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# Regular Elements of $B_X(D)$ Defined by the Class $\Sigma_1(X,10)-II$

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## Abstract

**This part of the paper is the continuotion of paper "Regular Elements of $B_X(D)$ Defined by the Class $\Sigma_1(X,10)-I$".**

## Keywords

**Semilattice, Semigroup, Binary Relation**

## 1. Introduction

This work is continuation of the paper "Regular Elements of $B_X(D)$ Defined by the Class $\Sigma_1(X,10)-I$" whose sections are labelled 1-2-3. In the introduction and the second section, some definitions and well known results are stated with references. In Section 3 we give a full description of regular elements of the semigroup $B_X(D)$ when an empty set is not included in *D* and $Z_9 \neq \varnothing$.

In the present work our aim is to identify regular elements of thesemigroup $B_X(D)$ when $\varnothing \in D$ and $Z_9 = \varnothing$.

The method used in this part does not differ from the method given in [1].

## 2. Regular Elements of the Complete Semigroups of Binary Relations of the Class $\Sigma_1(X,10)$, When $\varnothing \in D$ and $Z_9 = \varnothing$

We denoted the following semilattices by symbols:

1) $Q_1 = \{\varnothing\}$, where $\varnothing \in D$ (see diagram 1 of the **Figure 1**);

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2) $Q_2 = \{\varnothing, T\}$, where $T \in D$ (see diagram 2 of the **Figure 1**);

3) $Q_3 = \{\varnothing, T, T'\}$, where $T, T' \in D$ and $\varnothing \subset T \subset T'$ (see diagram 3 of the **Figure 1**);

4) $Q_4 = \{\varnothing, T, T', \breve{D}\}$, where $T, T' \in D$ and $\varnothing \subset T \subset T' \subset \breve{D}$ (see diagram 4 of the **Figure 1**);

5) $Q_5 = \{\varnothing, T, T', T \cup T'\}$ where $T, T' \in D$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, (see diagram 5 of the **Figure 1**);

6) $Q_6 = \{\varnothing, T, T', T \cup T', \breve{D}\}$, where $T, T' \in D$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $T \cup T' \subset \breve{D}$ (see diagram 6 of the **Figure 1**);

7) $Q_7 = \{\varnothing, Z_6, T, T', \breve{D}\}$, where $T, T' \in D$, $Z_6 \subset T$, $Z_6 \subset T'$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $T \cup T' = \breve{D}$ (see diagram 7 of the **Figure 1**);

8) $Q_8 = \{\varnothing, T, T', T \cup T', Z, \breve{D}\}$, where $\varnothing \subset T$, $\varnothing \subset T' \subset Z$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $(T \cup T') \setminus Z \neq \varnothing$, $Z \setminus (T \cup T') \neq \varnothing$, $T \cup T' \cup Z = \breve{D}$ (see diagram 8 of the **Figure 1**);

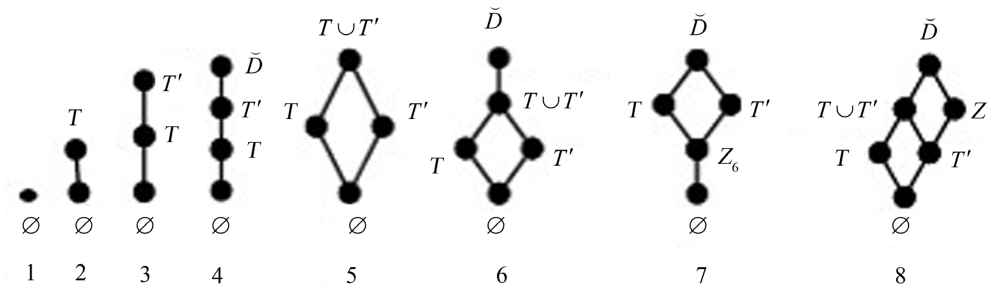


**Figure 1.** Diagram of all XI-subsemilattices of semi lattices of unions *D*.

Note that the semilattices 1)-8), which are given by diagram 1-8 of the **Figure 1** always are *XI*-semilattices (see [2], Lemma 1.2.3).

Remark that

$$Q_1\vartheta_{XI} = \{\varnothing\};$$

$$Q_2\vartheta_{XI} = \{\{\varnothing, \breve{D}\}, \{\varnothing, Z_8\}, \{\varnothing, Z_7\}, \{\varnothing, Z_6\}, \{\varnothing, Z_5\}, \{\varnothing, Z_4\}, \{\varnothing, Z_3\}, \{\varnothing, Z_2\}, \{\varnothing, Z_1\}\};$$

$\ldots$

$$Q_8\vartheta_{XI} = \{\{\varnothing, Z_8, Z_6, Z_3, Z_2, \breve{D}\}, \{\varnothing, Z_8, Z_6, Z_3, Z_1, \breve{D}\}, \{\varnothing, Z_7, Z_6, Z_3, Z_2, \breve{D}\}, \{\varnothing, Z_7, Z_6, Z_3, Z_1, \breve{D}\},$$
$$\{\varnothing, Z_6, Z_5, Z_3, Z_1, \breve{D}\}, \{\varnothing, Z_6, Z_5, Z_2, Z_1, \breve{D}\}, \{\varnothing, Z_6, Z_4, Z_3, Z_1, \breve{D}\}, \{\varnothing, Z_6, Z_4, Z_2, Z_1, \breve{D}\}\}.$$

**Lemma 1.** *Let* $\varphi$ *be an isomorphism between* $Q_i$ *and* $D_i'$ *semilattices,* $T \in Q_i$, $\overline{T} \in D_i'$ *and* $\varphi(T) = \overline{T}$. *If X is a finite set and* $|Q_i\vartheta_{XI}| = m_i$ $(i = 1, 2, \cdots, 8)$ *and* $Z_9 = \varnothing$, *then the following equalities are true*:

1) $|R(Q_1)| = 1$;

2) $|R(Q_2)| = m_2 \cdot \left(2^{|\overline{T}|} - 1\right) \cdot 2^{|X \setminus \overline{T}|}$;

3) $|R(Q_3)| = m_3 \cdot \left(2^{|\overline{T}|} - 1\right) \cdot \left(3^{|\overline{T}' \setminus \overline{T}|} - 2^{|\overline{T}' \setminus \overline{T}|}\right) \cdot 3^{|X \setminus \overline{T}'|}$;

4) $|R(Q_4)| = m_4 \cdot \left(2^{|\overline{T}|} - 1\right) \cdot \left(3^{|\overline{T}' \setminus \overline{T}|} - 2^{|\overline{T}' \setminus \overline{T}|}\right) \cdot \left(4^{|\breve{D} \setminus \overline{T}'|} - 3^{|\breve{D} \setminus \overline{T}'|}\right) \cdot 4^{|X \setminus \breve{D}|}$;

5) $|R(Q_5)| = 2 \cdot m_5 \cdot \left(2^{|\overline{T} \setminus \overline{T}'|} - 1\right) \cdot \left(2^{|\overline{T}' \setminus \overline{T}|} - 1\right) \cdot 4^{|X \setminus (\overline{T} \cup \overline{T}')|}$;

6) $|R(Q_6)| = 2 \cdot m_6 \cdot \left(2^{|\overline{T} \setminus \overline{T}'|} - 1\right) \cdot \left(2^{|\overline{T}' \setminus \overline{T}|} - 1\right) \cdot \left(5^{|\breve{D} \setminus (\overline{T} \cup \overline{T}')|} - 4^{|\breve{D} \setminus (\overline{T} \cup \overline{T}')|}\right) \cdot 5^{|X \setminus \breve{D}|}$;

7) $|R(Q_7)| = 2 \cdot m_7 \left(2^{|Z_6|} - 1\right) 2^{|(\overline{T} \cap \overline{T}') \setminus Z_6|} \left(3^{|\overline{T} \setminus \overline{T}'|} - 2^{|\overline{T} \setminus \overline{T}'|}\right) \left(3^{|\overline{T}' \setminus \overline{T}|} - 2^{|\overline{T}' \setminus \overline{T}|}\right) 5^{|X \setminus \breve{D}|}$;

8) $\left|R(Q_8)\right| = 2\cdot m_8 \cdot \left(2^{\left|\bar{T}\setminus\bar{Z}\right|}-1\right)\cdot\left(2^{\left|\bar{T}'\setminus\bar{T}\right|}-1\right)\cdot\left(3^{\left|\bar{Z}\setminus(\bar{T}\cup\bar{T}')\right|}-2^{\left|\bar{Z}\setminus(\bar{T}\cup\bar{T}')\right|}\right)\cdot 6^{\left|X\setminus\breve{D}\right|}$;

*Proof.* Let $Z_9 = \varnothing$. Then given Lemma immediately follows from ([1], Lemma 3). □

