Applied Mathematics, 2016, 7, 953-966
Published Online May 2016 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2016.79085

Scientific Research Publishing

Idempotent Elements of the Semigroups $B_X(D)$ Defined by Semilattices of the Class $\Sigma_3(X,8)$ When $Z_7=\varnothing$

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Received 18 March 2016; accepted 24 May 2016; published 27 May 2016

Abstract

In this paper, complete semigroup binary relation is defined by semilattices of the class $\Sigma_3(X,8)$. We give a full description of idempotent elements of given semigroup. For the case where X is a finite set and $Z_7=\varnothing$, we derive formulas by calculating the numbers of idempotent elements of the respective semigroup.

Keywords

Semilattice, Semigroup, Binary Relation, Idempotent Element

1. Introduction

Definition 1.1. Let $\varepsilon\in B_X(D)$. If $\varepsilon\circ\varepsilon=\varepsilon$ or $\alpha\circ\varepsilon=\alpha$ for any $\alpha\in B_X(D)$, then ε is called an idempotent element or called right unit of the semigroup $B_X(D)$ respectively.

Definition 1.2. We say that a complete X-semilattice of unions D is an XI-semilattice of unions if it satisfies the following two conditions:

a) $\wedge(D,D_t)\in D$ for any $t\in\breve{D}$;

b) $Z=\bigcup_{t\in Z}\wedge(D,D_t)$ for any nonempty element Z of D (see [1], Definition 1.14.2 or see [2], Definition 1.14.2).

Definition 1.3. Let D be an arbitrary complete X-semilattice of unions, $\alpha\in B_X(D)$. If

How to cite this paper: Tavdgiridze, G., Diasamidze, Ya. and Givradze, O. (2016) Idempotent Elements of the Semigroups $B_X(D)$ Defined by Semilattices of the Class $\Sigma_3(X,8)$ When $Z_7=\varnothing$. *Applied Mathematics*, **7**, 953-966.
http://dx.doi.org/10.4236/am.2016.79085

$$V[\alpha]=\begin{cases} V\left(X^*,\alpha\right), \text{ if } \varnothing\notin D, \\ V\left(X^*,\alpha\right), \text{ if } \varnothing\in V(X^*,\alpha), \\ V\left(X^*,\alpha\right)\cup\{\varnothing\}, \text{ if } \varnothing\notin V\left(X^*,\alpha\right) \text{ and } \varnothing\in D, \end{cases}$$

then it is obvious that any binary relation α of a semigroup $B_X(D)$ can always be written in the form $\alpha=\bigcup_{T\in V[\alpha]}\left(Y_T^{\alpha}\times T\right)$ the sequel, such a representation of a binary relation α will be called quasinormal.

Note that for a quasinormal representation of a binary relation α, not all sets $Y_T^{\alpha}\left(T\in V[\alpha]\right)$ can be different from an empty set. But for this representation the following conditions are always fulfilled:

a) $Y_T^{\alpha}\cap Y_{T'}^{\alpha}=\varnothing$, for any $T,T'\in D$ and $T\neq T'$;

b) $X=\bigcup_{T\in V[\alpha]}Y_T^{\alpha}$ (see [1], Definition 1.11 or see [2], Definition 1.11).

Theorem 1.1. Let D, $\Sigma(D)$, $E_X^{(r)}(D')$ and I denote respectively the complete X-semilattice of unions D, the set of all XI-subsemilattices of the semilattice D, the set of all right units of the semigroup $B_X(D')$ $(D'\in\Sigma(D))$ and the set of all idempotents of the semigroup $B_X(D)$. Then for the sets $E_X^{(r)}(D')$ and I the following statements are true:

a) if $\varnothing\in D$ and $\Sigma_{\varnothing}(D)=\left\{D'\in\Sigma(D)\,|\,\varnothing\in D'\right\}$, then

1) $E_X^{(r)}(D')\cap E_X^{(r)}(D'')=\varnothing$ for any elements D' and D'' of the set $\Sigma_{\varnothing}(D)$ that satisfy the condition $D'\neq D''$;

2) $I=\bigcup_{D'\in\Sigma_{\varnothing}(D)}E_X^{(r)}(D')$;

3) the equality $|I|=\sum_{D'\in\Sigma_{\varnothing}(D)}\left|E_X^{(r)}(D')\right|$ is fulfilled for the finite set X.

b) if $\varnothing\notin D$, then

1) $E_X^{(r)}(D')\cap E_X^{(r)}(D'')=\varnothing$ for any elements D' and D'' of the set $\Sigma(D)$ that satisfy the condition $D'\neq D''$;

2) $I=\bigcup_{D'\in\Sigma(D)}E_X^{(r)}(D')$;

3) the equality $|I|=\sum_{D'\in\Sigma(D)}\left|E_X^{(r)}(D')\right|$ is fulfilled for the finite set X (see [1] [2] Theorem 6.2.3).

2. Results

Lemma 2.1. Let $D\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. Then the following sets are all XI-subsemilattices of the given semilattice D:

1) $\{\varnothing\}$. (see diagram 1 of the **Figure 1**);

2) $\{\varnothing,Z_6\},\{\varnothing,Z_5\},\{\varnothing,Z_4\},\{\varnothing,Z_3\},\{\varnothing,Z_2\},\{\varnothing,Z_1\},\{\varnothing,\breve{D}\}$ (see diagram 2 of the **Figure 1**);

3) $\{\varnothing,Z_6,Z_4\},\{\varnothing,Z_6,Z_2\},\{\varnothing,Z_6,Z_1\},\{\varnothing,Z_6,\breve{D}\},\{\varnothing,Z_5,Z_4\},\{\varnothing,Z_5,Z_3\},\{\varnothing,Z_5,Z_2\},\{\varnothing,Z_5,Z_1\},$ $\{\varnothing,Z_5,\breve{D}\},\{\varnothing,Z_4,Z_2\},\{\varnothing,Z_4,Z_1\},\{\varnothing,Z_4,D\},\{\varnothing,Z_3,Z_1\},\{\varnothing,Z_3,\breve{D}\},\{\varnothing,Z_2,\breve{D}\},\{\varnothing,Z_1,\breve{D}\}$; (see diagram 3 of the **Figure 1**);

4) $\{\varnothing,Z_6,Z_4,Z_2\},\{\varnothing,Z_6,Z_4,Z_1\},\{\varnothing,Z_6,Z_4,\breve{D}\},\{\varnothing,Z_6,Z_2,D\},\{\varnothing,Z_6,Z_1,\breve{D}\},\{\varnothing,Z_5,Z_4,Z_2\},$ $\{\varnothing,Z_5,Z_4,Z_1\},\{\varnothing,Z_5,Z_4,\breve{D}\}$ $\{\varnothing,Z_5,Z_3,Z_1\},\{\varnothing,Z_5,Z_3,\breve{D}\},\{\varnothing,Z_5,Z_2,\breve{D}\},\{\varnothing,Z_5,Z_1,\breve{D}\},$ $\{Z_7,Z_4,Z_2,D\},\{Z_7,Z_4,Z_1,\breve{D}\},\{Z_7,Z_3,Z_1,\breve{D}\},$ (see diagram 4 of the **Figure 1**);

