New Implementation of Reproducing Kernel Method for Solving Functional-Differential Equations

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Abstract

In this paper, we apply the new algorithm of reproducing kernel method to give the approximate solution to some functional-differential equations. The numerical results demonstrate the accuracy of the proposed algorithm.

Keywords

Reproducing Kernel Hilbert Spaces, Functional-Differential Equations, Approximate Solutions

1. Introduction

The subject of differential equations is a wide field in pure and applied mathematics, engineering, physics, chemistry, biology, psychology and other fields. All of these disciplines are associated with the properties of differential equations of various types. Pure mathematics discusses existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions. Differential equations have a prominent role in modelling virtually each physical, technical and/or biological processes, from celestial motion, to bridge design, to interactions between neurons.

This paper investigates the approximate solution of the following linear functional-differential equation using new implementation of the reproducing kernel method (RKM):

$$y''(x) = g(x, y, y', y''), Ay(0) + By(1) = 0, Cy'(0) + Dy'(1) = 0, \ 0 < x < 1. \tag{1}$$

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where g is a function of its variables, A, B, C, D are real constants and unknown function $y(x)$ is continuous on the interval [0, 1]. These problems arise in many areas of applied mathematics, physics and engineering, such as fluid mechanics, gas dynamics, reaction diffusion process, nuclear physics, chemical reactor theory, geophysics, studies of atomic structures and etc. Several numerical techniques such as finite difference approximation [1], cubic splines [2] [3], B-splines [4], Adomian decomposition method [5], differential transformation method [6] and others [7] [8] have been proposed to obtain approximate solution of these problems by some authors. The application of RKM in linear and nonlinear problems has been developed by many researchers [9]-[12]. The RKM has been treated singular linear two-point boundary value problem, singular nonlinear two-point periodic boundary value problem, nonlinear system of boundary value problem, singular integral equations, nonlinear partial differential equations and etc. in recent years in [13]-[17].

As we know, Gram-Schmidt orthogonalization process is numerically unstable, in addition it may take a lot of time to produce numerical approximation. Here, instead of using orthogonal process, we successfully make use of the basic functions which are obtained by RKM.

This paper is organized as follows. In the next section, two reproducing kernel Hilbert space (RKHS) are introduced. Section 3 is devoted to solve Equation (1) by new implementation of RKM. Some numerical examples are presented in Section 4. Last section is a brief conclusion.

2. Reproducing Kernel Spaces

In this section, we follow the recent work of [18] [19] and present some useful materials.

Definition 1. For a nonempty set $\mathcal{X}$, let $\left(\mathcal{H}, \langle .,. \rangle_{\mathcal{H}}\right)$ be a Hilbert space of real-valued functions on some set $\mathcal{X}$. A function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is said to be the *reproducing kernel* of $\mathcal{H}$ if and only if

1. $k(x,.) \in \mathcal{H}$, $\forall x \in \mathcal{X}$,
2. $\langle u, k(x,.) \rangle_{\mathcal{H}} = u(x)$, $\forall u \in \mathcal{H}$, $\forall x \in \mathcal{X}$, (reproducing property).

Also, a Hilbert space of functions $\left(\mathcal{H}, \langle .,. \rangle_{\mathcal{H}}\right)$ that possesses a reproducing kernel k is a RKHS and we denote it by $\left(\mathcal{H}, \langle .,. \rangle_{\mathcal{H}}, k\right)$. In the following we often denote by k_x the function $k(x,.) : t \mapsto k(x,t)$.

Definition 2. $W_2^3[0,1] = \left\{ y \,\middle|\, y'' \text{ is an absolute continuous real-valued function on the interval}\right.$
$[0,1], y''' \in L^2[0,1], Ay(0) + By(1) = 0, Cy'(0) + Dy'(1) = 0 \}$. The inner product and the norm in the function space $W_2^3[0,1]$ are defined as follows:

$$\langle y, v \rangle_{W_2^3} = y'(0)v'(0) + y''(0)v''(0) + \int_0^1 y'''(x)v'''(x)\,\mathrm{d}x, \|y\|_{W_2^3} = \sqrt{\langle y, y \rangle_{W_2^3}}.$$

