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Rothe's Fixed Point Theorem and the Controllability of the Benjamin-Bona-Mahony Equation with Impulses and Delay

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Abstract

For many control systems in real life, impulses and delays are intrinsic phenomena that do not modify their controllability. So we conjecture that under certain conditions the abrupt changes and delays as perturbations of a system do not destroy its controllability. There are many practical examples of impulsive control systems with delays, such as a chemical reactor system, a financial system with two state variables, the amount of money in a market and the savings rate of a central bank, and the growth of a population diffusing throughout its habitat modeled by a reaction-diffusion equation. In this paper we apply the Rothe's Fixed Point Theorem to prove the interior approximate controllability of the following Benjamin-Bona-Mahony (BBM) type equation with impulses and delay

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = 1_\omega u(t,x) + f\left(t, z(t-r,x), u(t,x)\right), & \text{in } (0,\tau)\times\Omega, \\ z(t,x) = 0, \quad \text{on } (0,\tau)\times\partial\Omega, \\ z(s,x) = \phi(s,x), \quad s\in[-r,0], x\in\Omega, \\ z(t_k^+,x) = z(t_k^-,x) + I_k\left(t_k, z(t_k,x), u(t_k,x)\right), & k = 1,2,3,\cdots,p, \end{cases}$$

where $a \geq 0$ and $b > 0$ are constants, Ω is a domain in $\mathbb{R}^N$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u \in C\left(0,\tau; L_2(\Omega)\right)$, $\phi:[-r,0]\times\Omega\to\mathbb{R}$ are continuous functions and the nonlinear functions $f, I_k:[0,\tau]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are smooth enough functions satisfying some additional conditions.

Keywords

Interior Approximate Controllability, Benjamin-Bona-Mahony Equation with Impulses and Delay, Strongly Continuous Semigroup, Rothe's Fixed Point Theorem

1. Introduction

For many control systems in real life, impulses and delays are intrinsic phenomena that do not modify their controllability. So we conjecture that under certain conditions the abrupt changes and delays as perturbations of a system do not destroy its controllability. There are many practical examples of impulsive control systems with delays, such as a chemical reactor system, a financial system with two state variables, the amount of money in a market and the savings rate of a central bank, and the growth of a population diffusing throughout its habitat modeled by a reaction-diffusion equation. One may easily visualize situations in these examples where abrupt changes such as harvesting, disasters and instantaneous stocking may occur. These problems can be modeled by impulsive differential equations with delays, and one can find information about impulsive differential equations in Lakshmikantham [1] and Samoilenko and Perestyuk [2].

The controllability of impulsive evolution equations has been studied recently by several authors, but most of them study the exact controllability only. For example, D. N. Chalishajar [3] studied the exact controllability of impulsive partial neutral functional differential equations with infinite delay and S. Selvi and M. Mallika Arjunan [4] studied the exact controllability for impulsive differential systems with finite delay. For approximate controllability of impulsive semilinear evolution equation, Lizhen Chen and Gang Li [5] studied the approximate controllability of impulsive differential equations with nonlocal conditions, using measure of noncompactness and Monch Fixed Point Theorem, and assuming that the nonlinear term $f(t,z)$ does not depend on the control variable. Recently, in [6]-[10], the approximate controllability of semilinear evolution equations with impulses has been studied by applying Rothe's Fixed Point Theorem, showing that the influence of impulses do not destroy the controllability of some known systems like the heat equation, the wave equation, the strongly damped wave equation. More recently, in [11] the approximate controllability of the heat equation with impulses and delay has been studied.

The approximate controllability of the linear part of the Benjamin-Bona-Mahony (BBM) equation was proved in [12]. This result was used to study the controllability of the nonlinear BBM equations in [13], which could serve as a basis for studying the BBM equation under the influence of impulses and delays

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = 1_\omega u(t,x) + f\left(t, z(t-r,x), u(t,x)\right), & \text{in } (0,\tau)\times\Omega, \\ z(t,x) = 0, \quad \text{on } (0,\tau)\times\partial\Omega, \\ z(s,x) = \phi(s,x), \ s\in[-r,0], x\in\Omega, \\ z(t_k^+,x) = z(t_k^-,x) + I_k\left(t_k, z(t_k,x), u(t_k,x)\right), & k = 1,2,3,\cdots,p, \end{cases} \tag{1}$$

where $a \ge 0$ and $b > 0$ are constants, Ω is a domain in $\mathbb{R}^N$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u \in C\left(0,\tau; L_2(\Omega)\right)$, $\phi:[-r,0]\times\Omega\to\mathbb{R}$ are continuous functions. Here $r \ge 0$ is the delay and the nonlinear functions $f, I_k : [0,\tau]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are smooth enough and satisfy

$$\left|f(t,z,u)\right| \le a_0 |z|^{\alpha_0} + b_0 |u|^{\beta_0} + c_0, \quad u, z \in \mathbb{R}. \tag{2}$$

$$\left|I_k(t,z,u)\right| \le a_k |z|^{\alpha_k} + b_k |u|^{\beta_k} + c_k, \ k = 1,2,3,\cdots,p, \quad u, z \in \mathbb{R}. \tag{3}$$

$$\frac{1}{2} \le \alpha_k < 1, \ \frac{1}{2} \le \beta_k < 1, k = 0,1,2,3,\cdots,p,$$

and

$$z(t_k, x) = z(t_k^+, x) = \lim_{t \to t_k^+} z(t,x), \ z(t_k^-, x) = \lim_{t \to t_k^-} z(t,x).$$

One natural space to work evolution equations with delay and impulses is the Banach space

$$PC([-r,\tau];Z) = \left\{z : J = [-r,\tau] \to Z : z \in C(J';Z), \exists z(t_k^+,\cdot), z(t_k^-,\cdot) \text{ and } z(t_k,\cdot) = z(t_k^+,\cdot)\right\},$$

where $Z = L_2(\Omega)$ and $J' = [-r,\tau] \setminus \{t_1, t_2, \cdots, t_p\}$, endowed with the norm

$$\|z\| = \sup_{t\in[-r,\tau]} \left|z(t,\cdot)\right|_Z,$$

with

$$\|z\|_Z = \sqrt{\int_\Omega \|z(x)\|^2 \, \mathrm{d}x}, \ \forall z \in Z = L_2(\Omega).$$

We shall denote by C the space of continuous functions:

$$C = \left\{\phi : [-r,0] \to L_2(\Omega) = Z : \phi \text{ is continuous}\right\},$$

endowed with the norm

$$\|\phi\| = \sup_{-r\le s\le 0} \|\phi(s)\|_{L_2(\Omega)}, \ \text{and} \ \phi(s)(x) = \phi(s,x), \quad x \in \Omega.$$

Definition 1.1. (**Approximate Controllability**) *The system* (1) *is said to be approximately controllable on* $[0,\tau]$ *if for every* $\phi \in C$ *and* $z^1 \in Z = U = L_2(\Omega)$, $\varepsilon > 0$ *there exists* $u \in C(0,\tau;U)$ *such that the mild solution* $z(t)$ *of* (1) *corresponding to u verifies*:

$$z(0) = \phi(0) \ \text{and} \ \|z(\tau) - z_1\|_Z < \varepsilon.$$

where

$$\|z(\tau) - z^1\|_Z = \left(\int_\Omega \left|z(\tau,x) - z^1(x)\right|^2 \mathrm{d}x\right)^{1/2}$$

As a consequence of this result we obtain the interior approximate controllability of the semilinear heat equation by putting $a = 0$ and $b = 1$.