5) $\{\varnothing,Z_6,Z_4,Z_2,\breve{D}\},\{\varnothing,Z_6,Z_4,Z_1,\breve{D}\},\{\varnothing,Z_5,Z_4,Z_2,\breve{D}\},\{\varnothing,Z_5,Z_4,Z_1,\breve{D}\},\{\varnothing,Z_5,Z_3,Z_1,\breve{D}\},$ (see diagram

5 of the **Figure 1**);

6) $\{\varnothing, Z_6, Z_5, Z_4\}, \{\varnothing, Z_6, Z_3, Z_1\}, \{\varnothing, Z_4, Z_3, Z_1\}, \{\varnothing, Z_3, Z_2, \breve{D}\}, \{\varnothing, Z_2, Z_1, \breve{D}\}$ (see diagram 6 of the **Figure 1**);

7) $\{\varnothing, Z_6, Z_2, Z_1, \breve{D}\}, \{\varnothing, Z_5, Z_4, Z_3, Z_1,\}, \{\varnothing, Z_5, Z_3, Z_2, \breve{D}\}, \{\varnothing, Z_5, Z_2, Z_1, \breve{D}\}, \{\varnothing, Z_4, Z_2, Z_1, \breve{D}\}$, (see diagram 7 of the **Figure 1**);

8) $\{\varnothing, Z_6, Z_4, Z_2, Z_1, \breve{D}\}, \{\varnothing, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$; (see diagram 8 of the **Figure 1**);

9) $\{\varnothing, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$; (see diagram 9 of the **Figure 1**);

10) $\{\varnothing, Z_6, Z_5, Z_4, Z_2\}, \{\varnothing, Z_6, Z_5, Z_4, Z_1\}, \{\varnothing, Z_6, Z_5, Z_4, \breve{D}\}, \{\varnothing, Z_4, Z_3, Z_1, \breve{D}\}, \{\varnothing, Z_6, Z_3, Z_1, \breve{D}\}$, (see diagram 10 of the **Figure 1**);

11) $\{\varnothing, Z_6, Z_5, Z_4, Z_2, \breve{D}\}, \{\varnothing, Z_6, Z_5, Z_4, Z_1, \breve{D}\}$; (see diagram 11 of the **Figure 1**);

12) $\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1\}, \{\varnothing, Z_6, Z_3, Z_2, Z_1, \breve{D}\}, \{\varnothing, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ (see diagram 12 of the **Figure 1**);

13) $\{\varnothing, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$; (see diagram 13 of the **Figure 1**);

14) $\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$; (see diagram 14 of the **Figure 1**);

15) $\{\varnothing, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$; (see diagram 15 of the **Figure 1**);

16) $\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$; (see diagram 16 of the **Figure 1**);

Proof: This lemma immediately follows from the ([3], lemma 2.4).

Lemma is proved.

We denote the following semitattices $Q_i, i = (1, 2, \cdots 16)$ as follows:

1) $Q_1 = \{\varnothing\}$, where $\varnothing \in D$;

2) $Q_2 = \{\varnothing, T'\}$, where $\varnothing \neq T' \in D$;

3) $Q_3 = \{\varnothing, T', T''\}$, where $\varnothing \neq T' \subset T'' \in D$;

4) $Q_4 = \{\varnothing, T', T'', T'''\}$, where $\varnothing \neq T' \subset T'' \subset T''' \in D$;

5) $Q_5 = \{\varnothing, T, T', T'', \breve{D}\}$, where $\varnothing \neq T \subset T' \subset T'' \subset \breve{D} \in D$;

6) $Q_6 = \{\varnothing, T', T'', T' \cup T''\}$, where $T', T'' \in D$, $\varnothing \neq T'$, $\varnothing \neq T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$;

7) $Q_7 = \{\varnothing, T', T'', T''', T'' \cup T'''\}$, where, $\varnothing \neq T' \subset T''$, $\varnothing \neq T' \subset T'''$, $T'' \setminus T''' \neq \varnothing$, $T''' \setminus T'' \neq \varnothing$;

8) $Q_8 = \{\varnothing, T, Z_4, Z_2, Z_1, \breve{D}\}$, where $T \in \{Z_6, Z_5\}$;

9) $Q_9 = \{\varnothing, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$;

10) $Q_{10} = \{\varnothing, T', T'', T' \cup T'', T'''\}$, where $\varnothing \neq T'$, $\varnothing \neq T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$, $T' \cup T'' \subset T'''$;

11) $Q_{11} = \{\varnothing, Z_6, Z_5, Z_4, T, \breve{D}\}$, where $T \in \{Z_2, Z_1\}$;

12) $Q_{12} = \{\varnothing, T', T'', T' \cup T'', T''', T' \cup T'' \cup T'''\}$, where, $\varnothing \neq T'$, $\varnothing \neq T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$, $T'' \subset T'''$, $(T' \cup T'') \setminus T''' \neq \varnothing$, $T''' \setminus (T' \cup T'') \neq \varnothing$;

13) $Q_{13} = \{\varnothing, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$;

14) $Q_{14} = \{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$;

15) $Q_{15} = \{\varnothing, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$;

16) $Q_{16} = \{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$.

Theorem 2.1. Let $D \in \Sigma_3(X, 8)$, $Z_7 = \varnothing$ and $\alpha \in B_X(D)$. Binary relation α is an idempotent relation of the semigroup $B_X(D)$ iff binary relation α satisfies only one conditions of the following conditions:

1) $\alpha = \varnothing$;

2) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_{T'}^\alpha \times T')$, where $\varnothing \neq T' \in D$, $Y_{T'}^\alpha \notin \{\varnothing\}$, and satisfies the conditions: $Y_7^\alpha \supseteq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$;

3) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'')$, where $\varnothing \neq T' \subset T'' \in \breve{D}$, $Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$, and satisfies the conditions: $Y_7^\alpha \supseteq \varnothing$, $Y_7^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$;

4) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T'''}^\alpha \times T''')$, where $\varnothing \neq T' \subset T'' \subset T''' \in D$, $Y_{T'}^\alpha, Y_{T''}^\alpha, Y_{T'''}^\alpha \notin \{\varnothing\}$, and satisfies the conditions: $Y_7^\alpha \supseteq \varnothing$, $Y_7^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_7^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq T''$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$, $Y_{T'''}^\alpha \cap T''' \neq \varnothing$;

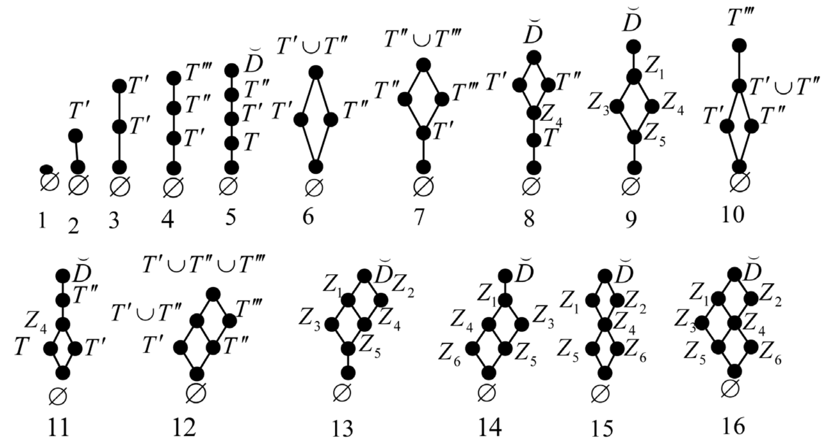

Figure 1. All Diagrams *XI*-subsemilattices of the semilattice *D*.

5) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T''}^\alpha\times T'')\cup(Y_0^\alpha\times\breve{D})$, where $Z_7\neq T\subset T'\subset T''\subset\breve{D}$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha,Y_0^\alpha\notin\{\varnothing\}$, and satisfies the conditions: $Y_7^\alpha\supseteq\varnothing$, $Y_7^\alpha\cup Y_T^\alpha\supseteq T$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_T^\alpha\cap T\neq\varnothing$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

6) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T''}^\alpha\times T'')\cup(Y_{T'\cup T''}^\alpha\times(T'\cup T''))$, where $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_7^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$;

7) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T''}^\alpha\times T'')\cup(Y_{T'''}^\alpha\times T''')\cup(Y_{T''\cup T'''}^\alpha\times(T''\cup T'''))$, where, $\varnothing\neq T'\subset T''$, $\varnothing\neq T'\subset T'''$, $T''\setminus T'''\neq\varnothing$, $T'''\setminus T''\neq\varnothing$, $Y_{T'}^\alpha,Y_{T''}^\alpha,Y_{T'''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq\varnothing$ $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T'''}^\alpha\supseteq T'''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_{T'''}^\alpha\cap T'''\neq\varnothing$;

8) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_T^\alpha\times T)\cup(Y_4^\alpha\times Z_4)\cup(Y_2^\alpha\times Z_2)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, where $T\in\{Z_6,Z_5\}$, $Y_T^\alpha,Y_4^\alpha,Y_2^\alpha,Y_1^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq\varnothing$, $Y_7^\alpha\cup Y_T^\alpha\supseteq T$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq Z_2$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_4^\alpha\cup Y_1^\alpha\supseteq Z_1$, $Y_T^\alpha\cap T\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$, $Y_2^\alpha\cap Z_2\neq\varnothing$, $Y_1^\alpha\cap Z_1\neq\varnothing$;

9) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_5^\alpha\times Z_5)\cup(Y_4^\alpha\times Z_4)\cup(Y_3^\alpha\times Z_3)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, where $Z_5\subset Z_3$, $Z_5\subset Z_4$, $Z_3\setminus Z_4\neq\varnothing$, $Z_4\setminus Z_3\neq\varnothing$, $Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq\varnothing$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_3^\alpha\cap Z_3\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

10) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T''}^\alpha\times T'')\cup(Y_{T'\cup T''}^\alpha\times(T'\cup T''))\cup(Y_{T'''}^\alpha\times T''')$, where, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $T'\cup T''\subset T'''$, $Y_{T'}^\alpha,Y_{T''}^\alpha,Y_{T'''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_7^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_{T'''}^\alpha\cap T'''\neq\varnothing$;

11) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_6^\alpha\times Z_6)\cup(Y_5^\alpha\times Z_5)\cup(Y_4^\alpha\times Z_4)\cup(Y_T^\alpha\times T)\cup(Y_0^\alpha\times\breve{D})$, where $T\in\{Z_2,Z_1\}$, $Y_6^\alpha,Y_5^\alpha,Y_T^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions:, $Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$ $Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_T^\alpha\supseteq T$, $Y_6^\alpha\cap Z_6\neq\varnothing$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_T^\alpha\cap T\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

12) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T''}^\alpha\times T'')\cup(Y_{T'\cup T''}^\alpha\times(T'\cup T''))\cup(Y_{T'''}^\alpha\times T''')\cup(Y_{T'\cup T''\cup T'''}^\alpha\times(T'\cup T''\cup T'''))$, where $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $T''\subset T'''$, $(T'\cup T'')\setminus T'''\neq\varnothing$, $T'''\setminus(T'\cup T'')\neq\varnothing$, $Y_{T'}^\alpha,Y_{T''}^\alpha,Y_{T'''}^\alpha,Y_{T'\cup T''\cup T'''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions:, $Y_7^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_7^\alpha\cup Y_{T''}^\alpha\supseteq T''$ $Y_7^\alpha\cup Y_{T''}^\alpha\cup Y_{T'''}^\alpha\supseteq T'''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_{T'''}^\alpha\cap T'''\neq\varnothing$;

13) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_5^\alpha\times Z_5)\cup(Y_4^\alpha\times Z_4)\cup(Y_3^\alpha\times Z_3)\cup(Y_2^\alpha\times Z_2)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, where $Z_5\subset Z_3$, $Z_5\subset Z_4$, $Z_3\setminus Z_4\neq\varnothing$, $Z_4\setminus Z_3\neq\varnothing$, $Z_4\subset Z_2$, $Z_1\setminus Z_2\neq\varnothing$, $Z_2\setminus Z_1\neq\varnothing$, $Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_2^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_1^\alpha\supseteq Z_1$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_3^\alpha\cap Z_3\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$, $Y_1^\alpha\cap Z_1\neq\varnothing$;

14) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_6^\alpha\times Z_6)\cup(Y_5^\alpha\times Z_5)\cup(Y_4^\alpha\times Z_4)\cup(Y_3^\alpha\times Z_3)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, where, $Z_6\subset Z_4$,

$Y_6^\alpha, Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_7^\alpha \cup Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq Z_3$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_3^\alpha \cap Z_3 \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

15) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where $Y_6^\alpha, Y_5^\alpha, Y_4^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \cup Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq Z_2$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_1^\alpha \supseteq Z_1$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_2^\alpha \cap Z_2 \neq \varnothing$, $Y_1^\alpha \cap Z_1 \neq \varnothing$;

16) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where, $Y_6^\alpha, Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varnothing$, $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_7^\alpha \cup Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq Z_3$, $Y_7^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq Z_2$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_3^\alpha \cap Z_3 \neq \varnothing$, $Y_2^\alpha \cap Z_2 \neq \varnothing$.

Proof. This Theorem immediately follows from the ([3], Theorem 2.1]).

Theorem is proved.