**Theorem 1.** *Let* $D = \left\{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\} \in \Sigma_1(X,10)$ *and* $Z_9 = \varnothing$. *Then a binary relation* $\alpha$ *of the semigroup* $B_X(D)$ *whose quasinormal representation has a form* $\alpha = \bigcup_{T\in V(D,\alpha)}\left(Y_T^\alpha \times T\right)$ *will be a regular element of this semigroup iff there exist a complete* $\alpha$*-isomorphism* $\varphi$ *of the semilattice* $V(D,\alpha)$ *on some subsemilattice* $D'$ *of the semilattice D which satisfies at least one of the following conditions*:

- $\alpha = \varnothing$;
- $\alpha = \left(Y_\varnothing^\alpha \times \varnothing\right) \cup \left(Y_T^\alpha \times T\right)$, for some $T \in D$ and $Y_T^\alpha \neq \varnothing$ which satisfies the condition $Y_T^\alpha \cap \varphi(T) \neq \varnothing$;
- $\alpha = \left(Y_\varnothing^\alpha \times \varnothing\right) \cup \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right)$, for some $T, T' \in D$, $\varnothing \subset T \subset T'$, and $Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$ which satisfies the conditions: $Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq \varphi(T)$, $Y_T^\alpha \cap \varphi(T) \neq \varnothing$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$;
- $\alpha = \left(Y_\varnothing^\alpha \times \varnothing\right) \cup \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$, for some $T, T' \in D$, $\varnothing \subset T \subset T' \subset \breve{D}$ and $Y_T^\alpha, Y_{T'}^\alpha, Y_0^\alpha \notin \{\varnothing\}$ which satisfies the conditions: $Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq \varphi(T)$, $Y_\varnothing^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cap \varphi(T) \neq \varnothing$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$, $Y_0^\alpha \cap \varphi(\breve{D}) \neq \varnothing$;
- $\alpha = \left(Y_\varnothing^\alpha \times \varnothing\right) \cup \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_{T\cup T'}^\alpha \times (T\cup T')\right)$, where $T, T' \in D$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq \varphi(T)$, $Y_\varnothing^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cap \varphi(T) \neq \varnothing$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$;
- $\alpha = \left(Y_\varnothing^\alpha \times \varnothing\right) \cup \left(Y_6^\alpha \times Z_6\right) \cup \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$, where, $Y_6^\alpha, Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$, $T, T' \in \{Z_3, Z_2, Z_1\}$, $T \neq T'$, $T \cup T' = \breve{D}$ and satisfies the conditions: $Y_\varnothing^\alpha \cup Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_\varnothing^\alpha \cup Y_6^\alpha \cup Y_T^\alpha \supseteq \varphi(T)$, $Y_\varnothing^\alpha \cup Y_6^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_6^\alpha \cap \varphi(Z_6) \neq \varnothing$, $Y_T^\alpha \cap \varphi(T) \neq \varnothing$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$.
- $\alpha = \left(Y_\varnothing^\alpha \times \varnothing\right) \cup \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_{T\cup T'}^\alpha \times (T\cup T')\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$, where $Y_T^\alpha, Y_{T'}^\alpha, Y_0^\alpha \notin \{\varnothing\}$, $T \in \{Z_8, Z_7, Z_6, Z_5\}$, $T' \in \{Z_7, Z_6, Z_5, Z_4\}$, $T \neq T'$, and satisfies the conditions: $Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq \varphi(T)$, $Y_\varnothing^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cap \varphi(T) \neq \varnothing$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$, $Y_0^\alpha \cap \varphi(\breve{D}) \neq \varnothing$;
- $\alpha = \left(Y_\varnothing^\alpha \times \varnothing\right) \cup \left(Y_T^\alpha \times T\right) \cup \left(Y_6^\alpha \times Z_6\right) \cup \left(Y_{T\cup Z_6}^\alpha \times (T\cup Z_6)\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$, where $Y_T^\alpha, Y_6^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$, $T \setminus Z_6 \neq \varnothing$, $Z_6 \setminus T \neq \varnothing$, $\varnothing \subset Z_6 \subset T'$ and satisfies the conditions: $Y_\varnothing^\alpha \cup Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq \varphi(T)$, $Y_\varnothing^\alpha \cup Y_6^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cap \varphi(Z_6) \neq \varnothing$, $Y_{T'}^\alpha \cap \varphi(T) \neq \varnothing$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$.

*Proof.* Let $Z_9 = \varnothing$. Then given Theorem immediately follows from ([1], Theorem 2). □

**Lemma 2.** *Let* $D = \left\{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\} \in \Sigma_1(X,10)$ *and* $Z_9 = \varnothing$. *Let* $R^*(Q_1)$ *be set of all regular elements of the semigroup* $B_X(D)$ *such that each element satisfies the condition a) of Theorem* 1. *Then* $\left|R^*(Q_1)\right| = 1$.

Now let a binary relation $\alpha$ of the semigroup $B_X(D)$ satisfy the condition b) of Theorem 1 (see diagram 2 of the **Figure 1**). In this case we have $Q_2 = \{\varnothing, T\}$, where $T \in D$ and $\varnothing \subset T$. By definition of the semilattice $D$ it follows that

$$Q_2\vartheta_{XI} = \left\{\{\varnothing, Z_8\}, \{\varnothing, Z_7\}, \{\varnothing, Z_6\}, \{\varnothing, Z_5\}, \{\varnothing, Z_4\}, \{\varnothing, Z_3\}, \{\varnothing, Z_2\}, \{\varnothing, Z_1\}, \left\{\varnothing, \breve{D}\right\}\right\}.$$

It is easy to see $\left|\Phi(Q_2, Q_2)\right| = 1$ and $\left|\Omega(Q_2)\right| = 9$. If

$$D_1' = \left\{\varnothing, \breve{D}\right\},\ D_2' = \{\varnothing, Z_8\},\ D_3' = \{\varnothing, Z_7\},\ D_4' = \{\varnothing, Z_6\},$$
$$D_5' = \{\varnothing, Z_5\},\ D_6' = \{\varnothing, Z_4\},\ D_7' = \{\varnothing, Z_3\},\ D_8' = \{\varnothing, Z_2\},\ D_9' = \{\varnothing, Z_1\},$$

then

$$R^*(Q_2)=\bigcup_{i=1}^{9}R(D'_i) \tag{1}$$

(see remark page 5 in [1]).

**Lemma 3.** *Let* $X$ *be a finite set,* $D=\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_1(X,10)$ *and* $Z_9=\varnothing$. *Let* $R^*(Q_2)$ *be set of all regular elements of the semigroup* $B_X(D)$ *such that each element satisfies the condition b) of Theorem* 1. *Then*

$$\left|R^*(Q_2)\right|=9\cdot\left(2^{|\breve{D}|}-1\right)\cdot 2^{|X\setminus\breve{D}|}.$$

*Proof.* Let $Z\in D$, $D'=\{\varnothing,Z\}$ and $\alpha\in R(D')$. Then quasinormal representation of a binary relation $\alpha$ has a form $\alpha=(Y_\varnothing^\alpha\times\varnothing)\cup(Y_T^\alpha\times T)$ for some $T\in D$, $Y_T^\alpha\neq\varnothing$ and by statement b) of Theorem 1 satisfies the conditions $Y_\varnothing^\alpha\supseteq\varnothing$ and $Y_T^\alpha\cap Z\neq\varnothing$. By definition of the semilattice $D$ we have $\breve{D}\supseteq Z$, *i.e.*, $Y_\varnothing^\alpha\supseteq\varnothing$ and $Y_T^\alpha\cap\breve{D}\neq\varnothing$. It follows that $\alpha\in R(D'_1)$. Therefore the inclusion $R(D')\subseteq R(D'_1)$ holds. By the Equality (1) we have

$$R^*(Q_2)=R(D'_1). \tag{2}$$

From this equality and by statement b) of Lemma 1 it immediately follows that

$$\left|R^*(Q_2)\right|=9\cdot\left(2^{|\breve{D}|}-1\right)\cdot 2^{|X\setminus\breve{D}|}.$$

□

Let binary relation $\alpha$ of the semigroup $B_X(D)$ satisfy the condition c) of Theorem 1 (see diagram 3 of the **Figure 1**). In this case we have $Q_2=\{\varnothing,T,T'\}$, where $T,T'\in D$ and $\varnothing\subset T\subset T'$. By definition of the semilattice $D$ it follows that

$$\begin{aligned}Q_3\vartheta_{XI}=\Big\{&\{\varnothing,Z_8,\breve{D}\},\{\varnothing,Z_7,\breve{D}\},\{\varnothing,Z_6,\breve{D}\},\{\varnothing,Z_5,\breve{D}\},\{\varnothing,Z_4,\breve{D}\},\{\varnothing,Z_3,\breve{D}\},\\&\{\varnothing,Z_2,\breve{D}\},\{\varnothing,Z_1,\breve{D}\},\{\varnothing,Z_8,Z_3\},\{\varnothing,Z_7,Z_3\},\{\varnothing,Z_6,Z_3\},\{\varnothing,Z_6,Z_2\},\\&\{\varnothing,Z_6,Z_1\},\{\varnothing,Z_5,Z_1\},\{\varnothing,Z_4,Z_1\}\Big\}.\end{aligned}$$