We also study the approximate controllability of the corresponding linear system

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = 1_\omega u(t,x), & \text{in } (0,\tau)\times\Omega, \\ z(t,x) = 0, & \text{on } (0,\tau)\times\partial\Omega, \end{cases} \tag{4}$$

by applying the classical Unique Continuation Principle for Elliptic Equations (see [14]) and the following lemma.

Lemma 1.1. (*see Lemma* 3.14 *from* [15], p. 62) *Let* $\{\alpha_j\}_{j\ge1}$ *and* $\{\beta_{i,j} : i = 1,2,\cdots,m\}_{j\ge1}$ *be sequences of real numbers such that*: $\alpha_1 > \alpha_2 > \alpha_3 \cdots$. *Then*

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$$\sum_{j=1}^{\infty} e^{\alpha_j t}\beta_{i,j} = 0, \quad \forall t \in [0,\tau], \quad i = 1,2,\cdots,m$$

if and only if

$$\beta_{i,j} = 0, \; i = 1,2,\cdots,m; \; j = 1,2,\cdots,\infty.$$

The approximate controllability of the system (1) follows from the approximate controllability of (4), the compactness of the semigroup generated by the associated linear operator, the conditions (2) and (3) satisfied by the nonlinear term f, I_k and the following results:

Proposition 1.1. *Let* (X,Σ,μ) *be a measure space with* $\mu(X) < \infty$ *and* $1 \le q < r < \infty$. *Then* $L_r(\mu) \subset L_q(\mu)$ *and*

$$\|f\|_q \le \mu(X)^{\frac{r-q}{rq}} \|f\|_r, \quad f \in L_r(\mu). \tag{5}$$

Theorem 1.1. (*Rothe's Fixed Theorem*, [16]-[18]) *Let E be a Banach space and* $B \subset E$ *be a closed convex subset such that the zero of E is contained in the interior of B. Consider* $\Phi: B \to E$ *be a continuous mapping with*

a) $\overline{\Phi(B)}$ is compact.

b) $\Phi(\partial B) \subset B$ (∂B, where ∂B denotes the boundary of *B*.

Then there is a point $x^* \in B$ such that

$$\Phi(x^*) = x^*.$$

2. Abstract Formulation of the Problem

In this section we choose a Hilbert space where system (1) can be written as an abstract differential equation with impulses and delay; to this end, we consider the following notations:

Let $Z = L^2(\Omega) = L^2(\Omega,\mathbb{R})$ and consider the linear unbounded operator $A: D(A) \subset Z \to Z$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H^2(\Omega,\mathbb{R}) \cap H_0^1(\Omega,\mathbb{R}).$$

The operator *A* has the following very well known properties (see N. I. Akhiezer and I. M. Glazman [19]): the spectrum of *A* consists of eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \quad \text{with } \lambda_j \to \infty, \tag{6}$$

each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace. Therefore:

a) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of A.

b) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{7}$$

where $\langle \cdot,\cdot \rangle$ is the inner product in *Z* and

$$E_j z = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \tag{8}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and

$$z=\sum_{j=1}^{\infty} E_j z, \quad z\in Z. \tag{9}$$

c) $-A$ generates the analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At}z=\sum_{j=1}^{\infty} e^{-\lambda_j t}E_j z. \tag{10}$$

Consequently, the system (1) can be written as abstract differential equations with impulses and delay in Z:

$$\begin{cases} z'+aAz'+bAz=1_\omega u(t)+f^e(t,z_t,u), \quad z\in Z, \quad t\in(0,\tau], \\ z(s)=\phi(s), \ s\in[-r,0], \\ z(t_k^+)=z(t_k^-)+I_k^e(t_k,z(t_k),u(t_k)), k=1,2,3,\cdots,p, \end{cases} \tag{11}$$

where $u\in C([0,\tau];U)$, $U=Z$, $B_\omega:U\to Z$, $B_\omega u=1_\omega u$ is a bounded linear operator, $z_t\in C([-r,0];Z)$ is defined by $z_t(s)=z(t+s), -r\le s\le 0$ and the functions $I_k^e:[0,\tau]\times Z\times U\to Z$, $f^e:[0,\tau]\times C\times U\to Z$ are defined by

$$I_k^e(t,z,u)(x)=I_k(t,z(x),u(x)), \quad f^e(t,\phi,u)(x)=f(t,\phi(-r,x),u(x)),$$

$$\forall x\in\Omega, k=1,2,\cdots,p.$$

On the other hand, from conditions (2) and (3) we get the following estimates.

Proposition 2.1. *Under the conditions* (2)-(3) *the functions* f^e, I_k^e, $k=1,2,\cdots,p$, *defined above satisfy* $\forall u,z\in Z=L_2(\Omega)$ *and* $\phi\in C$:

$$\|f^e(t,\phi,u)\|_Z\le \tilde{a}_0\|\phi(-r)\|_Z^{\alpha_0}+\tilde{b}_0\|u\|_Z^{\beta_0}+\tilde{c}_0 \tag{12}$$

$$\|I_k^e(t,z,u)\|_Z\le \tilde{a}_k\|z\|_z^{\alpha_k}+\tilde{b}_k\|u\|_Z^{\beta_k}+\tilde{c}_k, \quad k=1,2,3,\cdots,p. \tag{13}$$

Since $(I+aA)=a\left(A-\left(-\frac{1}{a}\right)I\right)$ and $-\frac{1}{a}\in\rho(A)$ ($\rho(A)$ is the resolvent set of A), then the operator:

$$I+aA:D(A)\to Z \tag{14}$$

is invertible with bounded inverse

$$(I+aA)^{-1}:Z\to D(A). \tag{15}$$

Therefore, the systems (11) and its linear part can be written as follows, for $z\in Z, t\in(0,\tau]$

$$\begin{cases} z'+b(I+aA)^{-1}Az=(I+aA)^{-1}1_\omega u(t)+(I+aA)^{-1}f^e(t,z_t,u), \\ z(s)=\phi(s), \ s\in[-r,0], \\ z(t_k^+)=z(t_k^-)+I_k^e(t_k,z(t_k),u(t_k)), k=1,2,3,\cdots,p. \end{cases} \tag{16}$$

$$\begin{cases} z'+b(I+aA)^{-1}Az=(I+aA)^{-1}1_\omega u(t), z\in Z, t\in(0,\tau], \\ z(0)=z^0. \end{cases} \tag{17}$$