Lemma 2.2. If X be a finite set, then the following equalities are true:

a) $|\mathrm{I}(Q_1)| = 1$;

b) $|\mathrm{I}(Q_2)| = (2^{|T'|} - 1) \cdot 2^{|X \setminus T'|}$;

c) $|\mathrm{I}(Q_3)| = (2^{|T'|} - 1) \cdot (3^{|T'' \setminus T'|} - 2^{|T'' \setminus T'|}) \cdot 3^{|X \setminus T''|}$;

d) $|\mathrm{I}(Q_4)| = (2^{|T'|} - 1) \cdot (3^{|T'' \setminus T'|} - 2^{|T'' \setminus T'|}) \cdot (4^{|T''' \setminus T''|} - 3^{|T''' \setminus T''|}) \cdot 4^{|X \setminus T'''|}$;

e) $|\mathrm{I}(Q_5)| = (2^{|T|} - 1) \cdot (3^{|T' \setminus T|} - 2^{|T' \setminus T|}) \cdot (4^{|T'' \setminus T'|} - 3^{|T'' \setminus T'|}) \cdot (5^{|\breve{D} \setminus T''|} - 4^{|\breve{D} \setminus T''|}) \cdot 5^{|X \setminus \breve{D}|}$;

f) $|\mathrm{I}(Q_6)| = (2^{|T' \setminus T''|} - 1) \cdot (2^{|T'' \setminus T'|} - 1) \cdot 4^{|X \setminus (T' \cup T'')|}$;

g) $|\mathrm{I}(Q_7)| = (2^{|T'|} - 1) \cdot 2^{|(T'' \cap T''') \setminus T'|} \cdot (3^{|T'' \setminus T'''|} - 2^{|T'' \setminus T'''|}) \cdot (3^{|T''' \setminus T''|} - 2^{|T''' \setminus T''|}) \cdot 5^{|X \setminus (T'' \cup T''')|}$;

h) $|\mathrm{I}(Q_8)| = (2^{|T|} - 1) \cdot (3^{|Z_4 \setminus T|} - 2^{|Z_4 \setminus T|}) \cdot 3^{|(Z_2 \cap Z_1) \setminus Z_4|} \cdot (4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 6^{|X \setminus \breve{D}|}$;

i) $|\mathrm{I}(Q_9)| = (2^{|Z_5|} - 1) \cdot 2^{|(Z_3 \cap Z_4) \setminus Z_5|} \cdot (3^{|Z_3 \setminus Z_4|} - 2^{|Z_3 \setminus Z_4|}) \cdot (3^{|Z_4 \setminus Z_3|} - 2^{|Z_4 \setminus Z_3|}) \cdot (6^{|\breve{D} \setminus (Z_3 \cup Z_4)|} - 5^{|\breve{D} \setminus (Z_3 \cup Z_4)|}) \cdot 6^{|X \setminus \breve{D}|}$;

j) $|\mathrm{I}(Q_{10})| = (2^{|T' \setminus T''|} - 1) \cdot (2^{|T'' \setminus T'|} - 1) \cdot (5^{|T''' \setminus (T' \cup T'')|} - 4^{|T''' \setminus (T' \cup T'')|}) \cdot 5^{|X \setminus T'''|}$;

k) $|\mathrm{I}(Q_{11})| = (2^{|T \setminus T'|} - 1) \cdot (2^{|T' \setminus T|} - 1) \cdot (5^{|T'' \setminus Z_4|} - 4^{|T'' \setminus Z_4|}) \cdot \left(6^{|\breve{D} \setminus T''|} - 5^{|\breve{D} \setminus T|''}\right) \cdot 6^{|X \setminus \breve{D}|}$;

l) $|\mathrm{I}(Q_{12})| = (2^{|T' \setminus T'''|} - 1) \cdot (2^{|T'' \setminus T|} - 1) \cdot (3^{|T''' \setminus (T' \cup T'')|} - 2^{|T''' \setminus (T' \cup T'')|}) \cdot 6^{|X \setminus (T' \cup T'' \cup T''')|}$;

m) $|\mathrm{I}(Q_{13})| = (2^{|Z_5|} - 1) \cdot 2^{|(Z_3 \cap Z_2) \setminus Z_5|} \cdot (3^{|Z_3 \setminus Z_4|} - 2^{|Z_3 \setminus Z_4|}) \cdot (3^{|Z_4 \setminus Z_2|} - 2^{|Z_4 \setminus Z_2|}) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 7^{|X \setminus \breve{D}|}$;

n) $|\mathrm{I}(Q_{14})| = (2^{|Z_5 \setminus Z_4|} - 1) \cdot (2^{|Z_6 \setminus Z_5|} - 1) \cdot (3^{|Z_4 \setminus Z_3|} - 2^{|Z_4 \setminus Z_3|}) \cdot (7^{|\breve{D} \setminus Z_1|} - 6^{|\breve{D} \setminus Z_1|}) \cdot 7^{|X \setminus \breve{D}|}$;

o) $|\mathrm{I}(Q_{15})| = (2^{|Z_5 \setminus Z_6|} - 1) \cdot (2^{|Z_6 \setminus Z_5|} - 1) \cdot 4^{|(Z_2 \cap Z_1) \setminus Z_4|} \cdot (5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}) \cdot (5^{|Z_1 \setminus Z_2|} - 4^{|Z_1 \setminus Z_2|}) \cdot 7^{|X \setminus \breve{D}|}$;

p) $|\mathrm{I}(Q_{16})| = (2^{|Z_6 \setminus Z_3|} - 1) \cdot 2^{|Z_5 \setminus Z_4|} \cdot (2^{|Z_5 \setminus Z_6|} - 1) \cdot (3^{|Z_3 \setminus Z_2|} - 2^{|Z_3 \setminus Z_2|}) \cdot (5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}) \cdot 8^{|X \setminus \breve{D}|}$.

Proof. This lemma immediately follows from the ([3], lemma 2.6).

Lemma is proved.

Lemma 2.3. Let $D \in \Sigma_3(X, 8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_1)|$ may be calculated by the formula $|I^*(Q_1)| = 1$.

Proof. By definition of the given semilattice D we have

$$Q_1 \vartheta_{XI} = \{\varnothing\}.$$

If the following equalities are hold

$$D_1' = \{\varnothing\},$$

then

$$\left|I^*(Q_1)\right| = \left|I(D_1')\right| = 1.$$

[See Theorem 1.1] Of this equality we have: $\left|I^*(Q_1)\right| = 1$.

[See statement a) of the Lemma 2.2.]

Lemma 2.4. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $\left|I^*(Q_2)\right|$ may be calculated by the formula

$$\left|I^*(Q_2)\right| = \left(2^{|\breve{D}|}-1\right)\cdot 2^{|X\setminus\breve{D}|} + \left(2^{|Z_6|}-1\right)\cdot 2^{|X\setminus Z_6|} + \left(2^{|Z_5|}-1\right)\cdot 2^{|X\setminus Z_5|} + \left(2^{|Z_4|}-1\right)\cdot 2^{|X\setminus Z_4|} + \left(2^{|Z_3|}-1\right)\cdot 2^{|X\setminus Z_3|} + \left(2^{|Z_2|}-1\right)\cdot 2^{|X\setminus Z_2|} + \left(2^{|Z_1|}-1\right)\cdot 2^{|X\setminus Z_1|}$$

Proof. By definition of the given semilattice D we have

$$Q_2\vartheta_{XI} = \left\{\{Z_7,\breve{D}\},\{Z_7,Z_6\},\{Z_7,Z_5\},\{Z_7,Z_4\},\{Z_7,Z_3\},\{Z_7,Z_2\},\{Z_7,Z_1\}\right\}$$

if

$$D_1' = \{Z_7,\breve{D}\},\ D_2' = \{Z_7,Z_6\},\ D_3' = \{Z_7,Z_5\},\ D_4' = \{Z_7,Z_4\},\ D_5' = \{Z_7,Z_3\},\ D_6' = \{Z_7,Z_2\}, D_7' = \{Z_7,Z_1\}.$$

