<\infty.$$

The aim of this paper is to discuss asymptotic behavior of solutions of class of third order neutral delay differential Equation (E) under the condition

$$\int_{t_0}^{\infty} r^{-1/\alpha}(s)<\infty. \tag{1}$$

By using Riccati transformation technique, we established sufficient conditions which insure that solution of class of third order neutral delay differential equation is oscillatory or tends to zero. The results of this study extend and generalize the previous results.

2. Main Results

In this section, we will establish some new oscillation criteria for solutions of (E).

Theorem 2.1. *Assume that conditions* (1) *and* (*H*) *are satisfied. If for some function* $\rho\in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1\geq t_0$ *and for* $t_3\geq t_2\geq t_1$, *one has*

$$\limsup_{t\to\infty}\int_{t_3}^{t}\left(\rho(s)q^*(s)(1-p)^{\alpha}G(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)\left(\rho'(s)\right)^{\alpha+1}}{\rho^{\alpha}(s)}\right)\mathrm{d}s=\infty, \tag{2}$$

where

$$G(s)=\left(\frac{\int_{t_2}^{g(s,c)}\left(\int_{t_1}^{v} r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\right)\mathrm{d}v}{\int_{t_1}^{s} r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\right)^{\alpha},\ q^*(s)=\int_c^d q(s,\xi)\mathrm{d}\xi, \tag{3}$$

and

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left[\frac{1}{r(u)}\int_{u}^{\infty}\int_{c}^{d}q(s,\xi)\,\mathrm{d}\xi\,\mathrm{d}s\right]^{\frac{1}{\alpha}}\mathrm{d}u\,\mathrm{d}v=\infty. \tag{4}$$

If

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\delta(s)q^{*}(s)(1-p)^{\alpha}\left(g(s,c)-t_1\right)^{\alpha}-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{1}{\left(\delta(s)r(s)\right)^{\alpha}}\right)\mathrm{d}s=\infty \tag{5}$$

where

$$\delta(t)=\int_{t}^{\infty}\frac{1}{r^{\frac{1}{\alpha}}(s)}\mathrm{d}s, \tag{6}$$

then every solution $x(t)$ of (E) is either oscillatory or converges to zero as $t\to\infty$.

Proof. Assume that $x(t)$ is a positive solution of (E). Based on the condition (1), there exist three possible cases

(1) $z(t)>0, z'(t)>0, z''(t)>0, \left(r(t)\left[z''(t)\right]^{\alpha}\right)'\le 0;$

(2) $z(t)>0, z'(t)<0, z''(t)>0, \left(r(t)\left[z''(t)\right]^{\alpha}\right)'\le 0;$

(3) $z(t)>0, z'(t)>0, z''(t)<0, \left(r(t)\left[z''(t)\right]^{\alpha}\right)'\le 0,$

for $t\ge t_1$, t_1 is large enough. We consider each of three cases separately. Suppose first that $z(t)$ has the property (1). We define the function $\omega(t)$ by

$$\omega(t)=\rho(t)\frac{r(t)\left(z''(t)\right)^{\alpha}}{\left(z'(t)\right)^{\alpha}}. \tag{7}$$

Then, $\omega(t)>0$ for $t\ge t_1$. Using $z'(t)>0$, we have

$$\begin{aligned}x(t)&=z(t)-\int_{a}^{b}p(t,\eta)x\left(\tau(t,\eta)\right)\mathrm{d}\eta\\&\ge z(t)-\int_{a}^{b}p(t,\eta)z\left(\tau(t,\eta)\right)\mathrm{d}\eta\\&\ge z(t)-z\left(\tau(t,b)\right)\int_{a}^{b}p(t,\eta)\mathrm{d}\eta\\&\ge(1-p)z(t).\end{aligned} \tag{8}$$

Since

$$z'(t)\ge\int_{t_1}^{t}\frac{\left(r(s)\left[z''(s)\right]^{\alpha}\right)^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)}\mathrm{d}s\ge\left(r(t)\left[z''(t)\right]^{\alpha}\right)^{\frac{1}{\alpha}}\int_{t_1}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}\mathrm{d}s,$$

we have that

$$\left(\frac{z'(t)}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s}\right)'\le 0. \tag{9}$$