It is easy to see $\left|\Phi(Q_3,Q_3)\right|=1$ and $\left|\Omega(Q_3)\right|=15$. If-1

$$\begin{aligned}&D'_1=\{\varnothing,Z_8,\breve{D}\},D'_2=\{\varnothing,Z_7,\breve{D}\},D'_3=\{\varnothing,Z_6,\breve{D}\},D'_4=\{\varnothing,Z_5,\breve{D}\},\\&D'_5=\{\varnothing,Z_4,\breve{D}\},D'_6=\{\varnothing,Z_3,\breve{D}\},D'_7=\{\varnothing,Z_2,\breve{D}\},D'_8=\{\varnothing,Z_1,\breve{D}\},\\&D'_9=\{\varnothing,Z_8,Z_3\},D'_{10}=\{\varnothing,Z_7,Z_3\},D'_{11}=\{\varnothing,Z_6,Z_3\},D'_{12}=\{\varnothing,Z_6,Z_2\},\\&D'_{13}=\{\varnothing,Z_6,Z_1\},D'_{14}=\{\varnothing,Z_5,Z_1\},D'_{15}=\{\varnothing,Z_4,Z_1\},\end{aligned}$$

then

$$R^*(Q_3)=\bigcup_{i=1}^{15}R(D'_i) \tag{3}$$

(see remark page 5 in [1] and Theorem 1).

**Lemma 4.** *Let* $X$ *be a finite set,* $D=\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_1(X,10)$ *and* $Z_9=\varnothing$. *Let* $R^*(Q_3)$ *be set of all regular elements of the semigroup* $B_X(D)$ *such that each element satisfies the condition c) of Theorem* 1. *Then*

$$\left|R^*(Q_3)\right|=\sum_{i=1}^{8}\left|R(D'_i)\right|-\sum_{(j,k)\in M_5(Q_3)}\left|R(D'_j)\cap R(D'_k)\right|$$

where

$$M_5(Q_3)=\{(2,6),(3,6),(3,7),(3,8),(4,8),(5,8)\}.$$

*Proof.* Let $D_i' = \{\varnothing, Y_i', Y_i''\}$ $(\varnothing \subset Y_i' \subset Y_i'')$ be arbitrary element of the set $Q_3\vartheta_{XI}$ and $\alpha \in R(D_i')$. Then quasinormal representation of a binary relation $\alpha$ has a form $\alpha = (Y_\varnothing^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T')$ for some $T, T' \in D$, $\varnothing \subset T \subset T'$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$ and by statement c) of Theorem 1 satisfies the conditions $Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq Y_i'$, $Y_T^\alpha \cap Y_i' \neq \varnothing$ and $Y_{T'}^\alpha \cap Y_i'' \neq \varnothing$. By definition of the semilattice $D$ we have $\breve{D} \supseteq Y_i''$. From this and by the condition $Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq Y_i'$, $Y_T^\alpha \cap Y_i' \neq \varnothing$, $Y_{T'}^\alpha \cap Y_i'' \neq \varnothing$ we have

$$Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq Y_i', Y_T^\alpha \cap Y_i' \neq \varnothing, Y_{T'}^\alpha \cap \breve{D} \neq \varnothing,$$

*i.e.* $\alpha \in R(D_j')$, where $D_j' = \{\varnothing, Y_i', \breve{D}\}$. It follows that $R(D_i') \subseteq R(D_j'')$, from the last inclusion and by definition of the semilattice $D$ we have $R(D_i') \subseteq R(D_j'')$ for all $(i, j) \in M_1(Q_3)$, where

$$M_1(Q_3) = \{(9,1),(10,2),(11,3),(12,3),(13,3),(14,4),(15,5)\}.$$

Therefore the following equality holds

$$R^*(Q_3) = \bigcup_{i=1}^{8} R(D_i'). \tag{4}$$

Now, let $D_i' = \{\varnothing, Y_i, \breve{D}\}, D_j' = \{\varnothing, Y_j, \breve{D}\} \subset \{D_1', D_2', \cdots, D_8'\}$, $D_i' \neq D_j'$ and $\alpha \in R(D_i') \cap R(D_j')$. Then for the binary relation $\alpha$ we have

$$Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq Y_i, Y_T^\alpha \cap Y_i \neq \varnothing, Y_{T'}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq Y_j, Y_T^\alpha \cap Y_j \neq \varnothing, Y_{T'}^\alpha \cap \breve{D} \neq \varnothing.$$

From the last condition it follows that $Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq Y_i \cup Y_j$.

1) $Y_i \cup Y_j = \breve{D}$. Then we have, that $(Y_\varnothing^\alpha \cup Y_T^\alpha) \cap Y_{T'}^\alpha \supseteq (Y_i \cup Y_j) \cap Y_{T'}^\alpha = \breve{D} \cap Y_{T'}^\alpha \neq \varnothing$. But the inequality $(Y_\varnothing^\alpha \cup Y_T^\alpha) \cap Y_{T'}^\alpha \neq \varnothing$ contradicts the condition that representation of binary relation $\alpha$ is quasinormal. So, the equality $R(D_i') \cap R(D_j') = \varnothing$ is true. From last equality and by definition of the semilattice $D$ we have $R(D_i') \cap R(D_j') = \varnothing$ for all $(i, j) \in M_2(Q_3)$, where

$$M_2(Q_3) = \{(1,4),(1,5),(1,7),(1,8),(2,4),(2,5),(2,7),(2,8),$$
$$(4,6),(4,7),(5,6),(5,7),(6,7),(6,8),(7,8)\}.$$

2) $D_i' = \{\varnothing, Y_i, \breve{D}\}, D_j' = \{\varnothing, Y_j, \breve{D}\}, D_k' = \{\varnothing, Y_i \cup Y_j, \breve{D}\} \subset \{D_1', D_2', \cdots, D_8'\}$, $D_i' \neq D_j'$, $D_i' \neq D_k'$, $D_j' \neq D_k'$, $\alpha \in R(D_i') \cap R(D_j')$ and $\alpha \in R(D_i') \cap R(D_j') \cap R(D_k')$ are true. Then we have

$$Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq Y_i, Y_T^\alpha \cap Y_i \neq \varnothing, Y_{T'}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq Y_j, Y_T^\alpha \cap Y_j \neq \varnothing, Y_{T'}^\alpha \cap \breve{D} \neq \varnothing,$$

and

$$Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq Y_i, Y_T^\alpha \cap Y_i \neq \varnothing, Y_{T'}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq Y_j, Y_T^\alpha \cap Y_j \neq \varnothing, Y_{T'}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq Y_i \cup Y_j, Y_T^\alpha \cap (Y_i \cup Y_j) \neq \varnothing, Y_{T'}^\alpha \cap \breve{D} \neq \varnothing$$

respectively, *i.e.*, $\alpha \in R(D_i') \cap R(D_j')$ or $\alpha \in R(D_i') \cap R(D_j') \cap R(D_k')$ if and only if

$$Y_\varnothing^\alpha \cup Y_T^\alpha \supseteq Y_i \cup Y_j, Y_T^\alpha \cap Y_i \neq \varnothing, Y_T^\alpha \cap Y_j \neq \varnothing, Y_{T'}^\alpha \cap \breve{D} \neq \varnothing.$$

Therefore, the equality $R(D_i') \cap R(D_j') = R(D_i') \cap R(D_j') \cap R(D_k')$ is true. From last equality and by definition of the semilattice $D$ we have: $R(D_i') \cap R(D_j') = R(D_i') \cap R(D_j') \cap R(D_k')$ for all $(i, j, k) \in M_3(Q_3)$, where

$$M_3(Q_3) = \{(1,2,6),(1,3,6),(2,3,6),(3,4,8),(3,5,8),(4,5,8)\}.$$

3) $D'_i=\{\varnothing,Y_i,\breve{D}\}, D'_j=\{\varnothing,Y_j,\breve{D}\}, D'_k=\{\varnothing,Y_k,\breve{D}\}, D'_t=\{\varnothing,Y_i\cup Y_j\cup Y_k,\breve{D}\}\subset\{D'_1,D'_2,\cdots,D'_8\}$, $D'_p\neq D'_q$, $p,q\in\{i,j,k,t\}$, $p\neq q$, $\alpha\in R(D'_i)\cap R(D'_j)\cap R(D'_k)$ and $\alpha\in R(D'_i)\cap R(D'_j)\cap R(D'_k)\cap R(D'_t)$ are true. Then we have

$$\begin{aligned}&Y_\varnothing^\alpha\cup Y_T^\alpha\supseteq Y_i, Y_T^\alpha\cap Y_i\neq\varnothing, Y_{T'}^\alpha\cap\breve{D}\neq\varnothing,\\&Y_\varnothing^\alpha\cup Y_T^\alpha\supseteq Y_j, Y_T^\alpha\cap Y_j\neq\varnothing, Y_{T'}^\alpha\cap\breve{D}\neq\varnothing,\\&Y_\varnothing^\alpha\cup Y_T^\alpha\supseteq Y_k, Y_T^\alpha\cap Y_k\neq\varnothing, Y_{T'}^\alpha\cap\breve{D}\neq\varnothing\end{aligned}$$