Moreover, $(I+aA)$ and $(I+aA)^{-1}$ can be written in terms of the eigenvalues of A:

$$(I+aA)z=\sum_{j=1}^{\infty}(1+a\lambda_j)E_j z \tag{18}$$

$$(I+aA)^{-1}z=\sum_{j=1}^{\infty}\frac{1}{1+a\lambda_j}E_j z. \tag{19}$$

Therefore, if we put $B=(I+aA)^{-1}$ and $F(t,\phi,u)=(I+aA)^{-1}f^e(t,\phi,u)$, systems (16) and (17) can be written in the form:

$$\begin{cases} z'+bBAz=BB_\omega u(t)+F(t,z_t,u), \\ z(s)=\phi(s),\ \ s\in[-r,0], \\ z(t_k^+)=z(t_k^-)+I_k^e(t_k,z(t_k),u(t_k)),\ k=1,2,3,\cdots,p. \end{cases} \tag{20}$$

$$\begin{cases} z'+bBAz=BB_\omega u(t), z\in Z, t\in(0,\tau], \\ z(0)=z^0, \end{cases} \tag{21}$$

and the functions F defined above satisfy:

$$\|F(t,\phi,u)\|_Z\le\hat{a}_0\|\phi(-r)\|_Z^{\alpha_0}+\hat{b}_0\|u\|_Z^{\beta_0}+\hat{c}_0. \tag{22}$$

Now, we formulate two simple propositions.

Proposition 2.2. ([12]) *The operators* bBA *and* $T(t)=e^{-bBAt}$ *are given by the following expressions*

$$bBAz=\sum_{j=1}^{\infty}\frac{b\lambda_j}{1+a\lambda_j}E_j z \tag{23}$$

$$T(t)z=e^{-bBAt}z=\sum_{j=1}^{\infty}e^{\frac{-b\lambda_j}{1+a\lambda_j}t}E_j z. \tag{24}$$

Moreover, the following estimate holds

$$\|T(t)\|\le e^{-\beta t},\quad t\ge 0, \tag{25}$$

where

$$\beta=\inf_{j\ge1}\left\{\frac{b\lambda_j}{1+a\lambda_j}\right\}=\frac{b\lambda_1}{1+a\lambda_1}. \tag{26}$$

Observe that, due to the above notation, systems (20)-(21) can be written as follows

$$\begin{cases} z'=-\mathcal{A}z+BB_\omega u(t)+F(t,z_t,u), \\ z(s)=\phi(s),\ \ s\in[-r,0], \\ z(t_k^+)=z(t_k^-)+I_k^e(t_k,z(t_k),u(t_k)),\ k=1,2,3,\cdots,p. \end{cases} \tag{27}$$

$$\begin{cases} z'=-\mathcal{A}z+BB_\omega u(t),\ z\in Z,\ t\in(0,\tau], \\ z(0)=z^0. \end{cases} \tag{28}$$

where $\mathcal{A}=bBA$.

3. Preliminaries on Controllability of the Linear Equation

In this section we prove the interior controllability of the linear system (28). To this end, notice that for an arbitrary $z^0 \in Z$ and $u \in L^2(0,\tau;U)$ the initial value problem

$$\begin{cases} z' = -\mathcal{A}z + BB_\omega u(t), & t \in (0,\tau], \\ z(0) = z^0, \end{cases} \tag{29}$$

admits only one mild solution given by

$$z(t) = T(t)z^0 + \int_0^t T(t-s)BB_\omega u(s)\mathrm{d}s, \; t \in [0,\tau]. \tag{30}$$

Definition 3.1. *For the system* (29) *we define the following concept*: *The controllability map* (*for* $\tau > 0$) $G: L^2(0,\tau;U) \to Z$ *is given by*

$$Gu = \int_0^\tau T(s)BB_\omega u(s)\mathrm{d}s, \tag{31}$$

whose adjoint operator $G^*: Z \to L^2(0,\tau;Z)$ is given by

$$(G^*z)(s) = B_\omega^* B^* T^*(s)z, \quad \forall s \in [0,\tau], \quad \forall z \in Z. \tag{32}$$

The following lemma holds in general for a linear bounded operator $G: W \to Z$ between Hilbert spaces W and Z.

Lemma 3.1. (*see* [15] [20] [21] *and* [22]) *The Equation* (28) *is approximately controllable on* $[0,\tau]$ *if and only if one of the following statements holds*:

a) $\overline{\mathrm{Rang}(G)} = Z$.

b) $Ker(G^*) = \{0\}$.

c) $\langle GG^*z, z\rangle > 0$, $z \neq 0$ in Z.

d) $\lim_{\alpha \to 0^+} \alpha(\alpha I + GG^*)^{-1} z = 0$.

e) $B_\omega^* B^* T^*(t)z = 0, \; \forall t \in [0,\tau], \; \Rightarrow z = 0$.

f) For all $z \in Z$ we have $Gu_\alpha = z - \alpha(\alpha I + GG^*)^{-1} z$, where

$$u_\alpha = G^*(\alpha I + GG^*)^{-1} z, \quad \alpha \in (0,1].$$

So, $\lim_{\alpha \to 0} Gu_\alpha = z$ and the error $E_\alpha z$ of this approximation is given by

$$E_\alpha z = \alpha(\alpha I + GG^*)^{-1} z, \quad \alpha \in (0,1].$$

Remark 3.1. *The Lemma* 3.1 *implies that the family of linear operators* $\Gamma_\alpha: Z \to L^2(0,\tau;U)$, *defined for* $0 < \alpha \le 1$ *by*

$$\Gamma_\alpha z = B_\omega^* B^* T^*(\cdot)(\alpha I + GG^*)^{-1} z = G^*(\alpha I + GG^*)^{-1} z, \tag{33}$$

is an approximate inverse for the right of the operator G in the sense that

$$\lim_{\alpha \to 0} G\Gamma_\alpha = I. \tag{34}$$

Proposition 3.4. (*see* [21]) *If* $\overline{\mathrm{Rang}(G)} = Z$, *then*

$$\sup_{\alpha > 0} \left\| \alpha(\alpha I + GG^*)^{-1} \right\| \le 1. \tag{35}$$

Theorem 3.1. *The system* (28) *is approximately controllable on* $[0,\tau]$. *Moreover, a sequence of controls steering the system* (28) *from initial state* z^0 *to an* ε *neighborhood of the final state* z^1 *at time* $\tau > 0$ *is given by the formula*

$$u_\alpha(t) = B_\omega^* B^* T^*(t)(\alpha I + GG^*)^{-1}(z^1 - T(\tau)z^0),$$

and the error of this approximation E_α is given by the expression

$$E_\alpha = \alpha\left(\alpha I + GG^*\right)^{-1}\left(z^1 - T(\tau)z^0\right).$$