Thus, we get

$$z(t)=z(t_2)+\int_{t_2}^{t}\left(\frac{z'(t)}{\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\right)\mathrm{d}s$$
$$\geq\frac{z'(t)}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\int_{t_2}^{t}\left(\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\right)\mathrm{d}s, \tag{10}$$

for $t\geq t_2>t_1$. Differentiating (7), we obtain

$$\omega'(t)=\rho'(t)\frac{r(t)(z''(t))^{\alpha}}{(z'(t))^{\alpha}}+\rho(t)\frac{\left(r(t)\left[z''(t)\right]^{\alpha}\right)'}{(z'(t))^{\alpha}}-\alpha\rho(t)\frac{r(t)(z''(t))^{\alpha+1}}{(z'(t))^{\alpha+1}}.$$

It follows from (E), (7) and (8) that

$$\omega'(t)\leq-\rho(t)\int_c^d q(t,\xi)\mathrm{d}\xi(1-p)^{\alpha}\frac{z^{\alpha}(g(t,c))}{(z'(t))^{\alpha}}+\frac{\rho'(t)}{\rho(t)}\omega(t)-\alpha\frac{\omega^{\frac{\alpha+1}{\alpha}}(t)}{(\rho(t)r(t))^{\frac{1}{\alpha}}},$$

that is

$$\omega'(t)\leq-\rho(t)\int_c^d q(t,\xi)\mathrm{d}\xi(1-p)^{\alpha}\frac{z^{\alpha}(g(t,c))}{(z'(g(t,c)))^{\alpha}}\frac{(z'(g(t,c)))^{\alpha}}{(z'(t))^{\alpha}}$$
$$+\frac{\rho'(t)}{\rho(t)}\omega(t)-\frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}\omega^{\frac{\alpha+1}{\alpha}},$$

which follows from (9) and (10) that

$$\omega'(t)\leq-\rho(t)\int_c^d q(t,\xi)\mathrm{d}\xi(1-p)^{\alpha}\left(\frac{\int_{t_2}^{g(t,c)}\left(\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\right)\mathrm{d}s}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\right)^{\alpha}$$
$$+\frac{\rho'(t)}{\rho(t)}\omega(t)-\frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}\omega^{\frac{\alpha+1}{\alpha}}(t).$$

Hence, we have

$$\omega'(t)\leq-\rho(t)\int_c^d q(t,\xi)\mathrm{d}\xi(1-p)^{\alpha}\left(\frac{\int_{t_2}^{g(t,c)}\left(\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u\right)\mathrm{d}s}{\int_{t_1}^{t}r^{-\frac{1}{\alpha}}(u)\mathrm{d}u}\right)^{\alpha}$$
$$+\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(t)(\rho'(t))^{\alpha+1}}{\rho^{\alpha}(t)}.$$

Integrating the last inequality from $t_3(>t_2)$ to t, we get

$$\omega(t_3)\geq\int_{t_3}^{t}\left(\rho(s)q^{*}(s)(1-p)^{\alpha}G(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)(\rho'(s))^{\alpha+1}}{\rho^{\alpha}(s)}\right)\mathrm{d}s, \tag{11}$$

which contradicts (2). Assume now that $z(t)$ has the property (2). Using the similar proof ([1], Lemma 2), we can get $\lim_{t\to\infty} x(t)=0$ due to condition (4). Thirdly, assume that $z(t)$ has the property (3). From $\left(r(t)\left[z''(t)\right]^{\alpha}\right)'<0$, $r(t)\left[z''(t)\right]^{\alpha}$ is decreasing. Thus we get

$$r(s)\left[z''(s)\right]^{\alpha}\le r(t)\left[z''(t)\right]^{\alpha}, s\ge t\ge t_1.$$

Dividing the above inequality by $r(s)$ and integrating it from t to l, we obtain

$$z'(l)\le z'(t)+r^{\frac{1}{\alpha}}(t)z''(t)\int_t^l r^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s.$$