Suppose that function $R_x \in W_2^3[0,1]$ satisfies the following generalized differential equations

$$\begin{cases} (-1)^3 \dfrac{\partial^6 R_x(t)}{\partial t^6} = \delta(t-x), \\ \dfrac{\partial R_x(0)}{\partial t} + \dfrac{\partial^4 R_x(0)}{\partial t^4} = 0, \dfrac{\partial^2 R_x(0)}{\partial t^2} - \dfrac{\partial^3 R_x(0)}{\partial t^3} = 0, \dfrac{\partial^4 R_x(1)}{\partial t^4} = 0, \dfrac{\partial^3 R_x(1)}{\partial t^3} = 0. \end{cases} \quad (2)$$

where δ is the Dirac delta function, therefore the following theorem holds.

Theorem 1 *Under the assumptions of Equation* (2), *Hilbert space* $W_2^3[0,1]$ *is a RKHS with the reproducing kernel function* R_x, *namely for any* $y \in W_2^3[0,1]$ *and each fixed* $x \in [0,1]$,

$$\langle y, R_x \rangle_{W_2^3} = y(x).$$

Proof. Applying integration by parts three times, since $R_x \in W_2^3[0,1]$, we have

$$\langle y, R_x \rangle_{W_2^3} = y'(0)\frac{\partial R_x(0)}{\partial t} + y''(0)\frac{\partial^2 R_x(0)}{\partial t^2} + \int_0^1 y'''(t)\frac{\partial^3 R_x(t)}{\partial t^3}\,\mathrm{d}t$$

$$= y'(0)\left[\frac{\partial R_x(0)}{\partial t} + \frac{\partial^4 R_x(0)}{\partial t^4}\right] + y''(0)\left[\frac{\partial^2 R_x(0)}{\partial t^2} - \frac{\partial^3 R_x(0)}{\partial t^3}\right]$$

$$+ y''(1)\frac{\partial^3 R_x(1)}{\partial t^3} - y'(1)\frac{\partial^4 R_x(1)}{\partial t^4} - \int_0^1 y(t)\frac{\partial^6 R_x(t)}{\partial t^6}\,\mathrm{d}t.$$

Therefore, Equation (2) implies that

$$\langle y, R_x \rangle_{W_2^3} = \int_0^1 y(t) \delta(t-x) \mathrm{d}t = y(x). \qquad \square$$

While $x \neq t$, function $R_x(t)$ is the solution of the following constant linear homogeneous differential equation with 6 order,

$$\frac{\partial^6 R_x(t)}{\partial t^6} = 0, \qquad (3)$$

with the boundary condition:

$$\begin{cases} \dfrac{\partial R_x(0)}{\partial t} + \dfrac{\partial^4 R_x(0)}{\partial t^4} = 0, & \dfrac{\partial^3 R_x(1)}{\partial t^3} = 0, \\ \dfrac{\partial^2 R_x(0)}{\partial t^2} - \dfrac{\partial^3 R_x(0)}{\partial t^3} = 0, & \dfrac{\partial^4 R_x(1)}{\partial t^4} = 0. \end{cases} \qquad (4)$$

We know that Equation (3) has characteristic equation $\lambda^6 = 0$, and the eigenvalue $\lambda = 0$ is a root whose multiplicity is 6. Hence, the general solution of Equation (2) is

$$R_x(t) = \begin{cases} \displaystyle\sum_{i=1}^{6} c_i(x) t^{i-1}, & t \leq x, \\ \displaystyle\sum_{i=1}^{6} d_i(x) t^{i-1}, & t > x. \end{cases} \qquad (5)$$

Now, we are ready to calculate the coefficients $c_i(x)$ and $d_i(x), i = 1, \cdots, 6$. Since

$$\frac{\partial^6 R_x(t)}{\partial t^6} = \delta(t-x),$$

we have

$$\begin{cases} \dfrac{\partial^k R_x(x^+)}{\partial t^k} = \dfrac{\partial^k R_x(x^-)}{\partial t^k}, & k = 0, \cdots, 4, \\ \dfrac{\partial^5 R_x(x^+)}{\partial t^5} - \dfrac{\partial^5 R_x(x^-)}{\partial t^5} = -1. \end{cases} \qquad (6)$$