Proof. It is enough to show that the restriction $G_\omega = G|_{L^2(0,\tau;L^2(\omega))}$ of G to the space $L^2\left(0,\tau;L^2(\omega)\right)$ has range dense, *i.e.*, $\overline{\text{Rang}(G_\omega)} = Z$ or $Ker\left(G_\omega^*\right) = \{0\}$. Consequently, $G_\omega : L^2\left(0,\tau;L^2(\omega)\right) \to Z$ takes the following form

$$G_\omega u = \int_0^\tau T(s)Bu(s)\,\text{d}s.$$

whose adjoint operator $G_\omega^* : Z \to L^2\left(0,\tau;L^2(\omega)\right)$ is given by

$$\left(G_\omega z\right)(s) = B^*T^*(s)z, \quad \forall s \in [0,\tau], \quad \forall z \in Z.$$

Since B is given by the formula

$$Bz = \sum_{j=1}^{\infty}\frac{1}{1+a\lambda_j}E_j z,$$

and $T(t)$ by (24), we get that $B = B^*$ and $T^*(t) = T(t)$.

Suppose that

$$B^*T^*(t)z = 0, \quad \forall t \in [0,\tau].$$

Then we have that

$$B^*T^*(t)z = \sum_{j=1}^{\infty}\frac{\text{e}^{-\gamma_j t}}{1+a\lambda_j}E_j z = 0,$$

where $\gamma_j = \dfrac{b\lambda_j}{1+a\lambda_j}$, which satisfies the conditions:

$$0 < \gamma_1 < \gamma_2 < \cdots < \gamma_j < \cdots. \tag{36}$$

Hence, following the proof of Lemma 1.1, we obtain that

$$E_j z(x) = \sum_{k=1}^{\gamma_j}\left\langle z,\phi_{j,k}\right\rangle\phi_{j,k}(x) = 0, \quad \forall x \in \omega, \quad j = 1,2,3,\cdots.$$

Now, putting $f(x) = \sum_{k=1}^{\gamma_j}\left\langle z,\phi_{j,k}\right\rangle\phi_{j,k}(x), \forall x \in \Omega$, we obtain that

$$\begin{cases}\left(\Delta + \lambda_j I\right)f \equiv 0 & \text{in } \Omega,\\ f(x) = 0 & \forall x \in \omega.\end{cases}$$

Then, from the classical Unique Continuation Principle for Elliptic Equations (see [14]), it follows that $f(x) = 0, \forall x \in \Omega$. So,

$$\sum_{k=1}^{\gamma_j}\left\langle z,\phi_{j,k}\right\rangle\phi_{j,k}(x) = 0, \quad \forall x \in \Omega.$$

On the other hand, $\left\{\phi_{j,k}\right\}$ is a complete orthonormal set in $Z = L_2(\Omega)$, which implies that $\left\langle z,\phi_{j,k}\right\rangle = 0$.

Therefore, $E_j z = 0, j = 1,2,3,\cdots$, which implies that $z = 0$. So, $\overline{\text{Rang}(G)} = Z$. Hence, the system (29) is approximately controllable on $[0,\tau]$, and the remainder of the proof follows from Lemma 3.1. □

Lemma 3.2. *Let S be any dense subspace of $L_2(0,\tau;U)$. Then, system* (29) *is ap-*

proximately controllable with control $u \in L_2(0,\tau;U)$ *if, and only if, it is approximately controllable with control* $u \in S$ *. i.e.,*

$$\overline{\text{Rang}(G)} = Z \Leftarrow \overline{\text{Rang}(G|_S)} = Z,$$

where $G|_S$ is the restriction of G to S.

Proof ($\Rightarrow$) Suppose $\overline{\text{Rang}(G)} = Z$ and $\overline{S} = L_2(0,\tau;U)$. Then, for a given $\epsilon > 0$ and $z \in Z$ there exits $u \in L_2(0,\tau;U)$ and a sequence $\{u_n\}_{n\geq 1} \subset S$ such that

$$\|Gu - z\| < \frac{\epsilon}{2} \text{ and } \lim_{n\to\infty} u_n = u.$$

Therefore, $\lim_{n\to\infty} Gu_n = Gu$ and $\|Gu_n - z\| < \epsilon$ for n big enough. Hence, $\overline{\text{Rang}(G|_S)} = Z$.

($\Leftarrow$) This side is trivial. □

Remark 3.2 *According to the previous Lemma, if the system is approximately controllable, it is approximately controllable with control functions in the following dense spaces of* $L_2(0,\tau;U)$:

$$S = C([0,\tau];U), \quad S = C^\infty(0,\tau;U), \quad S = PC(J).$$

Moreover, the operators G, $\mathcal{W}$ and Γ are well define in the space of continuous functions: $G : C([0,\tau];U) \to Z$ by

$$Gu = \int_0^\tau T(\tau - s) BB_\omega u(s) \text{ds}, \tag{37}$$

and $G^* : Z \to C([0,\tau];U)$ by

$$(G^* z)(s) = B_\omega^* B^*(s) T^*(\tau - s) z, \quad \forall s \in [0,\tau]. \quad \forall z \in Z. \tag{38}$$

Also, the Controllability Grammian operator is still the same $\mathcal{W} : Z \to Z$

$$\mathcal{W}z = GG^* z = \int_0^\tau T(\tau - s) BB_\omega B_\omega^* B^* T^*(\tau - s) z\text{ds}. \tag{39}$$

Finally, the operators $\Gamma_\alpha : Z \to C([0,\tau];U)$ defined for $0 < \alpha \leq 1$ by

$$\Gamma_\alpha z = B_\omega^* B^* T^*(\tau - \cdot)(\alpha I + \mathcal{W})^{-1} z = G^*(\alpha I + GG^*)^{-1} z, \tag{40}$$

is an approximate inverse for the right of the operator G in the sense that

$$\lim_{\alpha\to 0} G\Gamma_\alpha = I. \tag{41}$$

4. Main Result

In this section we prove the main result of this paper, the interior controllability of the semilinear BBM Equation with impulses and delay given by (1), which is equivalent to prove the approximate controllability of the system (27). To this end, observe that for all $\phi \in C$ and $u \in C(0,\tau;U)$ the initial value problem

$$\begin{cases} z' = -\mathcal{A}z + BB_\omega u(t) + F(t, z_t, u), \\ z(s) = \phi(s), \quad s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + I_k^e(t_k, z(t_k), u(t_k)), \; k = 1,2,3,\cdots, p, \end{cases} \tag{42}$$

admits only one mild solution given by the formula

$$\begin{aligned} z^u(t) &= T(t)z^0 + \int_0^t T(t-s)BB_\omega u(s)\mathrm{d}s \\ &\quad + \int_0^t T(t-s)F(s,z_s,(s))\mathrm{d}s,\ t\in[0,\tau] \\ &\quad + \sum_{0<t_k<t} T(t-t_k)I_k^e(t_k,z(t_k),u(t_k)),\ t\in[0,\tau], \\ z(t) &= \phi(t), t\in[-r,0]. \end{aligned} \tag{43}$$

Now, we are ready to present and prove the main result of this paper, which is the interior approximate controllability of the Benjamin-Bona-Mahony (1) with impulses and delay.