Letting $l\to\infty$, we get

$$0\le z'(t)+r^{\frac{1}{\alpha}}(t)z''(t)\int_t^\infty r^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s,$$

that is

$$-\frac{r^{\frac{1}{\alpha}}(t)z''(t)}{z'(t)}\int_t^\infty r^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s\le 1. \tag{12}$$

Define function ψ by

$$\psi(t)=\frac{r(t)\left(z''(t)\right)^{\alpha}}{\left(z'(t)\right)^{\alpha}}, t\ge t_1. \tag{13}$$

Then $\psi(t)<0$ for $t\ge t_1$. Hence, by (12) and (13), we obtain

$$-\delta(t)\psi^{\frac{1}{\alpha}}(t)\le 1. \tag{14}$$

Differentiating (13), we get

$$\psi'(t)=\frac{\left(r(t)\left[z''(t)\right]^{\alpha}\right)'}{\left(z'(t)\right)^{\alpha}}-\alpha r(t)\frac{\left(z''(t)\right)^{\alpha+1}}{\left(z'(t)\right)^{\alpha+1}}.$$

Using $z'(t)>0$, we have (8). From (E) and (8), we have

$$\psi'(t)\le-\int_c^d q(t,\xi)\,\mathrm{d}\xi(1-p)^{\alpha}\frac{z^{\alpha}\left(g(t,c)\right)}{\left(z'(t)\right)^{\alpha}}-\alpha r(t)\left(\frac{z''(t)}{z'(t)}\right)^{\alpha+1}. \tag{15}$$

In view of (3), we see that

$$z(t)\ge z'(t)(t-t_1). \tag{16}$$

Hence,

$$\left(\frac{z(t)}{(t-t_1)}\right)'\le 0,$$

which implies that

$$\frac{z\left(g(t,c)\right)}{z(t)}\ge\frac{\left(g(t,c)-t_1\right)}{(t-t_1)}. \tag{17}$$

By (13) and (15)-(17), we get

$$\psi'(t) \le -\int_c^d q(t,\xi)\,\mathrm{d}\xi (1-p)^\alpha \left(g(t,c)-t_1\right)^\alpha - \alpha r^{-\frac{1}{\alpha}}(t)\psi^{\frac{\alpha+1}{\alpha}}(t).$$

Multiplying the last inequality by $\delta(t)$ and integrating from $t_2(>t_1)$ to *t*, we obtain

$$0 \ge \psi(t)\delta(t) - \psi(t_2)\delta(t_2) + \int_{t_2}^t \delta(s) q^*(s)(1-p)^\alpha \left(g(s,c)-t_1\right)^\alpha \mathrm{d}s$$
$$+\int_{t_2}^t \frac{\alpha\psi^{\frac{\alpha+1}{\alpha}}(s)\delta(s)}{r^{\frac{1}{\alpha}}(s)}\mathrm{d}s - \int_{t_2}^t \frac{\psi(s)}{r(s)}\mathrm{d}s,$$

which follows that

$$1+\psi(t_2)\delta(t_2) \ge \int_{t_2}^t \left(\delta(s)q^*(s)(1-p)^\alpha\left(g(s,c)-t_1\right)^\alpha - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{1}{\left(\delta(s)r(s)\right)^\alpha}\right)\mathrm{d}s,$$

which contradicts (5). This completes the proof. □

3. Examples

The following examples illustrate applications of our result in this paper.

Example 3.1. *For* $t \ge 1$ *and* $\lambda > 0$, *consider the third-order differential equation*

$$\left(t^{\frac{4}{3}}\left(x(t)+\int_0^1 p_1 x\left(\frac{t}{2}\right)\mathrm{d}\eta\right)''\right)' + \int_0^1 \frac{2\lambda}{t^{\frac{5}{3}}}\xi x(t-\xi)\,\mathrm{d}\xi = 0. \quad (18)$$