Then

$$\left|I^*(Q_2)\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right| + \left|I(D_4')\right| + \left|I(D_5')\right| + \left|I(D_6')\right| + \left|I(D_7')\right|.$$

[See Theorem 1.1] Of this equality we have:

$$\left|I^*(Q_2)\right| = \left(2^{|\breve{D}\setminus Z_7|}-1\right)\cdot 2^{|X\setminus\breve{D}|} + \left(2^{|Z_6|}-1\right)\cdot 2^{|X\setminus Z_6|} + \left(2^{|Z_5|}-1\right)\cdot 2^{|X\setminus Z_5|} + \left(2^{|Z_4|}-1\right)\cdot 2^{|X\setminus Z_4|} + \left(2^{|Z_3|}-1\right)\cdot 2^{|X\setminus Z_3|} + \left(2^{|Z_2|}-1\right)\cdot 2^{|X\setminus Z_2|} + \left(2^{|Z_1|}-1\right)\cdot 2^{|X\setminus Z_1|}$$

[See statement b) of the Lemma 2.2.]

Lemma is proved.

Lemma 2.5. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $\left|I^*(Q_3)\right|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_3)\right| = {} & \left(2^{|Z_1|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|} + \left(2^{|Z_2|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_3|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|}\right)\cdot 3^{|X\setminus\breve{D}|} + \left(2^{|Z_4|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_5|}-2^{|\breve{D}\setminus Z_5|}\right)\cdot 3^{|X\setminus\breve{D}|} + \left(2^{|Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_6|}-2^{|\breve{D}\setminus Z_6|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_4|} + \left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_2|}\\
&+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_1|} + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_4|}\\
&+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_3|} + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_2|}\\
&+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_1|} + \left(2^{|Z_4|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\cdot 3^{|X\setminus Z_2|}\\
&+\left(2^{|Z_4|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot 3^{|X\setminus Z_1|} + \left(2^{|Z_3|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 3^{|X\setminus Z_1|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$Q_3\vartheta_{XI} = \{\{\varnothing, Z_1, \breve{D}\}, \{\varnothing, Z_2, \breve{D}\}, \{\varnothing, Z_3, \breve{D}\}, \{\varnothing, Z_4, D\}, \{\varnothing, Z_5, \breve{D}\}, \{\varnothing, Z_6, \breve{D}\}, \{\varnothing, Z_6, Z_4\}, \{\varnothing, Z_6, Z_2\}, \{\varnothing, Z_6, Z_1\}, \{\varnothing, Z_5, Z_4\}, \{\varnothing, Z_5, Z_3\}, \{\varnothing, Z_5, Z_2\}, \{\varnothing, Z_5, Z_1\}, \{\varnothing, Z_4, Z_2\}, \{\varnothing, Z_4, Z_1\}, \{\varnothing, Z_3, Z_1\}\}$$

If

$$D_1' = \{\varnothing, Z_1, \breve{D}\},\quad D_2' = \{\varnothing, Z_2, \breve{D}\},\quad D_3' = \{\varnothing, Z_3, \breve{D}\},\quad D_4' = \{\varnothing, Z_4, D\},\quad D_5' = \{\varnothing, Z_5, \breve{D}\},$$
$$D_6' = \{\varnothing, Z_6, \breve{D}\},\quad D_7' = \{\varnothing, Z_6, Z_4\},\quad D_8' = \{\varnothing, Z_6, Z_2\},\quad D_9' = \{\varnothing, Z_6, Z_1\},\quad D_{10}' = \{\varnothing, Z_5, Z_4\},$$
$$D_{11}' = \{\varnothing, Z_5, Z_3\},\ D_{12}' = \{\varnothing, Z_5, Z_2\},\ D_{13}' = \{\varnothing, Z_5, Z_1\},\ D_{14}' = \{\varnothing, Z_4, Z_2\},\ D_{15}' = \{\varnothing, Z_4, Z_1\},$$
$$D_{16}' = \{\varnothing, Z_3, Z_1\}$$

Then

$$\left|I^*(Q_3)\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right| + \left|I(D_4')\right| + \left|I(D_5')\right| + \left|I(D_6')\right| + \left|I(D_7')\right| + \left|I(D_8')\right| + \left|I(D_9')\right| + \left|I(D_{10}')\right| + \left|I(D_{11}')\right| + \left|I(D_{12}')\right| + \left|I(D_{13}')\right| + \left|I(D_{14}')\right| + \left|I(D_{15}')\right| + \left|I(D_{16}')\right|$$

[See Theorem 1.1]. Of this equality we have:

$$\begin{aligned}\left|I^*(Q_3)\right| = {} & \left(2^{|Z_1|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|} + \left(2^{|Z_2|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
& + \left(2^{|Z_3|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|}\right)\cdot 3^{|X\setminus\breve{D}|} + \left(2^{|Z_4|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
& + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_5|}-2^{|\breve{D}\setminus Z_5|}\right)\cdot 3^{|X\setminus\breve{D}|} + \left(2^{|Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_6|}-2^{|\breve{D}\setminus Z_6|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
& + \left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_4|} + \left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_2|}\\
& + \left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_1|} + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_4|}\\
& + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_3|} + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_2|}\\
& + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_1|} + \left(2^{|Z_4|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\cdot 3^{|X\setminus Z_2|}\\
& + \left(2^{|Z_4|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot 3^{|X\setminus Z_1|} + \left(2^{|Z_3|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 3^{|X\setminus Z_1|}\end{aligned}$$

[See statement c) of the Lemma 2.2.]

Lemma is proved.

Lemma 2.6. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $\left|I^*(Q_4)\right|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_4)\right| = {} & \left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
& + \left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
& + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
& + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_4|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
& + \left(2^{|Z_4|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_3|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
& + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_2|} + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\times\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\\
& \times 4^{|X\setminus Z_1|} + \left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_3|}-3^{|Z_1\setminus Z_3|}\right)\cdot 4^{|X\setminus Z_1|} + \left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\\
& \times\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\times 4^{|X\setminus Z_2|} + \left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_1|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$Q_4\vartheta_{XI}=\Big\{\{\varnothing,Z_6,Z_4,\breve{D}\},\ \varnothing\{Z_7,Z_6,Z_2,D\},\ \{\varnothing,Z_6,Z_1,\breve{D}\},\ \{\varnothing,Z_5,Z_4,\breve{D}\},\ \{\varnothing,Z_5,Z_3,\breve{D}\},$$
$$\{\varnothing,Z_5,Z_2,\breve{D}\},\ \{\varnothing,Z_5,Z_1,\breve{D}\},\ \{\varnothing,Z_4,Z_2,D\},\ \{\varnothing,Z_4,Z_1,\breve{D}\},\ \{\varnothing,Z_3,Z_1,\breve{D}\}$$
$$\{\varnothing,Z_5,Z_4,Z_2\},\ \{\varnothing,Z_5,Z_4,Z_1\},\ \{\varnothing,Z_5,Z_3,Z_1\},\ \{\varnothing,Z_6,Z_4,Z_2\},\ \{\varnothing,Z_6,Z_4,Z_1\}.$$