and

$$\begin{aligned}&Y_\varnothing^\alpha\cup Y_T^\alpha\supseteq Y_i, Y_T^\alpha\cap Y_i\neq\varnothing, Y_{T'}^\alpha\cap\breve{D}\neq\varnothing,\\&Y_\varnothing^\alpha\cup Y_T^\alpha\supseteq Y_j, Y_T^\alpha\cap Y_j\neq\varnothing, Y_{T'}^\alpha\cap\breve{D}\neq\varnothing,\\&Y_\varnothing^\alpha\cup Y_T^\alpha\supseteq Y_k, Y_T^\alpha\cap Y_k\neq\varnothing, Y_{T'}^\alpha\cap\breve{D}\neq\varnothing,\\&Y_\varnothing^\alpha\cup Y_T^\alpha\supseteq Y_i\cup Y_j\cup Y_k, Y_T^\alpha\cap(Y_i\cup Y_j\cup Y_k)\neq\varnothing, Y_{T'}^\alpha\cap\breve{D}\neq\varnothing\end{aligned}$$

respectively, *i.e.*, $\alpha\in R(D'_i)\cap R(D'_j)\cap R(D'_k)$ and $\alpha\in R(D'_i)\cap R(D'_j)\cap R(D'_k)\cap R(D'_t)$ if and only if

$$Y_\varnothing^\alpha\cup Y_T^\alpha\supseteq Y_i\cup Y_j\cup Y_k, Y_T^\alpha\cap Y_i\neq\varnothing, Y_T^\alpha\cap Y_j\neq\varnothing, Y_T^\alpha\cap Y_k\neq\varnothing, Y_{T'}^\alpha\cap\breve{D}\neq\varnothing.$$

Therefore, the equality $R(D'_i)\cap R(D'_j)\cap R(D'_k)=R(D'_i)\cap R(D'_j)\cap R(D'_k)\cap R(D'_t)$ is true. From last equality and by definition of the semilattice *D* we have:
$R(D'_i)\cap R(D'_j)\cap R(D'_k)=R(D'_i)\cap R(D'_j)\cap R(D'_k)\cap R(D'_t)$ for all $(i,j,k,t)\in M_4(Q_3)$, where

$$M_4(Q_3)=\{(1,2,3,6),(3,4,5,8)\}.$$

Now, by Equality (2) and by conditions 1), 2) and 3) it follows that the following equality is true

$$|R^*(Q_3)|=\sum_{i=1}^{8}|R(D'_i)|-\sum_{(j,k)\in M_5(Q_3)}|R(D'_j)\cap R(D'_k)|$$

where

$$M_5(Q_3)=\{(2,6),(3,6),(3,7),(3,8),(4,8),(5,8)\}.$$

□

**Lemma 5.** *Let* $D'=\{\varnothing,Y,\breve{D}\}$, $D''=\{\varnothing,Y',\breve{D}\}$, *where* $Y,Y'\in D$ *and* $Y'\supseteq Y$. *If quasinormal representation of binary relation* $\alpha$ *of the semigroup* $B_X(D)$ *has a form* $\alpha=(Y_\varnothing^\alpha\times\varnothing)\cup(Y_T^\alpha\times T)\cup(Y_0^\alpha\times\breve{D})$ *for some* $T\in D$, $\varnothing\subset T\subset\breve{D}$ *and* $Y_T^\alpha,Y_0^\alpha\notin\{\varnothing\}$, *then* $\alpha\in R(D')\cap R(D'')$ *iff*

$$Y_\varnothing^\alpha\cup Y_T^\alpha\supseteq Y', Y_T^\alpha\cap Y\neq\varnothing, Y_0^\alpha\cap\breve{D}\neq\varnothing.$$

*Proof.* If $\alpha\in R(D')\cap R(D'')$, then by statement c) of theorem 1 we have

$$\begin{aligned}&Y_\varnothing^\alpha\cup Y_T^\alpha\supseteq Y, Y_T^\alpha\cap Y\neq\varnothing, Y_0^\alpha\cap\breve{D}\neq\varnothing;\\&Y_\varnothing^\alpha\cup Y_T^\alpha\supseteq Y', Y_T^\alpha\cap Y'\neq\varnothing, Y_0^\alpha\cap\breve{D}\neq\varnothing.\end{aligned}\tag{5}$$

From the last condition we have

$$Y_\varnothing^\alpha\cup Y_T^\alpha\supseteq Y', Y_T^\alpha\cap Y\neq\varnothing, Y_0^\alpha\cap\breve{D}\neq\varnothing\tag{6}$$

since $Y'\supseteq Y$ by assumption. On the other hand, if the conditions of (6) holds, then the conditions of (5) follow, *i.e.* $\alpha\in R(D')\cap R(D'')$. □

**Lemma 6.** *Let* $D'=\{\varnothing,Y,\breve{D}\}, D''=\{\varnothing,Y',\breve{D}\}\subset\{D'_1,D'_2,\cdots,D'_8\}$, $Y'\supseteq Y$ *and X be a finite set. Then the following equality holds*

$$|R(D')\cap R(D'')|=15\cdot 2^{|Y'\setminus Y|}\cdot(2^{|Y|}-1)\cdot\left(3^{|\breve{D}\setminus Y'|}-2^{|\breve{D}\setminus Y'|}\right)\cdot 3^{|X\setminus\breve{D}|}.$$

*Proof.* Let $D'=\{\varnothing,Y,\breve{D}\}, D''=\{\varnothing,Y',\breve{D}\}\subset\{D'_1,D'_2,\cdots,D'_8\}$, where $Y'\supseteq Y$. Assume that $\alpha\in R(D')\cap R(D'')$ and a quasinormal representation of a regular binary relation $\alpha$ has a form

$\alpha=\left(Y_{\varnothing}^{\alpha}\times\varnothing\right)\cup\left(Y_{T}^{\alpha}\times T\right)\cup\left(Y_{0}^{\alpha}\times\breve{D}\right)$ for some $T\in D$, $\varnothing\subset T\subset\breve{D}$ and $Y_{T}^{\alpha},Y_{0}^{\alpha}\notin\{\varnothing\}$. Then according to Lemma 5, we have

$$Y_{\varnothing}^{\alpha}\cup Y_{T}^{\alpha}\supseteq Y', Y_{T}^{\alpha}\cap Y\neq\varnothing, Y_{0}^{\alpha}\cap\breve{D}\neq\varnothing,\cdots \tag{7}$$

Further, let $f_{\alpha}$ be a mapping from $X$ to the semilattice $D$ satisfying the conditions $f_{\alpha}(t)=t\alpha$ for all $t\in X$. $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$ are the restrictions of the mapping $f_{\alpha}$ on the sets $Y'$, $\breve{D}\setminus Y'$, $X\setminus\breve{D}$ respectively. It is clear that the intersection of elements of the set $\left\{Y', \breve{D}\setminus Y', X\setminus\breve{D}\right\}$ is an empty set, and

$Y'\cup\left(\breve{D}\setminus Y'\right)\cup\left(X\setminus\breve{D}\right)=X$. We are going to find properties of the maps $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$.

1) $t\in Y'$. Then by the properties of $D$ we have $Y'\subseteq Y_{\varnothing}^{\alpha}\cup Y_{T}^{\alpha}$, *i.e.*, $t\in Y_{\varnothing}^{\alpha}\cup Y_{T}^{\alpha}$ and $t\alpha\in\{\varnothing,T\}$ by definition of the sets $Y_{\varnothing}^{\alpha}$ and $Y_{T}^{\alpha}$. Therefore $f_{1\alpha}(t)\in\{\varnothing,T\}$ for all $t\in Y'$. By suppose we have that $Y_{T}^{\alpha}\cap Y\neq\varnothing$, *i.e.* $t'\alpha=T$ for some $t'\in Y$. Therefore $f_{1\alpha}(t')=T$ for some $t'\in Y$.

2) $t\in\breve{D}\setminus Y'$. Then by properties of $D$ we have $\breve{D}\setminus Y'\subseteq\breve{D}\subseteq X=Y_{\varnothing}^{\alpha}\cup Y_{T}^{\alpha}\cup Y_{0}^{\alpha}$, *i.e.*, $t\in Y_{\varnothing}^{\alpha}\cup Y_{T}^{\alpha}\cup Y_{0}^{\alpha}$ and $t\alpha\in\left\{\varnothing,T,\breve{D}\right\}$ by definition of the sets $Y_{\varnothing}^{\alpha}$, $Y_{T}^{\alpha}$ and $Y_{0}^{\alpha}$. Therefore $f_{3\alpha}(t)\in\left\{\varnothing,T,\breve{D}\right\}$ for all $t\in\breve{D}\setminus Y'$. By suppose we have, that $Y_{0}^{\alpha}\cap\breve{D}\neq\varnothing$, *i.e.* $t''\alpha=\breve{D}$ for some $t''\in\breve{D}$. If $t''\in Y'$. Then $t''\in Y'\subseteq Y_{\varnothing}^{\alpha}\cup Y_{T}^{\alpha}$. Therefore $t''\alpha\in\{\varnothing,T\}$ by definition of the set $Y_{\varnothing}^{\alpha}$ and $Y_{T}^{\alpha}$. We have contradiction to the equality $t''\alpha=\breve{D}$. Therefore $f_{3\alpha}(t'')=\breve{D}$ for some $t''\in\breve{D}\setminus Y'$.