Define the operator

$\mathcal{K}^\alpha : PC([-r,\tau];Z)\times C([0,\tau];U) \to PC([-r,\tau];Z)\times C([0,\tau];U)$ by the following formula:

$$(y,v) = (\mathcal{K}_1^\alpha(z,u),\mathcal{K}_2^\alpha(z,u)) = \mathcal{K}^\alpha(z,u)$$

where

$$\begin{aligned} y(t) = \mathcal{K}_1^\alpha(z,u)(t) &= T(t)\phi(0) + \int_0^t T(t-s)BB_\omega(\Gamma_\alpha\mathcal{L}(z,u))(s)\mathrm{d}s \\ &\quad + \int_0^t T(t-s)F(s,z_s,u(s))\mathrm{d}s \\ &\quad + \sum_{0<t_k<t} T(t-t_k)I_k^e(t_k,z(t_k),u(t_k)),\quad t\in[0,\tau], \\ y(t) &= \phi(t), t\in[-r,0], \end{aligned} \tag{44}$$

and

$$\begin{aligned} v(t) = \mathcal{K}_2^\alpha(z,u)(t) &= (\Gamma_\alpha\mathcal{L}(z,u))(t) \\ &= B_\omega^* T^*(\tau-t)(\alpha I+\mathcal{W})^{-1}\mathcal{L}(z,u),\ t\in[0,\tau], \end{aligned} \tag{45}$$

with $\mathcal{L} : PC([-r,\tau];Z)\times C([0,\tau];U)\to Z$ is given by

$$\begin{aligned} \mathcal{L}(z,u) &= z_1 - T(\tau)\phi(0) - \int_0^\tau T(\tau-s)F(s,z_s,u(s))\mathrm{d}s \\ &\quad - \sum_{0<t_k<\tau} T(\tau-t_k)I_k^e(t_k,z(t_k),u(t_k)). \end{aligned} \tag{46}$$

Theorem 4.1. *The nonlinear system* (1) *is approximately controllable on* $[0,\tau]$. *Moreover, a sequence of controls steering the system* (1) *from initial state* $\phi(0)$ *to an* ϵ *-neighborhood of the final state* z^1 *at time* $\tau>0$ *is given by*

$$u_\alpha(t) = B_\omega^* T^*(\tau-t)(\alpha I+\mathcal{W})^{-1}\mathcal{L}(z^\alpha,u_\alpha), t\in[0,\tau],$$

and the error of this approximation $E_\alpha z$ is given by

$$E_\alpha z = \alpha(\alpha I+\mathcal{W})^{-1}\mathcal{L}(z^\alpha,u_\alpha),$$

where

$$\begin{aligned} z^\alpha(t) &= T(t)\phi(0) + \int_0^t T(t-s)BB_\omega u_\alpha(s)\mathrm{d}s + \int_0^t T(t-s)F(s,z_s^\alpha,u_\alpha(s))\mathrm{d}s \\ &\quad + \sum_{0<t_k<t} T(t-t_k)I_k^e(t_k,z^\alpha(t_k),u_\alpha(t_k)),\ t\in[0,\tau], \end{aligned} \tag{47}$$

$$z^{\alpha}(t)=\phi(t),\, t\in[-r,0].$$

Proof. We shall prove this Theorem by claims. Before, we note that $\|B_{\omega}\|=1$ and $\|T(t)\|\le e^{-\gamma_1 t}, t\ge 0$.

Claim 1. The operator $\mathcal{K}^{\alpha}$ is continuous. In fact, it is enough to prove that the operators:

$$\mathcal{K}_1^{\alpha}: PC([-r,\tau];Z)\times C([0,\tau];U)\to PC([-r,\tau];Z)$$

and

$$\mathcal{K}_2^{\alpha}: PC([-r,\tau];Z)\times C([0,\tau];U)\to C([0,\tau];U),$$

define above are continuous. The continuity of $\mathcal{K}_1^{\alpha}$ follows from the continuity of the nonlinear functions $f^e(t,\phi,u)$, $I_k^e(t,z,u)$ and the following estimate

$$\begin{aligned}\left\|\mathcal{K}_1^{\alpha}(z,u)(t)-\mathcal{K}_1^{\alpha}(w,v)(t)\right\| &\le \int_0^t e^{-\gamma_1(t-s)}\left\|(\alpha I+\mathcal{W})^{-1}\right\|\left\|\mathcal{L}(z,u)-\mathcal{L}(w,v)\right\|ds\\ &+\int_0^t e^{-\gamma_1(t-s)}\left\|F(s,z_s,u(s))-F(s,w_s,v(s))\right\|ds\\ &+\sum_{0<t_k<t} e^{-\gamma_1(t-t_k)}\left\|I_k^e(t_k,z(t_k),u(t_k))-I_k^e(t_k,w(t_k),v(t_k))\right\|.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left\|\mathcal{L}(z,u)-\mathcal{L}(w,v)\right\| &\le \tau \sup_{s\in[0,\tau]}\left\|F(s,z_s,u(s))-F(s,w_s,v(s))\right\|\\ &+\sum_{0<t_k<\tau} e^{-\gamma_1(t-t_k)}\left\|I_k^e(t_k,z(t_k),u(t_k))-I_k^e(t_k,w(t_k),v(t_k))\right\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\mathcal{K}_1^{\alpha}(z,u)-\mathcal{K}_1^{\alpha}(w,v)\right\| &\le L_1 \sup_{s\in[0,\tau]}\left\|F(s,z_s,u(s))-F(s,w_s,v(s))\right\|\\ &+L_2\sum_{0<t_k<\tau}\left\|I_k^e(t_k,z(t_k),u(t_k))-I_k^e(t_k,w(t_k),v(t_k))\right\|,\end{aligned}$$

where $L_1=\tau\left(\tau\left\|(\alpha I+\mathcal{W})^{-1}\right\|+1\right)$ and $L_2=\left(1+\tau\left\|(\alpha I+\mathcal{W})^{-1}\right\|\right)$.

The continuity of the operator $\mathcal{K}_2^{\alpha}$ follows from the continuity of the operators $\mathcal{L}$ and Γ_{α} define above.

Claim 2. The operator $\mathcal{K}^{\alpha}$ is compact. In fact, let D be a bounded subset of $PC(J;Z)\times C(J;U)$. It follows that $\forall (z,u)\in D$, we have

$$\left\|F(\cdot,z_{(\cdot)},u)\right\|\le L_3,\quad \left\|(\alpha I+\mathcal{W})^{-1}\mathcal{L}(z,u)\right\|\le L_4,$$

$$\left\|\mathcal{L}(z,u)\right\|\le L_5, \left\|I_k^e(\cdot,z,u)\right\|\le l_k,\quad k=1,2,\cdots,p.$$

Therefore, $\mathcal{K}(D)$ is uniformly bounded.