Then, using Equations (4) and (6), the unknown coefficients of Equation (5) are uniquely obtained. Therefore,

$$R_x(t) = \begin{cases} (1/120)t^5 - (1/24)t^4 x + (1/12)t^3 x^2 + (1/4)t^2 x^2, t \leq x, \\ t^2\left((1/12)x^3 + (1/4)x^2\right) - (1/24)tx^4 + (1/120)x^5, t > x, \end{cases}$$

Definition 3. $W_2^1[0,1] = \{\, y \mid y$ is an absolute continuous real-valued function on the interval $[0,1]$, $y' \in L^2[0,1] \,\}$. The inner product and the norm in the function space $W_2^1[0,1]$ are defined as follows:

$$\langle y, v \rangle_{W_2^1} = y(0)v(0) + \int_0^1 y'(x)v'(x)\mathrm{d}x, \|y\|_{W_2^1} = \sqrt{\langle y, y \rangle_{W_2^1}}.$$

Theorem 2 *Hilbert space $W_2^1[0,1]$ is a reproducing kernel space with the reproducing kernel function*

$$r_x(t) = \begin{cases} 1+t, & t \leq x, \\ 1+x, & t > x. \end{cases}$$

Theorem 3 (*see* [20]) *Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of interval $[0,1]$ then $\{r_{x_i}(x)\}_{i=1}^{\infty}$ is a basis of $W_2^1[0,1]$.*

3. The New Implementation of the Method

In this section, we shall give the exact or approximate solution of Equation (1) in the reproducing kernel space

$W_2^3[0,1]$. We suppose that Equation (1) has a unique solution. We consider Equation (1) as

$$\mathbb{L}(x) = f(x, y, y', y''), \ 0 \le x \le 1, \tag{7}$$

where $\mathbb{L}y(x) = p(x)y''(x)$ and $f(x, y, y', y'') = p(x)g(x, y, y', y'')$, such that f is an analytical function. It is clear that $\mathbb{L}$ is the bounded linear operator of $W_2^3[0,1]$ into $W_2^1[0,1]$. We shall give the representation of analytical solution of Equation (7) in the space $W_2^3[0,1]$. Put $\varphi_i(x) = r_{x_i}(x)$ and $\psi_i(x) = \mathbb{L}^* \varphi_i(x), i = 1, 2, \cdots,$ where $r_{x_i}(x)$ is the reproducing kernel of $W_2^1[0,1]$ and $\mathbb{L}^*$ is the adjoint operator of $\mathbb{L}$.

Theorem 4 *Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of interval $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system of $W_2^3[0,1]$ and $\psi_i(x) = \mathbb{L}_t R_x(t)|_{x_i}$, where the subscript t in the operator $\mathbb{L}$ indicates that the operator $\mathbb{L}$ applies to the function of t.*

Proof. For each fixed $y \in W_2^3[0,1]$, let $\langle y, \psi_i \rangle_{W_2^3} = 0, i = 1, 2, \cdots,$ which means that

$$\langle y, \mathbb{L}^* \varphi_i \rangle_{W_2^3} = \langle \mathbb{L}y, \varphi_i \rangle_{W_2^1} = \mathbb{L}y(x_i) = 0, i = 1, 2, \cdots.$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, hence, $\mathbb{L}y(x) = 0$. Due to the existence of $\mathbb{L}^{-1}$, then $y \equiv 0$. Therefore, $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$. Now, note that

$$\psi_i(x) = \mathbb{L}^* \varphi_i(x) = \langle \mathbb{L}^* \varphi_i, R_x \rangle_{W_2^3} = \langle \varphi_i, \mathbb{L}R_x \rangle_{W_2^1}$$
$$= \langle R_{x_i}, \mathbb{L}R_x \rangle_{W_2^1} = \mathbb{L}_t R_x(t)|_{t=x_i} . \qquad \square$$

Usually, a normalized orthogonal system is constructed from $\{\psi_i(x)\}_{i=1}^{\infty}$ by using the Gram-Schmidt algorithm, and then the approximate solution will be obtained by calculating a truncated series based on these functions. However, Gram-Schmidt algorithm has some drawbacks such as numerical instability and high volume of computations. Here, to fix these flaws, we state the following Theorem in which the following notations are used.