3) $t\in X\setminus\breve{D}$. Then by definition quasinormal representation binary relation $\alpha$ and by property of $D$ we have $t\in X\setminus\breve{D}\subseteq X=Y_{\varnothing}^{\alpha}\cup Y_{T}^{\alpha}\cup Y_{0}^{\alpha}$, *i.e.* $t\alpha\in\left\{\varnothing,T,\breve{D}\right\}$ by definition of the sets $Y_{\varnothing}^{\alpha},Y_{T}^{\alpha}$ and $Y_{0}^{\alpha}$. Therefore $f_{4\alpha}(t)\in\left\{\varnothing,T,\breve{D}\right\}$ for all $t\in X\setminus\breve{D}$. Therefore for every binary relation $\alpha\in R(D')\cap R(D'')$ there exists ordered system $(f_{1\alpha},f_{2\alpha},f_{3\alpha})$. It is obvious that for disjoint binary relations there exists disjoint ordered systems. Further, let

$$f_{1}:Y'\to\{\varnothing,T\}, f_{2}:\breve{D}\setminus Y'\to\left\{\varnothing,T,\breve{D}\right\}, f_{3}:X\setminus\breve{D}\to\left\{\varnothing,T,\breve{D}\right\}$$

be such mappings, which satisfy the conditions: $f_{1}(t)\in\{\varnothing,T\}$ for all $t\in Y'$ and $f_{1}(t')=T'$ for some $t'\in Y$; $f_{2}(t)\in\left\{\varnothing,T,\breve{D}\right\}$ for all $t\in\breve{D}\setminus Y'$ and $f_{2}(t'')=\breve{D}$ for some $t''\in\breve{D}\setminus Y'$; $f_{3}(t)\in\left\{\varnothing,T,\breve{D}\right\}$ for all $t\in X\setminus\breve{D}$. Now we define a map *f* from *X* to the semilattice *D*, which satisfies the condition:

$$f(t)=\begin{cases} f_{1}(t), \text{if } t\in Y'\setminus Z_{9},\\ f_{2}(t), \text{if } t\in\breve{D}\setminus Y',\\ f_{3}(t), \text{if } t\in X\setminus\breve{D}.\end{cases}$$

Further, let $\beta=\bigcup_{x\in X}\left(\{x\}\times f(x)\right)$, $Y_{\varnothing}^{\beta}=\{t\mid t\beta=\varnothing\}$, $Y_{T}^{\beta}=\{t\mid t\beta=T\}$ and $Y_{0}^{\beta}=\left\{t\mid t\beta=\breve{D}\right\}$. Then binary relation $\beta$ may be represented by

$$\beta=\left(Y_{\varnothing}^{\beta}\times\varnothing\right)\cup\left(Y_{T}^{\beta}\times T\right)\cup\left(Y_{0}^{\beta}\times\breve{D}\right)$$

and satisfy the conditions:

$$Y_{\varnothing}^{\beta}\cup Y_{T}^{\beta}\supseteq Y', Y_{T}^{\beta}\cap Y\neq\varnothing, Y_{0}^{\beta}\cap\breve{D}\neq\varnothing$$

(By suppose $f_{1}(t')=T'$ for some $t'\in Y$ and $f_{2}(t'')=\breve{D}$ for some $t''\in\breve{D}\setminus Y'$), *i.e.*, by lemma 5 we have that $\beta\in R(D')\cap R(D'')$. Therefore for every binary relation $\alpha\in R(D')\cap R(D'')$ and ordered system $(f_{1\alpha},f_{2\alpha},f_{3\alpha})$ there exists one to one mapping. By Lemma 1 and by Theorem 1 in [1] the number of the mappings $f_{1\alpha},f_{2\alpha},f_{3\alpha},f_{4\alpha}$ are respectively:

$$1, 2^{|(Y'\setminus Y)|}\cdot\left(2^{|Y|}-1\right), 3^{|\breve{D}\setminus Y'|}-2^{|\breve{D}\setminus Y'|}, 3^{|X\setminus\breve{D}|}.$$

Note that the number $2^{|Y'\setminus Y|}\cdot\left(2^{|Y|}-1\right)\cdot\left(3^{|\breve{D}\setminus Y'|}-2^{|\breve{D}\setminus Y'|}\right)\cdot 3^{|X\setminus\breve{D}|}$ does not depend on choice of chains $T\subset T'\subset T''$ $(T,T',T''\in D)$ of the semilattice *D*. Since the number of such different chains of the semilattice *D* is equal to 15, for arbitrary $T,T'\in D$ where $\varnothing\subset T\subset T'$, the number of regular elements of the set $R(D')\cap R(D'')$ is equal to

$$\left|R(D')\cap R(D'')\right| = 15\cdot 2^{|Y'\setminus Y|}\cdot\left(2^{|Y|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Y'\right|}-2^{\left|\breve{D}\setminus Y'\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}.$$

□

Therefore, we obtain:

$$\begin{aligned}
\left|R(D_1')\cap R(D_6')\right| &= 15\cdot 2^{|Z_3\setminus Z_8|}\cdot\left(2^{|Z_8|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_3\right|}-2^{\left|\breve{D}\setminus Z_3\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|},\\
\left|R(D_2')\cap R(D_6')\right| &= 15\cdot 2^{|Z_3\setminus Z_7|}\cdot\left(2^{|Z_7|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_3\right|}-2^{\left|\breve{D}\setminus Z_3\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|},\\
\left|R(D_3')\cap R(D_6')\right| &= 15\cdot 2^{|Z_3\setminus Z_6|}\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_3\right|}-2^{\left|\breve{D}\setminus Z_3\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|},\\
\left|R(D_3')\cap R(D_7')\right| &= 15\cdot 2^{|Z_2\setminus Z_6|}\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_2\right|}-2^{\left|\breve{D}\setminus Z_2\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|},\\
\left|R(D_3')\cap R(D_8')\right| &= 15\cdot 2^{|Z_1\setminus Z_6|}\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_1\right|}-2^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|},\\
\left|R(D_4')\cap R(D_8')\right| &= 15\cdot 2^{|Z_1\setminus Z_5|}\cdot\left(2^{|Z_5|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_1\right|}-2^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|},\\
\left|R(D_5')\cap R(D_8')\right| &= 15\cdot 2^{|Z_1\setminus Z_4|}\cdot\left(2^{|Z_4|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_1\right|}-2^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}.
\end{aligned} \tag{8}$$

**Lemma 7.** *Let* $X$ *be a finite set,* $D=\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_1(X,10)$ *and* $Z_9=\varnothing$. *Let* $R^*(Q_3)$ *be set of all regular elements of the semigroup* $B_X(D)$ *such that each element satisfies the condition c) of Theorem* 1. *Then*

$$\begin{aligned}
\left|R^*(Q_3)\right| &= 15\cdot\left(2^{|Z_8|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_8\right|}-2^{\left|\breve{D}\setminus Z_8\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&+15\cdot\left(2^{|Z_7|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_7\right|}-2^{\left|\breve{D}\setminus Z_7\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&+15\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_6\right|}-2^{\left|\breve{D}\setminus Z_6\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&+15\cdot\left(2^{|Z_5|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_5\right|}-2^{\left|\breve{D}\setminus Z_5\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&+15\cdot\left(2^{|Z_4|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_4\right|}-2^{\left|\breve{D}\setminus Z_4\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&+15\cdot\left(2^{|Z_3|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_3\right|}-2^{\left|\breve{D}\setminus Z_3\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&+15\cdot\left(2^{|Z_2|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_2\right|}-2^{\left|\breve{D}\setminus Z_2\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&+15\cdot\left(2^{|Z_1|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_1\right|}-2^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&-15\cdot 2^{|Z_3\setminus Z_8|}\cdot\left(2^{|Z_8|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_3\right|}-2^{\left|\breve{D}\setminus Z_3\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&-15\cdot 2^{|Z_3\setminus Z_7|}\cdot\left(2^{|Z_7|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_3\right|}-2^{\left|\breve{D}\setminus Z_3\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&-15\cdot 2^{|Z_3\setminus Z_6|}\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_3\right|}-2^{\left|\breve{D}\setminus Z_3\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&15\cdot 2^{|Z_2\setminus Z_6|}\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_2\right|}-2^{\left|\breve{D}\setminus Z_2\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&-22\cdot 2^{|Z_1\setminus Z_6|}\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_1\right|}-2^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&15\cdot 2^{|Z_1\setminus Z_5|}\cdot\left(2^{|Z_5|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_1\right|}-2^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}\\
&-15\cdot 2^{|Z_1\setminus Z_4|}\cdot\left(2^{|Z_4|}-1\right)\cdot\left(3^{\left|\breve{D}\setminus Z_1\right|}-2^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 3^{\left|X\setminus\breve{D}\right|}.
\end{aligned}$$