Now, consider the following estimate:

$$\begin{aligned}&\left|\mathcal{K}^{\alpha}(z,u)(t_2)-\mathcal{K}^{\alpha}(z,u)(t_1)\right|\\ &=\left\|\mathcal{K}_1^{\alpha}(z,u)(t_2)-\mathcal{K}_1^{\alpha}(z,u)(t_1)\right\|+\left\|\mathcal{K}_2^{\alpha}(z,u)(t_2)-\mathcal{K}_2^{\alpha}(z,u)(t_1)\right\|.\end{aligned}$$

Without lose of generality we assume that $0<t_1<t_2$. On the other hand we have:

$$
\begin{aligned}
&\left\|\mathcal{K}_1^\alpha(z,u)(t_2)-\mathcal{K}_1^\alpha(z,u)(t_1)\right\| \\
&\le \left\|T(t_2)-T(t_1)\right\|\left\|\phi\right\|+\int_0^{t_1}\left\|T(t_2-s)-T(t_1-s)\right\|\left\|\mathcal{L}(z,u)(s)\right\|\mathrm{d}s \\
&\quad+\int_{t_1}^{t_2}\left\|T(t_2-s)\right\|\left\|\mathcal{L}(z,u)(s)\right\|\mathrm{d}s \\
&\quad+\int_0^{t_1}\left\|T(t_2-s)-T(t_1-s)\right\|\left\|F(s,z_s,u(s))\right\|\mathrm{d}s \\
&\quad+\int_{t_1}^{t_2}\left\|T(t_2-s)\right\|\left\|F(s,z_s,u(s))\right\|\mathrm{d}s \\
&\quad+\sum_{0<t_k<t_1}\left\|T(t_2-t_k)-T(t_1-t_k)\right\|\left\|I_k^e(t_k,z(t_k),u(t_k))\right\| \\
&\quad+\sum_{t_1<t_k<t_2}\left\|T(t_2-t_k)I_k^e(t_k,z(t_k),u(t_k))\right\|,
\end{aligned}
$$

and

$$
\left\|\mathcal{K}_2^\alpha(z,u)(t_2)-\mathcal{K}_2^\alpha(z,u)(t_1)\right\|\le\left\|T^*(\tau-t_2)-T^*(\tau-t_1)\right\|\left\|(\alpha I+\mathcal{W})^{-1}\mathcal{L}(z,u)\right\|.
$$

Since $T(t)$ is a compact operator for $t>0$, then we know that the function $0<t\to T(t)$ is uniformly continuous. So,

$$
\lim_{|t_2-t_1|\to 0}\left\|T(t_2)-T(t_1)\right\|=0.
$$

Consequently, if we take a sequence $\{\phi_j : j=1,2,\cdots\}$ on $\mathcal{K}^\alpha(D)$, this sequence is uniformly bounded and equicontinuous on the interval $[-r,t_1]$ and, by Arzela theorem, there is a subsequence $\{\phi_j^1 : j=1,2,\cdots\}$ of $\{\phi_j : j=1,2,\cdots\}$, which is uniformly convergent on $[-r,t_1]$.

Consider the sequence $\{\phi_j^1 : j=1,2,\cdots\}$ on the interval $(t_1,t_2]$. On this interval the sequence $\{\phi_j^1 : j=1,2,\cdots\}$ is uniformly bounded and equicontinuous, and for the same reason, it has a subsequence $\{\phi_j^2\}$ uniformly convergent on $[-r,t_2]$.

Continuing this process for the intervals $(t_2,t_3]$, $(t_3,t_4]$, ⋯, $(t_p,\tau]$, we see that the sequence $\{\phi_j^{p+1} : j=1,2,\cdots\}$ converges uniformly on the interval $[-r,\tau]$. This means that $\overline{\mathcal{K}^\alpha(D)}$ is compact, which implies that the operator $\mathcal{K}^\alpha$ is compact.

Claim 3.

$$
\lim_{\left\|\left|(z,u)\right|\right\|\to\infty}\frac{\left\|\left|\mathcal{K}^\alpha(z,u)\right|\right\|}{\left\|\left|(z,u)\right|\right\|}=0,
$$

where $\left\|\left|(z,u)\right|\right\|=\|z\|+\|u\|$ is the norm in the space $PC([-r,\tau];Z)\times C(0,\tau;Z)$. In fact, consider the following estimates:

$$
\left\|\mathcal{L}(z,u)\right\|\le M_1+M_2\left\{\bar{a}_0\|z\|^{\alpha_0}+\bar{b}_0\|u\|^{\beta_0}+\bar{c}_0\right\}+M_3\sum_{0<t_k<\tau}\left\{\bar{a}_k\|z\|^{\alpha_k}+\bar{b}_k\|u\|^{\beta_k}+\bar{c}_k\right\},
$$

where

$$
M_1=\left\|z^1\right\|+\mathrm{e}^{-\gamma_1\tau}\left\|\phi(0)\right\|,\quad M_2=1-\gamma_1\left(\mathrm{e}^{-\gamma_1\tau}-1\right)\ \text{and}\ M_3=\mathrm{e}^{-\gamma_1\tau}.
$$

$$
\begin{aligned}
\left\|\mathcal{K}_2^\alpha(z,u)\right\|\le{}& M_3M_1\left\|(\alpha I+\mathcal{W})^{-1}\right\|+M_3M_2\left\|(\alpha I+\mathcal{W})^{-1}\right\|\left\{\bar{a}_0\|z\|^{\alpha_0}+\bar{b}_0\|u\|^{\beta_0}+\bar{c}_0\right\} \\
&+M_3M_2\left\|(\alpha I+\mathcal{W})^{-1}\right\|\sum_{0<t_k<\tau}\left\{\bar{a}_k\|z\|^{\alpha_k}+\bar{b}_k\|u\|^{\beta_k}+\bar{c}_k\right\},
\end{aligned}
$$