$$\hat{a} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_N \end{bmatrix}, \bar{a} = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_N \end{bmatrix}, \bar{F} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_N \end{bmatrix}, B = \begin{bmatrix} \beta_{11} & 0 & \cdots & 0 \\ \beta_{21} & \beta_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N1} & \beta_{N2} & \cdots & \beta_{NN} \end{bmatrix},$$

where $\hat{f}_i = \langle f, \psi_i \rangle, \beta_{ii} > 0, i = 1, \cdots, N.$ And

$$\Psi = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1N} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{N1} & \psi_{N2} & \cdots & \psi_{NN} \end{bmatrix}, \tilde{\Psi} = \begin{bmatrix} \tilde{\psi}_{11} & \tilde{\psi}_{12} & \cdots & \tilde{\psi}_{1N} \\ \tilde{\psi}_{21} & \tilde{\psi}_{22} & \cdots & \tilde{\psi}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\psi}_{N1} & \tilde{\psi}_{N2} & \cdots & \tilde{\psi}_{NN} \end{bmatrix},$$

where $\psi_{ij} = \langle \mathbb{L}\psi_j, \psi_i \rangle, \tilde{\psi}_{ij} = \langle \mathbb{L}\tilde{\psi}_j, \tilde{\psi}_i \rangle, i, j = 1, \cdots, N.$

Theorem 5 *Suppose that $\{\psi_i(x)\}_{i=1}^{\infty}$ is a linearly independent set in $W_2^3[0,1]$ and $\{\tilde{\psi}_i(x)\}_{i=1}^{\infty}$ be a normalized orthogonal system in $W_2^3[0,1]$, such that $\tilde{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x)$. If*
$$y(x) = \sum_{i=1}^{\infty} \bar{a}_i \psi_i(x) \simeq y_N(x) = \sum_{i=1}^{N} \bar{a}_i \psi_i(x) = \sum_{i=1}^{N} \hat{a}_i \tilde{\psi}_i(x) \ then \ \Psi\bar{a} = \bar{F}.$$

Proof. Suppose that $y \in W_2^3[0,1]$ then $y(x) = \sum_{i=1}^{\infty} \bar{a}_i \psi_i(x) = \sum_{i=1}^{\infty} \hat{a}_i \tilde{\psi}_i(x).$ Now, by truncating N-term of the two series, because of $y_N(x) = \sum_{i=1}^{\infty} \bar{a}_i \psi_i(x) = \sum_{i=1}^{\infty} \hat{a}_i \tilde{\psi}_i(x)$ and since $\tilde{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x)$ so

$$\sum_{i=1}^{N} \bar{a}_i \psi_i(x) = \sum_{i=1}^{N} \hat{a}_i \tilde{\psi}_i(x) = \sum_{i=1}^{N} \hat{a}_i \sum_{k=1}^{i} \beta_{ik} \psi_k(x) = \sum_{k=1}^{N} \left(\sum_{i=k}^{N} \hat{a}_i \beta_{ik} \right) \psi_k(x).$$

Due to the linear independence of $\{\psi_i(x)\}_{i=1}^{\infty}$, $\bar{a}_k = \sum_{i=k}^{N} \hat{a}_i \beta_{ik}, k = 1, \cdots, N$ therefore

$$\bar{a} = B^{\mathrm{T}} \hat{a}. \tag{8}$$

Equation (7), imply $\mathbb{L} y_N(x) = f(x)$. For $i = 1, \cdots, N$ we have

$$\langle \mathbb{L} y_N, \tilde{\psi}_i \rangle = \langle f, \tilde{\psi}_i \rangle \Rightarrow \sum_{j=1}^{N} \hat{a}_j \langle \mathbb{L} \tilde{\psi}_j, \tilde{\psi}_i \rangle = \langle f, \tilde{\psi}_i \rangle$$

$$\Rightarrow \sum_{j=1}^{N} \hat{a}_j \sum_{k=1}^{i} \beta_{ik} \sum_{l=1}^{j} \beta_{jl} \langle \mathbb{L} \psi_l, \psi_k \rangle = \sum_{k=1}^{i} \beta_{ik} \langle f, \psi_k \rangle$$