*Proof*. Let $Z_9 = \varnothing$. Then the given Lemma immediately follows from Lemma 4 and from the Equalities (3). □

Now let binary relation $\alpha$ of the semigroup $B_X(D)$ satisfy the condition d) of Theorem 1 (see diagram 4 of the **Figure 1**). In this case we have $Q_4 = \{\varnothing, T, T', \breve{D}\}$, where $T, T' \in D$ and $\varnothing \subset T \subset T' \subset \breve{D}$. By definition of the semilattice $D$ it follows that

$$Q_4\vartheta_{XI} = \Big\{\{\varnothing, Z_8, Z_3, \breve{D}\}, \{\varnothing, Z_7, Z_3, \breve{D}\}, \{\varnothing, Z_6, Z_3, \breve{D}\}, \{\varnothing, Z_6, Z_2, \breve{D}\}, \{\varnothing, Z_6, Z_1, \breve{D}\}, \{\varnothing, Z_5, Z_1, \breve{D}\}, \{\varnothing, Z_4, Z_1, \breve{D}\}\Big\}.$$

It is easy to see $|\Phi(Q_4, Q_4)| = 1$ and $|\Omega(Q_4)| = 7$. If

$$D_1' = \{\varnothing, Z_8, Z_3, \breve{D}\},\ D_2' = \{\varnothing, Z_7, Z_3, \breve{D}\},\ D_3' = \{\varnothing, Z_6, Z_3, \breve{D}\},$$
$$D_4' = \{\varnothing, Z_6, Z_2, \breve{D}\},\ D_5' = \{\varnothing, Z_6, Z_1, \breve{D}\},\ D_6' = \{\varnothing, Z_5, Z_1, \breve{D}\},$$
$$D_7' = \{\varnothing, Z_4, Z_1, \breve{D}\};$$

then

$$R^*(Q_4) = \bigcup_{i=1}^{7} R(D_i') \tag{9}$$

(see Definition [1], Definition 4 and [1], Theorem 2).

**Lemma 8.** *Let* $X$ *be a finite set,* $D = \{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_1(X, 10)$ *and* $Z_9 = \varnothing$. *Let* $R^*(Q_4)$ *be set of all regular elements of the semigroup* $B_X(D)$ *such that each element satisfies the condition d) of Theorem* 1. *Then*

$$\begin{aligned}
|R^*(Q_4)| &= 7\cdot\left(2^{|Z_8|}-1\right)\cdot\left(3^{|Z_3\setminus Z_8|}-2^{|Z_3\setminus Z_8|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+7\cdot\left(2^{|Z_7|}-1\right)\cdot\left(3^{|Z_3\setminus Z_7|}-2^{|Z_3\setminus Z_7|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+7\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_3\setminus Z_6|}-2^{|Z_3\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+7\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+7\cdot\left(2^{|Z_6|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+7\cdot\left(2^{|Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+7\cdot\left(2^{|Z_4|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}.
\end{aligned}$$

*Proof*. Let $Z_9 = \varnothing$. Then the given Lemma immediately follows from ([1], Lemma 10). □

Now let binary relation $\alpha$ of the semigroup $B_X(D)$ satisfy the condition e) of Theorem 1 (see diagram 5 of the **Figure 1**). In this case we have $Q_5 = \{\varnothing, T, T', T \cup T'\}$, where $T, T' \in D$ and $T \setminus T' \neq \varnothing$ and $T' \setminus T \neq \varnothing$. By definition of the semilattice $D$ it follows that

$$\begin{aligned}
Q_5\vartheta_{XI} = \Big\{&\{\varnothing, Z_8, Z_7, Z_3\}, \{\varnothing, Z_8, Z_6, Z_3\}, \{\varnothing, Z_8, Z_5, \breve{D}\}, \{\varnothing, Z_8, Z_4, \breve{D}\}, \{\varnothing, Z_8, Z_2, \breve{D}\},\\
&\{\varnothing, Z_8, Z_1, \breve{D}\}, \{\varnothing, Z_7, Z_6, Z_3\}, \{\varnothing, Z_7, Z_5, \breve{D}\}, \{\varnothing, Z_7, Z_4, \breve{D}\}, \{\varnothing, Z_7, Z_2, \breve{D}\},\\
&\{\varnothing, Z_7, Z_1, \breve{D}\}, \{\varnothing, Z_6, Z_5, Z_1\}, \{\varnothing, Z_6, Z_4, Z_1\}, \{\varnothing, Z_5, Z_4, Z_1\}, \{\varnothing, Z_5, Z_3, \breve{D}\},\\
&\{\varnothing, Z_5, Z_2, \breve{D}\}, \{\varnothing, Z_4, Z_3, \breve{D}\}, \{\varnothing, Z_4, Z_2, \breve{D}\}, \{\varnothing, Z_3, Z_2, \breve{D}\}, \{\varnothing, Z_3, Z_1, \breve{D}\},\\
&\{\varnothing, Z_2, Z_1, \breve{D}\}.
\end{aligned}$$

It is easy to see $|\Phi(Q_5, Q_5)| = 2$ and $|\Omega(Q_5)| = 21$. If

$$D_1' = \{\varnothing, Z_8, Z_7, Z_3\}, D_2' = \{\varnothing, Z_7, Z_8, Z_3\}, D_3' = \{\varnothing, Z_8, Z_6, Z_3\}, D_4' = \{\varnothing, Z_6, Z_8, Z_3\},$$
$$D_5' = \{\varnothing, Z_8, Z_5, \breve{D}\}, D_6' = \{\varnothing, Z_5, Z_8, \breve{D}\}, D_7' = \{\varnothing, Z_8, Z_4, \breve{D}\}, D_8' = \{\varnothing, Z_4, Z_8, \breve{D}\},$$
$$D_9' = \{\varnothing, Z_8, Z_2, \breve{D}\}, D_{10}' = \{\varnothing, Z_2, Z_8, \breve{D}\}, D_{11}' = \{\varnothing, Z_8, Z_1, \breve{D}\}, D_{12}' = \{\varnothing, Z_1, Z_8, \breve{D}\},$$
$$D_{13}' = \{\varnothing, Z_7, Z_6, Z_3\}, D_{14}' = \{\varnothing, Z_6, Z_7, Z_3\}, D_{15}' = \{\varnothing, Z_7, Z_5, \breve{D}\}, D_{16}' = \{\varnothing, Z_5, Z_7, \breve{D}\},$$
$$D_{17}' = \{\varnothing, Z_7, Z_4, \breve{D}\}, D_{18}' = \{\varnothing, Z_4, Z_7, \breve{D}\}, D_{19}' = \{\varnothing, Z_7, Z_2, \breve{D}\}, D_{20}' = \{\varnothing, Z_2, Z_7, \breve{D}\},$$
$$D_{21}' = \{\varnothing, Z_7, Z_1, \breve{D}\}, D_{22}' = \{\varnothing, Z_1, Z_7, \breve{D}\}, D_{23}' = \{\varnothing, Z_6, Z_5, Z_1\}, D_{24}' = \{\varnothing, Z_5, Z_6, Z_1\},$$
$$D_{25}' = \{\varnothing, Z_6, Z_4, Z_1\}, D_{26}' = \{\varnothing, Z_4, Z_6, Z_1\}, D_{27}' = \{\varnothing, Z_5, Z_4, Z_1\}, D_{28}' = \{\varnothing, Z_4, Z_5, Z_1\},$$
$$D_{29}' = \{\varnothing, Z_5, Z_3, \breve{D}\}, D_{30}' = \{\varnothing, Z_3, Z_5, \breve{D}\}, D_{31}' = \{\varnothing, Z_5, Z_2, \breve{D}\}, D_{32}' = \{\varnothing, Z_2, Z_5, \breve{D}\},$$
$$D_{33}' = \{\varnothing, Z_4, Z_3, \breve{D}\}, D_{34}' = \{\varnothing, Z_3, Z_4, \breve{D}\}, D_{35}' = \{\varnothing, Z_4, Z_2, \breve{D}\}, D_{36}' = \{\varnothing, Z_2, Z_4, \breve{D}\},$$
$$D_{37}' = \{\varnothing, Z_3, Z_2, \breve{D}\}, D_{38}' = \{\varnothing, Z_2, Z_3, \breve{D}\}, D_{39}' = \{\varnothing, Z_3, Z_1, \breve{D}\}, D_{40}' = \{\varnothing, Z_1, Z_3, \breve{D}\},$$
$$D_{41}' = \{\varnothing, Z_2, Z_1, \breve{D}\}, D_{42}' = \{\varnothing, Z_1, Z_2, \breve{D}\}$$

then

$$R^*(Q_5) = \bigcup_{i=1}^{42} R(D_i') \tag{10}$$

(see [1], Definition 4 and [1], Theorem 1).