Let $\rho(t)=1$, $\alpha=1$, $a=0$, $b=1$, $c=0$, $d=1$, $r(t)=t^{\frac{4}{3}}$, $p(t,n)=p_1$ such that $0 \le \int_0^1 p_1 \mathrm{d}\eta \le p < 1$, $\tau(t,\eta)=\frac{t}{2}$, $q(t,\xi)=2\lambda\xi \big/ t^{\frac{5}{3}}$, $g(t,\xi)=t-\xi$. Note that,

$$\int_{t_0}^\infty r^{-\frac{1}{\alpha}}(s)\mathrm{d}s = \int_1^\infty s^{-\frac{4}{3}}\mathrm{d}s = 3 < \infty, \delta(t) = 3t^{-\frac{1}{3}},$$

and

$$\int_1^\infty \int_v^\infty u^{-\frac{4}{3}} \int_u^\infty \int_0^1 \frac{2\lambda\xi}{s^{\frac{5}{3}}}\mathrm{d}\xi\mathrm{d}s\mathrm{d}u\mathrm{d}v = \infty.$$

Furthermore

$$\limsup_{t\to\infty} \int_{t_3}^t \left(\rho(s)q^*(s)(1-p)^\alpha G(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)\left(\rho'(s)\right)^{\alpha+1}}{\rho^\alpha(s)}\right)\mathrm{d}s$$
$$=\limsup_{t\to\infty}\int_{t_3}^t \left(\frac{\lambda}{6}(1-p)s^{-\frac{5}{3}}\left(\frac{9s^{\frac{2}{3}} - 6t_1^{-\frac{1}{3}}s + \beta}{s^{-\frac{1}{3}} - t_1^{-\frac{1}{3}}}\right)\right)\mathrm{d}s = \infty,$$

such that $q^*(s)$, $G(s)$ are defined as in (3) and $\beta = 6t_1^{-\frac{1}{3}}t_2 - 9t_2^{\frac{2}{3}}$,

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\delta(s)q^*(s)(1-p)^{\alpha}(g(t,c)-t_1)^{\alpha}-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{1}{(\delta(s)r(s))^{\alpha}}\right)\mathrm{d}s$$

$$=\limsup_{t\to\infty}\int_{t_2}^{t}\left(3\lambda(1-p)s^{-2}(s-t_1)-\frac{1}{12}s^{-1}\right)\mathrm{d}s=\infty.$$

Using our result, every solution of (18) is either oscillatory or converges to zero as $t\to\infty$ if $\lambda>1/36(1-p)$.

Example 3.2. *For* $t\ge 1$ *and* $\mu>0$, *consider the third-order differential equation*

$$\left(t^5\left[\left(x(t)+\int_1^2\frac{\eta}{t+1}x\left(\frac{t+\eta}{3}\right)\mathrm{d}\eta\right)''\right]^3\right)'+\int_0^1\frac{2\mu}{3}t^{-2}\xi x^3\left(t-\frac{\xi}{2}\right)\mathrm{d}\xi=0. \quad (19)$$

Let $\rho(t)=1$, $\alpha=3$, $a=1$, $b=2$, $c=0$, $d=1$, $r(t)=t^5$, $p(t,n)=\frac{\eta}{t+1}$ such that $0\le\int_1^2(\eta/(t+1))\mathrm{d}\eta\le\frac{3}{4}<1$, $\tau(t,\eta)=(t+\eta)/2$, $q(t,\xi)=\frac{2\mu}{3}t^{-2}\xi$, $g(t,\xi)=\left(t-\frac{\xi}{2}\right)$. Note that,

$$\int_{t_0}^{\infty}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s=\int_1^{\infty}s^{-\frac{5}{3}}\mathrm{d}s=\frac{3}{2}<\infty,\delta(t)=\frac{3}{2}t^{-\frac{2}{3}},$$

and

$$\int_1^{\infty}\int_v^{\infty}\left[u^{-5}\int_u^{\infty}\int_0^1\frac{2\mu\xi}{3s^2}\mathrm{d}\xi\mathrm{d}s\right]^{\frac{1}{3}}\mathrm{d}u\mathrm{d}v=\infty.$$