$$\Rightarrow \sum_{j=1}^{N} \hat{a}_j \sum_{k=1}^{i} \sum_{l=1}^{j} \beta_{ik} \langle \mathbb{L} \psi_l, \psi_k \rangle \beta_{lj}^{\mathrm{T}} = \sum_{k=1}^{i} \beta_{ik} \langle f, \psi_k \rangle$$

$$\Rightarrow \sum_{j=1}^{N} \hat{a}_j \left(B \Psi B^{\mathrm{T}} \right)_{ij} = \sum_{k=1}^{i} \beta_{ik} \langle f, \psi_k \rangle,$$

$$\Rightarrow \left(B \Psi B^{\mathrm{T}} \right) \hat{a} = B \bar{F}.$$

Equation (8), imply $B \Psi \bar{a} = B \bar{F}$, hence

$$\Psi \bar{a} = \bar{F}. \qquad \qquad \square$$

It is necessary to mention that here we solve the system $\Psi \bar{a} = \bar{F}$. which obtained without using the Gram-Schmidt algorithm.

4. Numerical Examples

To illustrate the effectiveness of the proposed method, some numerical examples are considered in this section. The numerical results in **Table 1** and **Table 2** show that the approximate solution and its derivatives up to second order, converge to the exact solution and its derivatives respectively. The examples are computed by using Maple 18.

Example 1. Consider the following second-order ordinary differential equation with singular coefficients

$$\left(1 - \frac{x}{2} \right) y''(x) + \frac{3}{2} \left(\frac{1}{x} - 1 \right) y'(x) + \left(\frac{x}{2} - 1 \right)(x) = g(x), \quad 0 < x < 1, \tag{9}$$

$$y(0) = 0, \quad y'(0) = 0,$$

Table 1. Absolute errors for Example 1 with CPU time 46.582 in seconds.

x	$\left\| y(x) - y_{200}(x) \right\|$	$\left\| y'(x) - y'_{200}(x) \right\|$	$\left\| y''(x) - y''_{200}(x) \right\|$
0.1	$2.2548986321 \times 10^{-7}$	$8.2346060164 \times 10^{-8}$	$9.4469099176 \times 10^{-7}$
0.2	$2.1962174673 \times 10^{-7}$	$1.8836657717 \times 10^{-7}$	$1.4753989278 \times 10^{-6}$
0.3	$1.8757713167 \times 10^{-7}$	$4.5646925657 \times 10^{-7}$	$2.0671564234 \times 10^{-6}$
0.4	$1.2771103383 \times 10^{-7}$	$7.4243132897 \times 10^{-7}$	$2.2157991465 \times 10^{-6}$
0.5	$3.9095978004 \times 10^{-8}$	$1.0282644732 \times 10^{-6}$	$2.0956249245 \times 10^{-6}$
0.6	$7.7341240318 \times 10^{-8}$	$1.2962303524 \times 10^{-6}$	$1.7744164060 \times 10^{-6}$
0.7	$2.1908577292 \times 10^{-7}$	$1.5324625973 \times 10^{-6}$	$1.2965365761 \times 10^{-6}$
0.8	$3.8244585501 \times 10^{-7}$	$1.7272791325 \times 10^{-6}$	$6.9710360283 \times 10^{-7}$
0.9	$5.6296625266 \times 10^{-7}$	$1.8750776130 \times 10^{-6}$	$5.2390846268 \times 10^{-9}$

where $g(x) = 5 - \dfrac{29x}{2} + \dfrac{13x^2}{2} + \dfrac{3x^3}{2} - \dfrac{x^4}{2}$. The exact solution is $y(x) = x^2 - x^3$.

Example 2. We consider the following second-order delay differential equation

$$y''(x) = y'\left(\frac{x}{2}\right) - \frac{1}{2}y''\left(\frac{x}{2}\right) - x + 3, \quad 0 < x < 1,$$

$$y(0) = 1, \quad y'(0) = 0,$$

(10)

with exact solution $y(x) = 1 + x^2$.

Remark 1. The RKM is tested on these problems with gird points $x_i = \dfrac{i}{N}, i = 1, \cdots, N$ for $N = 20, 200$. The numerical results for Example 1 and 2 are listed in **Table 1**, **Table 2** and **Figure 1**, **Figure 2**. The results shown in **Table 1**, **Table 2**, indicate that the approximate solution and its derivatives up to second order, converge to the exact solution and its derivatives respectively.