**Lemma 9.** *Let X be a finite set,* $D = \{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_1(X, 10)$ *and* $Z_9 = \varnothing$. *Let* $R^*(Q_5)$ *be set of all regular elements of the semigroup* $B_X(D)$ *such that each element satisfies the condition e) of Theorem* 1. *Then*

$$\left|R^*(Q_5)\right| = \sum_{i=1}^{42} \left|R(D_i')\right| - \sum_{(k,j)\in M_4'} \left|R(D_k') \cap R(D_j')\right|,$$

where

$$\begin{aligned} M_4'(Q_5) = \{&(3,9),(3,11),(4,10),(4,12),(5,11),(5,30),(5,39),(6,12),(6,29),\\ &(6,40),(7,11),(7,34),(7,39),(8,12),(8,33),(8,40),(9,37),(10,38),\\ &(11,39),(12,40),(13,19),(13,21),(14,20),(14,22),(15,21),(15,30),\\ &(15,39),(16,22),(16,29),(16,40),(17,21),(17,34),(17,39),(18,22),\\ &(18,33),(18,40),(19,37),(20,38),(21,39),(22,40),(23,30),(23,32),\\ &(23,39),(23,41),(24,29),(24,31),(25,34),(25,36),(26,33),(26,35),\\ &(29,40),(30,39),(31,42),(32,41),(33,40),(34,39),(35,42),(36,41)\}. \end{aligned}$$

*Proof.* Let $Z_9 = \varnothing$. Then the given Lemma immediately follows from ([1], Lemma 13). □

**Lemma 10.** *Let* $D_i' = \{\varnothing, Y_i, Y_i', Y_i \cup Y_i'\}$ *and* $D_j' = \{\varnothing, Y_j, Y_j', Y_j \cup Y_j'\}$ *be arbitrary elements of the set* $\{D_1', D_2', \cdots, D_{42}'\}$, *where* $D_i' \neq D_j'$, $Y_j \supseteq Y_i$ *and* $Y_j' \supseteq Y_i'$. *Then the following equality holds*

$$\left|R(D_i') \cap R(D_j')\right| = 21 \cdot 2^{\left|(Y_j \setminus (Y_i \cup Y_j'))\right|} \cdot \left(2^{\left|Y_i \setminus Y_j'\right|} - 1\right) \cdot 2^{\left|(Y_j' \setminus (Y_i' \cup Y_j))\right|} \cdot \left(2^{\left|Y_i' \setminus Y_j\right|} - 1\right) \cdot 4^{\left|X \setminus (Y_j \cup Y_j')\right|}.$$

*Proof.* Let $Z_9 = \varnothing$. Then the given Lemma immediately follows from definition semilattice $D$ and by ([1], Lemma 13). □

**Lemma 11.** *Let X be a finite set,* $D = \{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_1(X, 10)$ *and* $Z_9 = \varnothing$. *Let* $R^*(Q_5)$ *be set of all regular elements of the semigroup* $B_X(D)$ *such that each element satisfies the condition e) of Theorem* 1. *Then* $\left|R^*(Q_5)\right| = \sum_{i=1}^{42} \left|R(D_i')\right| - \sum_{(k,j)\in M_4'} \left|R(D_k') \cap R(D_j')\right|$, *where*

$$\begin{aligned}
\sum_{i=1}^{42}\left|R\left(D_i'\right)\right| &= 42\cdot\left(2^{|Z_8\setminus Z_7|}-1\right)\cdot\left(2^{|Z_7\setminus Z_8|}-1\right)\cdot 4^{|X\setminus Z_3|}+42\cdot\left(2^{|Z_8\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot 4^{|X\setminus Z_3|}\\
&+42\cdot\left(2^{|Z_8\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_8|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_8\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_8|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+42\cdot\left(2^{|Z_8\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_8|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_8|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+42\cdot\left(2^{|Z_7\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 4^{|X\setminus Z_3|}+42\cdot\left(2^{|Z_7\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+42\cdot\left(2^{|Z_7\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_7\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_7|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+42\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_7|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot 4^{|X\setminus Z_1|}\\
&+42\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot 4^{|X\setminus Z_1|}+42\cdot\left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot 4^{|X\setminus Z_1|}\\
&+42\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_5|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_5\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_5|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+42\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_4\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_4|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+42\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_3\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&+42\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot 4^{|X\setminus \breve{D}|}.
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{(k,j)\in M_4'}\left|R\left(D_k'\right)\cap R\left(D_j'\right)\right|\\
&=42\cdot\left(2^{|Z_8\setminus Z_2|}-1\right)\cdot 2^{|Z_2\setminus Z_3|}\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot 2^{|Z_1\setminus Z_3|}\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_5\setminus Z_8|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_8\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_4\setminus Z_8|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot\left(2^{|Z_8\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot\left(2^{|Z_8\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot\left(2^{|Z_7\setminus Z_2|}-1\right)\cdot 2^{|Z_2\setminus Z_3|}\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot 2^{|Z_1\setminus Z_3|}\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_7\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot\left(2^{|Z_7\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot\left(2^{|Z_7\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot 2^{|Z_3\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot 2^{|Z_2\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_2|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot 2^{|Z_3\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot 2^{|Z_2\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_2|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_1|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_5\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\
&\quad+42\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_1|}-1\right)\cdot 4^{|X\setminus \breve{D}|}+42\cdot\left(2^{|Z_4\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus \breve{D}|}.
\end{aligned}$$

*Proof*. Let $Z_9=\varnothing$. Then the given Lemma immediately follows from Lemma 9 and 10. □

Let *f* be a binary relation $\alpha$ of the semigroup $B_X(D)$ satisfy the condition g) of Theorem 1 (see diagram 7 of the **Figure 1**). In this case we have $Q_6=\{Z_9,Z_6,T,T',\breve{D}\}$ where $T,T'\in\{Z_3,Z_2,Z_1\}$, $T\setminus T'\neq\varnothing$ and $T'\setminus T\neq\varnothing$. By definition of the semilattice *D* it follows that

$$Q_6\vartheta_{XI}=\{\{\varnothing,Z_6,Z_3,Z_2,\breve{D}\},\{\varnothing,Z_6,Z_3,Z_1,\breve{D}\},\{\varnothing,Z_6,Z_2,Z_1,\breve{D}\}\}.$$

It is easy to see $|\Phi(Q_6,Q_6)|=2$ and $|\Omega(Q_6)|=3$. If

$$D_1'=\{Z_9,Z_6,Z_3,Z_2,\breve{D}\},\ D_2'=\{Z_9,Z_6,Z_2,Z_3,\breve{D}\},\ D_3'=\{Z_9,Z_6,Z_3,Z_1,\breve{D}\},$$
$$D_4'=\{Z_9,Z_6,Z_1,Z_3,\breve{D}\},\ D_5'=\{Z_9,Z_6,Z_2,Z_1,\breve{D}\},\ D_6'=\{Z_9,Z_6,Z_1,Z_2,\breve{D}\}.$$

Then

$$R^*(Q_6)=\bigcup_{i=1}^{6}R(D_i') \tag{11}$$

(see Definition [1], Definition 4 and [1], Theorem 2).

**Lemma 12.** *Let X be a finite set,* $D=\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_1(X,10)$ *and* $Z_9=\varnothing$. *Let* $R^*(Q_6)$ *be set of all regular elements of the semigroup* $B_X(D)$ *such that each element satisfies the condition f) of Theorem* 1. *Then*

$$\begin{aligned}|R^*(Q_6)|&=6\left(2^{|Z_6\setminus Z_9|}-1\right)2^{|(Z_3\cap Z_2)\setminus Z_6|}\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)5^{|X\setminus\breve{D}|}\\&+6\left(2^{|Z_6\setminus Z_9|}-1\right)2^{|(Z_3\cap Z_1)\setminus Z_6|}\left(3^{|Z_3\setminus Z_1|}-2^{|Z_3\setminus Z_1|}\right)\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)5^{|X\setminus\breve{D}|}\\&+6\left(2^{|Z_6\setminus Z_9|}-1\right)2^{|(Z_2\cap Z_1)\setminus Z_6|}\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)5^{|X\setminus\breve{D}|}\end{aligned}$$

*Proof*. Let $Z_9=\varnothing$. Then the given Lemma immediately follows from ([1], Lemma 15). □