and

$$\begin{aligned}\left\|\mathcal{K}_1^\alpha(z,u)\right\| &\le M_3\left\{\|z_0\|+M_1M_2\left\|(\alpha I+\mathcal{W})^{-1}\right\|\right\}\\ &+M_2\left\{1+M_2M_3\left\|(\alpha I+\mathcal{W})^{-1}\right\|\right\}\left\{\bar{a}_0\|z\|^{\alpha_0}+\bar{b}_0\|u\|^{\beta_0}+c_0\right\}\\ &+M_3\left\{1+M_2M_3\left\|(\alpha I+\mathcal{W})^{-1}\right\|\right\}\sum_{0<t_k<\tau}\left\{\bar{a}_k\|z\|^{\alpha_k}+\bar{b}_k\|u\|^{\beta_k}+\bar{c}_k\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left|\left|\left|\mathcal{K}^\alpha(z,u)\right|\right|\right|=\left\|\mathcal{K}_1^\alpha(z,u)\right\|+\left\|\mathcal{K}_2^\alpha(z,u)\right\|\le M_4\\ &+\left\{M_3M_2\left\|(\alpha I+\mathcal{W})^{-1}\right\|\{1+2M_2\}\right\}\left\{\bar{a}_0\|z\|^{\alpha_0}+\bar{b}_0\|u\|^{\beta_0}+\bar{c}_0\right\}\\ &+\left\{M_3M_2\left\|(\alpha I+\mathcal{W})^{-1}\right\|\{1+M_3\}+M_3\right\}\sum_{0<t_k<\tau}\left\{\bar{a}_k\|z\|^{\alpha_k}+\bar{b}_k\|u\|^{\beta_k}+\bar{c}_k\right\},\end{aligned}$$

where M_4 is given by:

$$M_4=M_3\left\{\|z_0\|+(M_2+1)M_1\left\|(\alpha I+\mathcal{W})^{-1}\right\|\right\}.$$

Hence

$$\begin{aligned}\frac{\left|\left|\left|\mathcal{K}^\alpha(z,u)\right|\right|\right|}{\left|\left|\left|(z,u)\right|\right|\right|}&\le M_4\|z\|+\|u\|+\left\{M_3M_2\left\|(\alpha I+\mathcal{W})^{-1}\right\|\{1+M_2\}\right\}\\ &\times\left\{\bar{a}_0\|z\|^{\alpha_0-1}+\bar{b}_0\|u\|^{\beta_0-1}+\frac{\bar{c}_0}{\|z\|+\|u\|}\right\}\\ &+\left\{M_3M_2\left\|(\alpha I+\mathcal{W})^{-1}\right\|\{1+M_3\}+M_3\right\}\\ &\times\sum_{0<t_k<\tau}\left\{\bar{a}_k\|z\|^{\alpha_k-1}+\bar{b}_k\|u\|^{\beta_k-1}+\frac{\bar{c}_k}{\|z\|+\|u\|}\right\},\end{aligned}$$

and

$$\lim_{\left|\left|\left|(z,u)\right|\right|\right|\to\infty}\frac{\left|\left|\left|\mathcal{K}^\alpha(z,u)\right|\right|\right|}{\left|\left|\left|(z,u)\right|\right|\right|}=0. \tag{48}$$

Claim 4. The operator $\mathcal{K}^\alpha$ has a fixed point. In fact, for a fixed $0<\rho<1$, there exists $R>0$ big enough such that

$$\left|\left|\left|\mathcal{K}^\alpha(z,u)\right|\right|\right|\le\rho\left|\left|\left|(z,u)\right|\right|\right|,\ \left|\left|\left|(z,u)\right|\right|\right|\ge R.$$

Hence, if we denote by $B(0,R)$ the ball of center zero and radius $R>0$, we get that $\mathcal{K}^\alpha(\partial B(0,R))\subset B(0,R)$. Since $\mathcal{K}^\alpha$ is compact and maps the sphere $\partial B(0,R)$ into the interior of the ball $B(0,R)$, we can apply Rothe's fixed point Theorem 1.1 to ensure the existence of a fixed point $(z^\alpha,u_\alpha)\in B(0,R)\subset PC([0,\tau];Z)\times C([0,\tau];U)$ such that

$$(z^\alpha,u_\alpha)=\mathcal{K}^\alpha(z^\alpha,u_\alpha). \tag{49}$$

Claim 5. The sequence $\left\{(z^\alpha,u_\alpha)\right\}_{\alpha\in(0,1]}$ is bounded. In fact, for the purpose of contradiction, let us assume that $\left\{\left(z^{\alpha},u_{\alpha}\right)\right\}_{\alpha\in(0,1]}$ is unbounded. Then, there exits a subsequence $\left\{\left(z_{\alpha_n},u_{\alpha_n}\right)\right\}_{\alpha\in(0,1]}\subset\left\{\left(z^{\alpha},u_{\alpha}\right)\right\}_{\alpha\in(0,1]}$ such that

$$\lim_{n\to\infty}\left\|\left|\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|=\infty.$$

On the other hand, from (48) we know for all $\alpha\in(0,1]$ that

$$\lim_{n\to\infty}\frac{\left\|\left|\mathcal{K}^{\alpha}\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|}=0.$$

Particularly, we have the following situation:

$$\begin{array}{cccccc}
\frac{\left\|\left|\mathcal{K}^{\alpha_1}\left(z_{\alpha_1},u_{\alpha_1}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_1},u_{\alpha_1}\right)\right|\right\|} & \frac{\left\|\left|\mathcal{K}^{\alpha_1}\left(z_{\alpha_2},u_{\alpha_2}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_2},u_{\alpha_2}\right)\right|\right\|} & \frac{\left\|\left|\mathcal{K}^{\alpha_1}\left(z_{\alpha_3},u_{\alpha_3}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_3},u_{\alpha_3}\right)\right|\right\|} & \cdots & \frac{\left\|\left|\mathcal{K}^{\alpha_1}\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|} & \to 0.\\
\frac{\left\|\left|\mathcal{K}^{\alpha_2}\left(z_{\alpha_1},u_{\alpha_1}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_1},u_{\alpha_1}\right)\right|\right\|} & \frac{\left\|\left|\mathcal{K}^{\alpha_2}\left(z_{\alpha_2},u_{\alpha_2}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_2},u_{\alpha_2}\right)\right|\right\|} & \frac{\left\|\left|\mathcal{K}^{\alpha_2}\left(z_{\alpha_3},u_{\alpha_3}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_3},u_{\alpha_3}\right)\right|\right\|} & \cdots & \frac{\left\|\left|\mathcal{K}^{\alpha_2}\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|} & \to 0.\\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\frac{\left\|\left|\mathcal{K}^{\alpha_k}\left(z_{\alpha_1},u_{\alpha_1}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_1},u_{\alpha_1}\right)\right|\right\|} & \frac{\left\|\left|\mathcal{K}^{\alpha_k}\left(z_{\alpha_2},u_{\alpha_2}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_2},u_{\alpha_2}\right)\right|\right\|} & \frac{\left\|\left|\mathcal{K}^{\alpha_k}\left(z_{\alpha_3},u_{\alpha_3}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_3},u_{\alpha_3}\right)\right|\right\|} & \cdots & \frac{\left\|\left|\mathcal{K}^{\alpha_k}\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|} & \to 0.
\end{array}$$

Now, applying Cantor's diagonalization process, we obtain that

$$\lim_{n\to\infty}\frac{\left\|\left|\mathcal{K}^{\alpha_n}\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|}=0,$$

and from (49) we have that

$$\frac{\left\|\left|\mathcal{K}^{\alpha_n}\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|}{\left\|\left|\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|}=1,$$

which is evidently a contradiction. Then, the claim is true and there exists $\gamma>0$ such that

$$\left\|\left|\left(z_{\alpha_n},u_{\alpha_n}\right)\right|\right\|\le\gamma,\ \ (0<\alpha\le 1).$$