Table 2. Absolute errors for Example 2 with CPU time 14.321 in seconds.

| x	$\left|y(x) - y_{200}(x)\right|$	$\left|y'(x) - y'_{200}(x)\right|$	$\left|y''(x) - y''_{200}(x)\right|$
0.1	$5.2264693462 \times 10^{-8}$	$1.0611304470 \times 10^{-6}$	$3.3094355000 \times 10^{-6}$
0.2	$1.0436888331 \times 10^{-7}$	$6.4357294000 \times 10^{-8}$	$5.0328956000 \times 10^{-7}$
0.3	$1.3870091453 \times 10^{-7}$	$7.4194858790 \times 10^{-7}$	$7.9044009800 \times 10^{-6}$
0.4	$2.4136835086 \times 10^{-7}$	$1.1820612788 \times 10^{-6}$	$3.9655752000 \times 10^{-7}$
0.5	$3.5146566701 \times 10^{-7}$	$9.5106389175 \times 10^{-7}$	$3.7509744025 \times 10^{-6}$
0.6	$4.2789167831 \times 10^{-7}$	$5.8832386536 \times 10^{-7}$	$3.1381459680 \times 10^{-6}$
0.7	$4.7405615700 \times 10^{-7}$	$3.6855082200 \times 10^{-7}$	$1.1318180400 \times 10^{-6}$
0.8	$5.0902815873 \times 10^{-7}$	$3.6716689840 \times 10^{-7}$	$1.0503155200 \times 10^{-6}$
0.9	$5.5388322306 \times 10^{-7}$	$5.5412781980 \times 10^{-7}$	$2.5048595280 \times 10^{-6}$
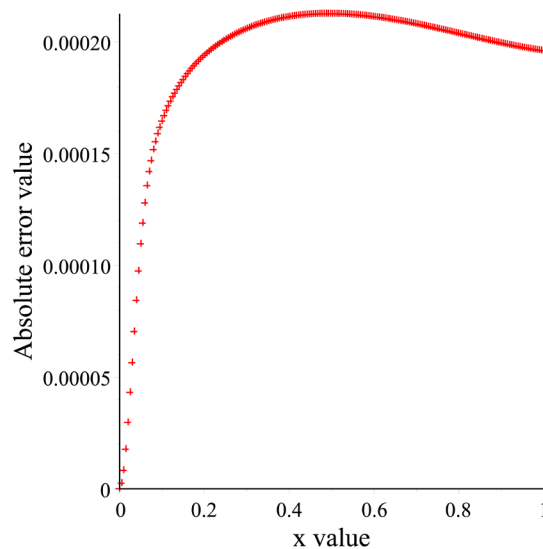
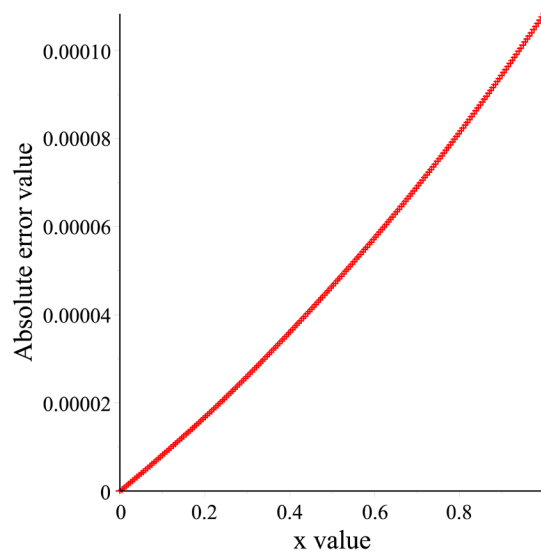**Figure 1.** The absolute error between exact and approximate solutions for Example 1 with $N = 20$.

Figure 2. The absolute error between exact and approximate solutions for Example 2 with $N = 20$.

5. Conclusion

In this paper, it is shown that the RKM without Gram-Schmidt algorithm is quiet efficient and well suited for finding the approximate solutions to some two-order initial value problems. We obtained the numerical solutions, with high accuracy and moderate CPU time. The numerical results obtained here, indicate the high performance of this method for approximating solution of functional-differential equations.

Acknowledgements

We thank the Editor and the referees for their comments.

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