Furthermore

$$\limsup_{t\to\infty}\int_{t_3}^{t}\left(\rho(s)q^*(s)(1-p)^{\alpha}G(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)(\rho'(s))^{\alpha+1}}{\rho^{\alpha}(s)}\right)\mathrm{d}s$$

$$=\limsup_{t\to\infty}\mathrm{supp}\int_{t_3}^{t}\left(\frac{\mu}{(3)^4(4)^3}s^{-2}\left(\frac{9s^{\frac{1}{3}}-3t_1^{-\frac{2}{3}}s+\beta}{s^{-\frac{2}{3}}-t_1^{-\frac{2}{3}}}\right)^3\right)\mathrm{d}s=\infty,$$

such that $q^*(s)$, $G(s)$ are defined as in (3) and $\beta=3t_1^{-\frac{2}{3}}t_2-9t_2^{\frac{1}{3}}$,

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\delta(s)q^*(s)(1-p)^{\alpha}(g(t,c)-t_1)^{\alpha}-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{1}{(\delta(s)r(s))^{\alpha}}\right)\mathrm{d}s$$

$$=\limsup_{t\to\infty}\int_{t_2}^{t}\left(\frac{\mu}{128}s^{-\frac{8}{3}}(s-t_1)^3-\frac{1}{864}s^{-13}\right)\mathrm{d}s=\infty.$$

Using our result, every solution of (19) is either oscillatory or converges to zero as $t\to\infty$ if $\lambda>0$ for some $t_1\in\left(0,\frac{1}{12}\right)$.

Example 3.3. *For* $t\ge 1$ *and* $\gamma>0$, *consider the third-order differential equation*

$$\left(t^{\frac{5}{4}}\left(x(t)+\int_1^2 \frac{1}{3} x(t-\eta)\mathrm{d}\eta\right)''\right)' + \int_0^1 \frac{3\gamma}{t^{\frac{7}{4}}}\xi x(t)\mathrm{d}\xi = 0. \tag{20}$$

Let $\rho(t)=1$, $\alpha=1$, $a=1$, $b=2$, $c=0$, $d=1$, $r(t)=t^{\frac{5}{4}}$, $p(t,n)=\frac{1}{3}$ such that $0 \le \int_0^1 \frac{1}{3}\mathrm{d}\eta \le \frac{1}{3} < 1$, $\tau(t,\eta)=t-\eta$, $q(t,\xi)=3\gamma\xi \big/ t^{\frac{7}{4}}$, $g(t,\xi)=t$. Note that,

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s = \int_1^{\infty} s^{-\frac{5}{4}}\mathrm{d}s = 4 < \infty, \delta(t) = 4t^{-\frac{1}{4}},$$

and

$$\int_1^{\infty}\int_v^{\infty} u^{-\frac{5}{4}} \int_u^{\infty}\int_0^1 \frac{3\gamma\xi}{2s^{\frac{7}{4}}}\mathrm{d}\xi\mathrm{d}s\mathrm{d}u\mathrm{d}v = \infty.$$

Furthermore

$$\limsup_{t\to\infty} \int_{t_3}^{t}\left(\rho(s)q^*(s)(1-p)^{\alpha}G(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)(\rho'(s))^{\alpha+1}}{\rho^{\alpha}(s)}\right)\mathrm{d}s$$

$$= \limsup_{t\to\infty} \int_{t_3}^{t}\left(\frac{\gamma}{3}s^{-\frac{7}{4}}\left(\frac{4s^{\frac{3}{4}} - 3t_1^{-\frac{1}{4}}s + \beta}{s^{-\frac{1}{4}} - t_1^{-\frac{1}{4}}}\right)\right)\mathrm{d}s = \infty,$$

such that $q^*(s)$, $G(s)$ are defined as in (3) and $\beta = 3t_1^{-\frac{1}{4}}t_2 - 4t_2^{\frac{3}{4}}$,

$$\limsup_{t\to\infty} \int_{t_2}^{t}\left(\delta(s)q^*(s)(1-p)^{\alpha}(g(t,c)-t_1)^{\alpha} - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{1}{(\delta(s)r(s))^{\alpha}}\right)\mathrm{d}s$$

$$= \limsup_{t\to\infty} \int_{t_2}^{t}\left(4\gamma s^{-2}(s-t_1) - \frac{1}{16}s^{-1}\right)\mathrm{d}s = \infty.$$

Using our result, every solution of (20) is either oscillatory or converges to zero as $t\to\infty$ if $\lambda > \frac{1}{64}$.

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