Therefore, without loss of generality, we can assume that the sequence $\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)$ converges to $y\in Z$. So, if

$$u_{\alpha}=\Gamma_{\alpha}\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)=G^{*}\left(\alpha I+GG^{*}\right)^{-1}\mathcal{L}\left(z^{\alpha},u_{\alpha}\right).$$

Then,

$$\begin{aligned}
Gu_{\alpha}&=G\Gamma_{\alpha}\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)=GG^{*}\left(\alpha I+GG^{*}\right)^{-1}\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)\\
&=\left(\alpha I+GG^{*}-\alpha I\right)\left(\alpha I+GG^{*}\right)^{-1}\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)\\
&=\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)-\alpha\left(\alpha I+GG^{*}\right)^{-1}\mathcal{L}\left(z^{\alpha},u_{\alpha}\right).
\end{aligned}$$

Hence,

$$Gu_{\alpha}-\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)=-\alpha\left(\alpha I+GG^{*}\right)^{-1}\mathcal{L}\left(z^{\alpha},u_{\alpha}\right).$$

To conclude the proof of this Theorem, it enough to prove that

$$\lim_{\alpha\to 0}\left\{-\alpha\left(\alpha I+GG^{*}\right)^{-1}\right\}\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)=0.$$

From Lemma 3.2.d) we get that

$$\begin{aligned}&\lim_{\alpha\to 0}\left\{\alpha\left(\alpha I+GG^{*}\right)^{-1}\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)\right\}\\&=\lim_{\alpha\to 0}\alpha\left(\alpha I+GG^{*}\right)^{-1}y+\lim_{\alpha\to 0}\alpha\left(\alpha I+GG^{*}\right)^{-1}\left(\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)-y\right)\\&=\lim_{\alpha\to 0}-\alpha\left(\alpha I+GG^{*}\right)^{-1}\left(\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)-y\right).\end{aligned}$$

Now, from Proposition 3.1, we get that

$$\left\|\alpha\left(\alpha I+GG^{*}\right)^{-1}\left(\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)-y\right)\right\|\le\left\|\left(\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)-y\right)\right\|.$$

Therefore, since $\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)$ converges to *y*, we get that

$$\lim_{\alpha\to 0}\left\{-\alpha\left(\alpha I+GG^{*}\right)^{-1}\left(\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)-y\right)\right\}=0.$$

Consequently,

$$\lim_{\alpha\to 0}\left\{-\alpha\left(\alpha I+GG^{*}\right)^{-1}\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)\right\}=0.$$

Then,

$$\lim_{\alpha\to 0}\left\{Gu_{\alpha}-\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)\right\}=0.$$

Therefore,

$$\lim_{\alpha\to 0}\left\{T(\tau)\phi(0)+\int_0^{\tau}T(\tau-s)BB_{\omega}u_{\alpha}(s)\mathrm{d}s+\int_0^{\tau}T(\tau-s)F\left(s,z_s^{\alpha},u_{\alpha}(s)\right)\mathrm{d}s\right.$$
$$\left.+\sum_{0<t_k<\tau}T(\tau-t_k)I_k^e\left(z^{\alpha}(t_k),u_{\alpha}(t_k)\right)\right\}=z^1,$$

and the proof of the theorem is completed. □

As a consequence of the foregoing theorem we can prove the following characterization:

Theorem 4.2. *The Impulsive Semilinear System* (1) *is approximately controllable if for all states* z_0 *and a final state* z^1 *and* $\alpha\in(0,1]$ *the operator* $\mathcal{K}^{\alpha}$ *given by* (44)-(46) *has a fixed point and the sequence* $\left\{\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)\right\}_{\alpha\in(0,1]}$ *converges. i.e.,*

$$\left(z^{\alpha},u_{\alpha}\right)=\mathcal{K}^{\alpha}\left(z^{\alpha},u_{\alpha}\right),$$
$$\lim_{\alpha\to 0}\mathcal{L}\left(z^{\alpha},u_{\alpha}\right)=y\in Z.$$

5. Conclusions

Our technique can be applied to those control systems whose linear parts generate a compact semigroup and are under the influence of impulses and delays, as well as the following examples which represent research problems.

Problem 1. *It appears that our technique can also be applied to prove the interior controllability of the strongly damped wave equation with impulses and delay*

$$\begin{cases} \dfrac{\partial^2 w(t,x)}{\partial t^2} + \eta(-\Delta)^{1/2}\dfrac{\partial w(t,x)}{\partial t} + \gamma(-\Delta)w = 1_\omega u(t,x) + f\left(t, z(t-r,x), u(t,x)\right), \\ w = 0, \quad \text{on } (0,\tau)\times\partial\Omega, \\ w(s,x) = \phi(s,x), \quad w_t(s,x) = \psi(s,x), s\in[-r,0], x\in\Omega, \\ w_t(t_k^+,x) = w_t(t_k^-,x) + I_k\left(t_k, w(t_k,x), w_t(t_k,x), u(t_k,x)\right), \quad x\in\Omega, \end{cases}$$

in the space $Z_{1/2} = D\left((-\Delta)^{1/2}\right)\times L_2(\Omega)$*, where* Ω *is a bounded domain in* $\mathbb{R}^n$*,* ω *is an open nonempty subset of* Ω*,* 1_ω *denotes the characteristic function of the set* ω*, the distributed control* $u\in L_2(0,\tau;L_2(\Omega))$*,* $\psi,\phi:[-r,0]\times\Omega\to\mathbb{R}$ *are continuous functions, and* η*,* γ *are positive numbers.*

Problem 2. *Our technique may also be applied to a system given by partial differential equations modeling the structural damped vibrations of a string or a beam with impulses and delay*

$$\begin{cases} \dfrac{\partial^2 y(t,x)}{\partial t^2} - 2\beta\Delta\dfrac{\partial y(t,x)}{\partial t} + \Delta^2 y = 1_\omega u(t,x) + f\left(t, y(t-r,x), u(t,x)\right), \quad \text{on } (0,\tau)\times\Omega, \\ y = \Delta y = 0, \quad \text{on } (0,\tau)\times\partial\Omega, \\ y(s,x) = \phi(s,x), \quad y_t(s,x) = \psi(s,x), \quad s\in[-r,0], x\in\Omega, \\ y_t(t_k^+,x) = y_t(t_k^-,x) + I_k\left(t_k, y(t_k,x), y_t(t_k,x), u(t_k,x)\right), \quad x\in\Omega. \end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u\in L_2(0,\tau;L_2(\Omega))$, $\psi,\phi:[-r,0]\times\Omega\to\mathbb{R}$ are continuous functions and $y_0, y_1\in L_2(\Omega)$.

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Competing Interests

The authors declare that there is not competing of interests.

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