If

$$D_1'=\{\varnothing,Z_6,Z_4,\breve{D}\},D_2'=\{\varnothing,Z_6,Z_2,D\},D_3'=\{\varnothing,Z_6,Z_1,\breve{D}\},D_4'=\{\varnothing,Z_5,Z_4,\breve{D}\},$$
$$D_5'=\{\varnothing,Z_5,Z_3,\breve{D}\},D_6'=\{\varnothing,Z_5,Z_2,\breve{D}\},D_7'=\{\varnothing,Z_5,Z_1,\breve{D}\},D_8'=\{\varnothing,Z_4,Z_2,D\},$$
$$D_9'=\{\varnothing,Z_4,Z_1,\breve{D}\},D_{10}'=\{\varnothing,Z_3,Z_1,\breve{D}\},D_{11}'=\{\varnothing,Z_6,Z_4,Z_2\},D_{12}'=\{\varnothing,Z_6,Z_4,Z_1\}$$
$$D_{13}'=\{\varnothing,Z_5,Z_4,Z_2\},D_{14}'=\{\varnothing,Z_5,Z_4,Z_1\},D_{15}'=\{\varnothing,Z_5,Z_3,Z_1\}.$$

Then

$$\left|I^*(Q_4)\right|=\left|I(D_1')\right|+\left|I(D_2')\right|+\left|I(D_3')\right|+\left|I(D_4')\right|+\left|I(D_5')\right|+\left|I(D_6')\right|+\left|I(D_7')\right|+\left|I(D_8')\right|$$
$$+\left|I(D_9')\right|+\left|I(D_{10}')\right|+\left|I(D_{11}')\right|+\left|I(D_{12}')\right|+\left|I(D_{13}')\right|+\left|I(D_{14}')\right|+\left|I(D_{15}')\right|$$

[See Theorem 1.1] Of this equality we have:

$$\left|I^*(Q_4)\right|=\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_4|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_4|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_3|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_2|}+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\times\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)$$
$$\times 4^{|X\setminus Z_1|}+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_3|}-3^{|Z_1\setminus Z_3|}\right)\cdot 4^{|X\setminus Z_1|}+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)$$
$$\times\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\times 4^{|X\setminus Z_2|}+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_1|}$$

[See statement d) of the Lemma 2.2.]

Lemma is proved.

Lemma 2.7. Let $D\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. If X is a finite set, then the number $\left|I^*(Q_5)\right|$ may be calculated by the formula

$$\left|I^*(Q_5)\right|=\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_3|}-3^{|Z_1\setminus Z_3|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}$$

Proof. By definition of the given semilattice D we have

$$Q_5\vartheta_{XI}=\Big\{\{\varnothing,Z_6,Z_4,Z_2,\breve{D}\},\{\varnothing,Z_6,Z_4,Z_1,\breve{D}\},\{\varnothing,Z_5,Z_4,Z_2,\breve{D}\},$$
$$\{\varnothing,Z_5,Z_4,Z_1,\breve{D}\},\{\varnothing,Z_5,Z_3,Z_1,\breve{D}\}\Big\}.$$

If

$$D_1' = \{\varnothing, Z_6, Z_4, Z_2, \breve{D}\},\ D_2' = \{\varnothing, Z_6, Z_4, Z_1, \breve{D}\},\ D_3' = \{\varnothing, Z_5, Z_4, Z_2, \breve{D}\},$$
$$D_4' = \{\varnothing, Z_5, Z_4, Z_1, \breve{D}\},\ D_5' = \{\varnothing, Z_5, Z_3, Z_1, \breve{D}\}.$$

Then

$$|I^*(Q_5)| = |I(D_1')| + |I(D_2')| + |I(D_3')| + |I(D_4')| + |I(D_5')|$$

[See Theorem 1.1] Of this equality we have:

$$\begin{aligned}|I^*(Q_5)| &= \left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_3|}-3^{|Z_1\setminus Z_3|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\end{aligned}$$

[See statement e) of the Lemma 2.2.]

Lemma is proved.

Lemma 2.8. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_6)|$ may be calculated by the formula

$$\begin{aligned}|I^*(Q_6)| &= \left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot 4^{|X\setminus Z_4|}+\left(2^{|Z_3\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|}+\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|}\\
&+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$Q_6\vartheta_{XI} = \left\{\{\varnothing, Z_2, Z_1, \breve{D}\}, \{\varnothing, Z_6, Z_5, Z_4\}, \{\varnothing, Z_6, Z_3, Z_1\}, \{\varnothing, Z_4, Z_3, Z_1\}, \varnothing\{Z_7, Z_3, Z_2, \breve{D}\}\right\}$$

$$D_1' = \{\varnothing, Z_2, Z_1, \breve{D}\},\ D_2' = \{\varnothing, Z_6, Z_5, Z_4\},\ D_3' = \{\varnothing, Z_6, Z_3, Z_1\},\ D_4' = \{\varnothing, Z_4, Z_3, Z_1\},\ D_5' = \{\varnothing, Z_3, Z_2, \breve{D}\}$$

$$|I^*(Q_6)| = |I(D_1')| + |I(D_2')| + |I(D_3')| + |I(D_4')| + |I(D_5')|$$

[See Theorem 1.1] Of this equality we have:

$$\begin{aligned}|I^*(Q_6)| &= \left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot 4^{|X\setminus Z_4|}+\left(2^{|Z_3\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|}+\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|}\\
&+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\end{aligned}$$

[See statement f) of the Lemma 2.2.]

Lemma is proved.

Lemma 2.9. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_7)|$ may be calculated by the formula

$$\begin{aligned}|I^*(Q_7)| &= \left(2^{|Z_6|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_6|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_5|}-1\right)\cdot 2^{|(Z_3\cap Z_4)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot 5^{|X\setminus Z_1|}\\
&+\left(2^{|Z_5|}-1\right)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_5|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_5|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_4|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$Q_7\vartheta_{XI} = \left\{\left\{\varnothing, Z_4, Z_2, Z_1, \breve{D}\right\}, \left\{\varnothing, Z_6, Z_2, Z_1, \breve{D}\right\}, \left\{\varnothing, Z_5, Z_2, Z_1, \breve{D}\right\}, \left\{\varnothing, Z_5, Z_3, Z_2, \breve{D}\right\}, \left\{\varnothing, Z_5, Z_4, Z_3, Z_1\right\}\right\}$$

If

$$D_1' = \left\{\varnothing, Z_4, Z_2, Z_1, \breve{D}\right\},\ D_2' = \left\{\varnothing, Z_6, Z_2, Z_1, \breve{D}\right\},\ D_3' = \left\{\varnothing, Z_5, Z_2, Z_1, \breve{D}\right\},$$
$$D_4' = \left\{\varnothing, Z_5, Z_3, Z_2, \breve{D}\right\},\ D_5' = \left\{\varnothing, Z_5, Z_4, Z_3, Z_1\right\}$$

$$\left|I^*(Q_7)\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right| + \left|I(D_4')\right| + \left|I(D_5')\right|$$

[See Theorem 1.1] Of this equality we have:

$$\begin{aligned}\left|I^*(Q_7)\right| &= \left(2^{|Z_6|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_6|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_5|}-1\right)\cdot 2^{|(Z_3\cap Z_4)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot 5^{|X\setminus Z_1|}\\ &+\left(2^{|Z_5|}-1\right)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_5|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_5|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_4|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\end{aligned}$$

[See statement g) of the Lemma 2.2.]

Lemma is proved.

Lemma 2.10. Let $D\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. If X is a finite set, then the number $\left|I^*(Q_8)\right|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_8)\right| &= \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot 3^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot 3^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$Q_8\vartheta_{XI} = \left\{\left\{\varnothing, Z_6, Z_4, Z_2, Z_1, \breve{D}\right\}, \left\{\varnothing, Z_5, Z_4, Z_2, Z_1, \breve{D}\right\}\right\}$$

If

$$D_1' = \left\{\varnothing, Z_6, Z_4, Z_2, Z_1, \breve{D}\right\},\ D_2' = \left\{\varnothing, Z_5, Z_4, Z_2, Z_1, \breve{D}\right\}$$

$$\left|I^*(Q_8)\right| = \left|I(D_1')\right| + \left|I(D_2')\right|$$

[See Theorem 1.1] Of this equality we have:

$$\begin{aligned}\left|I^*(Q_8)\right| &= \left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot 3^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot 3^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}\end{aligned}$$

[See statement h) of the Lemma 2.2.]

Lemma is proved.

Lemma 2.11. Let $D\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. If X is a finite set, then the number $\left|I^*(Q_9)\right|$ may be calculated by the formula

$$\left|I^*(Q_9)\right| = \left(2^{|Z_5|}-1\right)\cdot 2^{|(Z_3\cap Z_4)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot\left(6^{|\breve{D}\setminus Z_1|}-5^{|\breve{D}\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|};$$

Proof. By definition of the given semilattice D we have $Q_9\vartheta_{XI} = \left\{\left\{\varnothing, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\right\}\right\}$.

If the following equality is hold $D_1' = \{\varnothing, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$ then $|I^*(Q_9)| = |I(D_1')|$.
[See Theorem 1.1] Of this equality we have:

$$\left|I^*(Q_9)\right| = \left(2^{|Z_5|}-1\right)\cdot 2^{|(Z_3\cap Z_4)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot\left(6^{|\breve{D}\setminus Z_1|}-5^{|\breve{D}\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|};$$

[See statement i) of the Lemma 2.2.]
Lemma is proved.

Lemma 2.12. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_{10})|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_{10})\right| &= \left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_4|}-4^{|\breve{D}\setminus Z_4|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|Z_2\setminus Z_4|}-4^{|Z_2\setminus Z_4|}\right)\cdot 5^{|X\setminus Z_2|}\\ &+\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|Z_1\setminus Z_4|}-4^{|Z_1\setminus Z_4|}\right)\cdot 5^{|X\setminus Z_1|}\\ &+\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$\begin{aligned}Q_{10}\vartheta_{XI} = \{&\{\varnothing, Z_6, Z_5, Z_4, \breve{D}\},\ \{\varnothing, Z_6, Z_3, Z_1, \breve{D}\}, \{\varnothing, Z_4, Z_3, Z_1, \breve{D}\},\\ &\{\varnothing, Z_6, Z_5, Z_4, Z_2\},\ \{\varnothing, Z_6, Z_5, Z_4, Z_1\}\end{aligned}$$

If

$$\begin{aligned}&D_1' = \{\varnothing, Z_6, Z_5, Z_4, \breve{D}\},\ D_2' = \{\varnothing, Z_6, Z_3, Z_1, \breve{D}\}, D_3' = \{\varnothing, Z_4, Z_3, Z_1, \breve{D}\},\\ &D_4' = \{\varnothing, Z_6, Z_5, Z_4, Z_2\},\ D_5' = \{\varnothing, Z_6, Z_5, Z_4, Z_1\}\end{aligned}$$

$$\left|I^*(Q_{10})\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right| + \left|I(D_4')\right| + \left|I(D_5')\right|$$

[See Theorem 1.1] Of this equality we have:

$$\begin{aligned}\left|I^*(Q_{10})\right| &= \left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_4|}-4^{|\breve{D}\setminus Z_4|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|Z_2\setminus Z_4|}-4^{|Z_2\setminus Z_4|}\right)\cdot 5^{|X\setminus Z_2|}\\ &+\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|Z_1\setminus Z_4|}-4^{|Z_1\setminus Z_4|}\right)\cdot 5^{|X\setminus Z_1|}\end{aligned}$$

[See statement j) of the Lemma 2.2.]
Lemma is proved.

Lemma 2.13. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_{11})|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_{11})\right| &= \left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|Z_2\setminus Z_4|}-4^{|Z_2\setminus Z_4|}\right)\cdot\left(6^{|\breve{D}\setminus Z_2|}-5^{|\breve{D}\setminus Z_2|}\right)\cdot 6^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|Z_1\setminus Z_4|}-4^{|Z_1\setminus Z_4|}\right)\cdot\left(6^{|\breve{D}\setminus Z_1|}-5^{|\breve{D}\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$Q_{11}\vartheta_{XI} = \left\{\left\{\varnothing, Z_6, Z_5, Z_4, Z_2, \breve{D}\right\}, \left\{\varnothing, Z_6, Z_5, Z_4, Z_1, \breve{D}\right\}\right\}$$

If

$$D_1' = \left\{\varnothing, Z_6, Z_5, Z_4, Z_2, \breve{D}\right\}, D_2' = \left\{\varnothing, Z_6, Z_5, Z_4, Z_1, \breve{D}\right\},$$

$$\left|I^*(Q_{11})\right| = \left|I(D_1')\right| + \left|I(D_2')\right|$$

[See Theorem 1.1] Of this equality we have:

$$\begin{aligned}\left|I^*(Q_{11})\right| &= \left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|Z_2\setminus Z_4|}-4^{|Z_2\setminus Z_4|}\right)\cdot\left(6^{|\breve{D}\setminus Z_2|}-5^{|\breve{D}\setminus Z_2|}\right)\cdot 6^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|Z_1\setminus Z_4|}-4^{|Z_1\setminus Z_4|}\right)\cdot\left(6^{|\breve{D}\setminus Z_1|}-5^{|\breve{D}\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\end{aligned}$$

[See statement k) of the Lemma 2.2.]

Lemma is proved.

Lemma 2.14. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $\left|I^*(Q_{12})\right|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_{12})\right| &= \left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot 6^{|X\setminus Z_1|}\\ &+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$Q_{12}\vartheta_{XI} = \left\{\left\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1\right\}, \left\{\varnothing, Z_6, Z_3, Z_2, Z_1, \breve{D}\right\}, \left\{\varnothing, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\},\right.$$

$$D_1' = \left\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1\right\}, D_2' = \left\{\varnothing, Z_6, Z_3, Z_2, Z_1, \breve{D}\right\}, D_3' = \left\{\varnothing, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}$$

$$\left|I^*(Q_{12})\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right|$$

[See Theorem 1.1] Of this equality we have:

$$\begin{aligned}\left|I^*(Q_{12})\right| &= \left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot 6^{|X\setminus Z_1|}\\ &+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\end{aligned}$$

[See statement l) of the Lemma 2.2.]

Lemma is proved.

Lemma 2.15. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $\left|I^*(Q_{13})\right|$ may be calculated by the formula

$$\left|I^*(Q_{13})\right| = \left(2^{|Z_5|}-1\right)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(3^{|Z_4\setminus Z_2|}-2^{|Z_4\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 7^{|X\setminus \breve{D}|}.$$

Proof. By definition of the given semilattice D we have $Q_{13}\vartheta_{XI} = \left\{\left\{\varnothing, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}\right\}$. If the following equality is hold $D_1' = \left\{\varnothing, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}$ then $\left|I^*(Q_{13})\right| = \left|I(D_1')\right|$.

[See Theorem 1.1] Of this equality we have:

$$\left|I^*(Q_{13})\right| = \left(2^{|Z_5\setminus\varnothing|}-1\right)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(3^{|Z_4\setminus Z_2|}-2^{|Z_4\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 7^{|X\setminus \breve{D}|};$$

[See statement m) of the Lemma 2.2.]

Lemma is proved.

Lemma 2.16. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $\left|I^*(Q_{14})\right|$ may be calculated by the formula

$$\left|I^*(Q_{14})\right| = \left(2^{|Z_5 \setminus Z_4|} - 1\right) \cdot \left(2^{|Z_6 \setminus Z_5|} - 1\right) \cdot \left(3^{|Z_4 \setminus Z_3|} - 2^{|Z_4 \setminus Z_3|}\right) \cdot \left(7^{|\breve{D} \setminus Z_1|} - 6^{|\breve{D} \setminus Z_1|}\right) \cdot 7^{|X \setminus \breve{D}|}.$$

Proof. By definition of the given semilattice D we have $Q_{14}\vartheta_{XI} = \left\{\left\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\right\}\right\}$. If the following equality is hold $D_1' = \left\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\right\}$ then $\left|I^*(Q_{14})\right| = \left|I(D_1')\right|$.

[See Theorem 1.1] Of this equality we have:

$$\left|I^*(Q_{14})\right| = \left(2^{|Z_5 \setminus Z_4|} - 1\right) \cdot \left(2^{|Z_6 \setminus Z_5|} - 1\right) \cdot \left(3^{|Z_4 \setminus Z_3|} - 2^{|Z_4 \setminus Z_3|}\right) \cdot \left(7^{|\breve{D} \setminus Z_1|} - 6^{|\breve{D} \setminus Z_1|}\right) \cdot 7^{|X \setminus \breve{D}|};$$

[See statement n) of the Lemma 2.2).]

Lemma is proved.

Lemma 2.17. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $\left|I^*(Q_{15})\right|$ may be calculated by the formula

$$\left|I^*(Q_{15})\right| = \left(2^{|Z_5 \setminus Z_6|} - 1\right) \cdot \left(2^{|Z_6 \setminus Z_5|} - 1\right) \cdot 4^{|(Z_1 \cap Z_2) \setminus Z_4|} \cdot \left(5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}\right) \cdot \left(5^{|Z_1 \setminus Z_2|} - 4^{|Z_1 \setminus Z_2|}\right) \cdot 7^{|X \setminus \breve{D}|}.$$

Proof. By definition of the given semilattice D we have $Q_{15}\vartheta_{XI} = \left\{\left\{\varnothing, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\right\}\right\}$. If the following equality is hold $D_1' = \left\{\varnothing, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\right\}$ then $\left|I^*(Q_{15})\right| = \left|I(D_1')\right|$.

[See Theorem 1.1] Of this equality we have:

$$\left|I^*(Q_{15})\right| = \left(2^{|Z_5 \setminus Z_6|} - 1\right) \cdot \left(2^{|Z_6 \setminus Z_5|} - 1\right) \cdot 4^{|(Z_1 \cap Z_2) \setminus Z_4|} \cdot \left(5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}\right) \cdot \left(5^{|Z_1 \setminus Z_2|} - 4^{|Z_1 \setminus Z_2|}\right) \cdot 7^{|X \setminus \breve{D}|};$$

[See statement o) of the Lemma 2.2).]

Lemma is proved.

Lemma 2.18. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $\left|I^*(Q_{16})\right|$ may be calculated by the formula

$$\left|I^*(Q_{16})\right| = \left(2^{|Z_6 \setminus Z_3|} - 1\right) \cdot 2^{|(Z_3 \cap Z_2) \setminus Z_4|} \cdot \left(2^{|Z_5 \setminus Z_6|} - 1\right) \cdot \left(3^{|Z_3 \setminus Z_2|} - 2^{|Z_3 \setminus Z_2|}\right) \cdot \left(5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}\right) \cdot 8^{|X \setminus \breve{D}|}.$$

Proof. By definition of the given semilattice D we have $Q_{16}\vartheta_{XI} = \left\{\left\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}\right\}$. If the following equality is hold $D_1' = \left\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}$ then $\left|I^*(Q_{16})\right| = \left|I(D_1')\right|$.

[See Theorem 1.1] Of this equality we have:

$$\left|I^*(Q_{16})\right| = \left(2^{|Z_6 \setminus Z_3|} - 1\right) \cdot 2^{|(Z_3 \cap Z_2) \setminus Z_4|} \cdot \left(2^{|Z_5 \setminus Z_6|} - 1\right) \cdot \left(3^{|Z_3 \setminus Z_2|} - 2^{|Z_3 \setminus Z_2|}\right) \cdot \left(5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}\right) \cdot 8^{|X \setminus \breve{D}|}.$$

[See statement p) of the Lemma 2.2).]

Lemma is proved.

Theorem 2.2. Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $\left|I(D)\right|$ may be calculated by the formula

$$\begin{aligned} \left|I(D)\right| = {} & \left|I^*(Q_1)\right| + \left|I^*(Q_2)\right| + \left|I^*(Q_3)\right| + \left|I^*(Q_4)\right| + \left|I^*(Q_5)\right| + \left|I^*(Q_6)\right| + \left|I^*(Q_7)\right| + \left|I^*(Q_8)\right| \\ & + \left|I^*(Q_9)\right| + \left|I^*(Q_{10})\right| + \left|I^*(Q_{11})\right| + \left|I^*(Q_{11})\right| + \left|I^*(Q_{13})\right| + \left|I^*(Q_{14})\right| + \left|I^*(Q_{15})\right| + \left|I^*(Q_{16})\right| \end{aligned}$$

Proof. This Theorem immediately follows from the Theorem 2.1.

Theorem is proved.

Example 2.1. Let $X = \{1,2,3,4\}$, $\breve{D} = \{1,2,3,4\}$, $Z_1 = \{2,3,4\}$, $Z_2 = \{1,3,4\}$, $Z_3 = \{2,4\}$, $Z_4 = \{3,4\}$, $Z_5 = \{4\}$, $Z_6 = \{3\}$, $Z_7 = \{\varnothing\}$, $\left|I(D)\right| = 448$.

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