Now let *g* be a binary relation $\alpha$ of the semigroup $B_X(D)$ satisfy the condition f) of Theorem 1 (see diagram 6 of the **Figure 1**). In this case we have $Q_7=\{\varnothing,T,T',T\cup T',\breve{D}\}$, where $T,T'\in\{Z_8,Z_7,Z_6,Z_5,Z_4\}$, $T\setminus T'\neq\varnothing$ and $T'\setminus T\neq\varnothing$. By definition of the semilattice *D* it follows that

$$\begin{aligned}Q_7\vartheta_{XI}=\{&\{\varnothing,Z_8,Z_7,Z_3,\breve{D}\},\{\varnothing,Z_8,Z_6,Z_3,\breve{D}\},\{\varnothing,Z_7,Z_6,Z_3,\breve{D}\},\\&\{\varnothing,Z_6,Z_5,Z_1,\breve{D}\},\{\varnothing,Z_6,Z_4,Z_1,\breve{D}\},\{\varnothing,Z_5,Z_4,Z_1,\breve{D}\}\}.\end{aligned}$$

It is easy to see $|\Phi(Q_7,Q_7)|=2$ and $|\Omega(Q_7)|=6$. If

$$D_1'=\{\varnothing,Z_8,Z_7,Z_3,\breve{D}\},\ D_2'=\{\varnothing,Z_7,Z_8,Z_3,\breve{D}\},\ D_3'=\{\varnothing,Z_8,Z_6,Z_3,\breve{D}\},$$
$$D_4'=\{\varnothing,Z_6,Z_8,Z_3,\breve{D}\},\ D_5'=\{\varnothing,Z_7,Z_6,Z_3,\breve{D}\},\ D_6'=\{\varnothing,Z_6,Z_7,Z_3,\breve{D}\},$$
$$D_7'=\{\varnothing,Z_6,Z_5,Z_1,\breve{D}\},\ D_8'=\{\varnothing,Z_5,Z_6,Z_1,\breve{D}\},\ D_9'=\{\varnothing,Z_6,Z_4,Z_1,\breve{D}\},$$
$$D_{10}'=\{\varnothing,Z_4,Z_6,Z_1,\breve{D}\},\ D_{11}'=\{\varnothing,Z_5,Z_4,Z_1,\breve{D}\},\ D_{12}'=\{\varnothing,Z_4,Z_5,Z_1,\breve{D}\}$$

then

$$R^*(Q_7)=\bigcup_{i=1}^{12}R(D_i') \tag{12}$$

(see [1], Definition 4 and [1], Theorem 2).

**Lemma 13.** *Let X be a finite set,* $D=\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_1(X,10)$ *and* $Z_9=\varnothing$. *Let* $R^*(Q_7)$ *be set of all regular elements of the semigroup* $B_X(D)$ *such that each element satisfies the condition g) of Theorem* 1. *Then*

$$\left|R^*(Q_7)\right| = 12\cdot\left(2^{|Z_8\setminus Z_7|}-1\right)\cdot\left(2^{|Z_7\setminus Z_8|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_3|}-4^{|\breve{D}\setminus Z_3|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+12\cdot\left(2^{|Z_8\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_3|}-4^{|\breve{D}\setminus Z_3|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+12\cdot\left(2^{|Z_7\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_3|}-4^{|\breve{D}\setminus Z_3|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+12\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+12\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+12\cdot\left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}.$$

*Proof*. Let $Z_9=\varnothing$. Then the given Lemma immediately follows from ([1], Lemma 16). □

Let $h$ be a binary relation $\alpha$ of the semigroup $B_X(D)$ satisfy the condition $h$) of Theorem 1 (see diagram 8 of the **Figure 1**). In this case we have $Q_8=\{\varnothing,T,Z_6,T\cup Z_6,T',\breve{D}\}$, Where $T\in\{Z_8,Z_7,Z_5,Z_4\}$, $T'\in\{Z_3,Z_2,Z_1\}$. By definition of the semilattice $D$ it follows that

$$Q_8\vartheta_{XI}=\{\{\varnothing,Z_8,Z_6,Z_3,Z_2,\breve{D}\},\{\varnothing,Z_8,Z_6,Z_3,Z_1,\breve{D}\},\{\varnothing,Z_7,Z_6,Z_3,Z_2,\breve{D}\},$$
$$\{\varnothing,Z_7,Z_6,Z_3,Z_1,\breve{D}\},\{\varnothing,Z_6,Z_5,Z_3,Z_1,\breve{D}\},\{\varnothing,Z_6,Z_5,Z_2,Z_1,\breve{D}\},$$
$$\{\varnothing,Z_6,Z_4,Z_3,Z_1,\breve{D}\},\{\varnothing,Z_6,Z_4,Z_2,Z_1,\breve{D}\}\}.$$

It is easy to see $\left|\Phi(Q_8,Q_8)\right|=1$ and $\left|\Omega(Q_8)\right|=8$. If

$$D_1'=\{\varnothing,Z_8,Z_6,Z_3,Z_2,\breve{D}\},\ D_2'=\{\varnothing,Z_8,Z_6,Z_3,Z_1,\breve{D}\},$$
$$D_3'=\{\varnothing,Z_7,Z_6,Z_3,Z_2,\breve{D}\},\ D_4'=\{\varnothing,Z_7,Z_6,Z_3,Z_1,\breve{D}\},$$
$$D_5'=\{\varnothing,Z_6,Z_5,Z_3,Z_1,\breve{D}\},\ D_6'=\{\varnothing,Z_6,Z_5,Z_2,Z_1,\breve{D}\},$$
$$D_7'=\{\varnothing,Z_6,Z_4,Z_3,Z_1,\breve{D}\},\ D_8'=\{\varnothing,Z_6,Z_4,Z_2,Z_1,\breve{D}\}.$$

Then

$$R^*(Q_8)=\bigcup_{i=1}^{8}R(D_i') \tag{13}$$

(see [1], Definition 4 and [1], Theorem 2).

**Lemma 14.** *Let $X$ be a finite set, $D=\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_1(X,10)$ and $Z_9=\varnothing$. Let $R^*(Q_8)$ be set of all regular elements of the semigroup $B_X(D)$ such that each element satisfies the condition h) of Theorem* 1. *Then*

$$\left|R^*(Q_8)\right| = 8\cdot\left(2^{|Z_8\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 6^{|X\setminus\breve{D}|}$$
$$+8\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 6^{|X\setminus\breve{D}|}$$
$$+8\cdot\left(2^{|Z_7\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 6^{|X\setminus\breve{D}|}$$
$$+8\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 6^{|X\setminus\breve{D}|}$$
$$+8\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_1|}-2^{|Z_3\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}$$
$$+8\cdot\left(2^{|Z_5\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}$$
$$+8\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(3^{|Z_3\setminus Z_1|}-2^{|Z_3\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}$$
$$+8\cdot\left(2^{|Z_4\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}.$$

*Proof*. Let $Z_9 = \varnothing$. Then the given Lemma immediately follows from ([1], Lemma 17). □

Let us assume that

$$s_2 = \sum_{i=1}^{8} \left| R^*(Q_i) \right|.$$

**Theorem 2.** *Let* $D \in \Sigma_1(X,10)$, $Z_9 = \varnothing$. *If X is a finite set and* $R_D$ *is a set of all regular elements of the semigroup* $B_X(D)$, *then* $|R_D| = s_2$.

*Proof*. This Theorem immediately follows from ([1], Theorem 2) and Theorem 1. □

**Example 1.** *Let* $X = \{1,2,3,4,5,6,7\}$,

$$P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\}, P_4 = \{4\}, P_5 = \{5\}, P_7 = \{6\}, P_8 = \{7\}, P_0 = P_6 = P_9 = \varnothing.$$

Then $\breve{D} = \{1,2,3,4,5,6,7\}$, $Z_1 = \{2,3,4,5,6,7\}$, $Z_2 = \{1,3,4,5,6,7\}$, $Z_3 = \{1,2,4,5,6,7\}$, $Z_4 = \{2,3,5,6,7\}$, $Z_5 = \{2,3,4,6,7\}$, $Z_6 = \{4,5,6,7\}$, $Z_7 = \{1,2,4,5,7\}$, $Z_8 = \{1,2,4,5,6\}$ and $Z_9 = \varnothing$.

$$D = \{\{1,2,3,4,5,6,7\},\{2,3,4,5,6,7\},\{1,3,4,5,6,7\},\{1,2,4,5,6,7\},$$
$$\{2,3,5,6,7\},\{2,3,4,6,7\},\{4,5,6,7\},\{1,2,4,5,7\},\{1,2,4,5,6\},\varnothing\}.$$

We have $Z_9 = \varnothing$, $|R^*(Q_1)| = 1$, $|R^*(Q_2)| = 1143$, $|R^*(Q_3)| = 9990$, $|R^*(Q_4)| = 2443$, $|R^*(Q_5)| = 3150$, $|R^*(Q_6)| = 540$, $|R^*(Q_7)| = 168$, $|R(Q_8)| = 64$, $R_D = 17499$.

**Theorem 3.** *Let* $D \in \Sigma_1(X,10)$. *Then the set* $R_D$ *of all regular elements of the semigroup* $B_X(D)$ *is a subsemigroup of this semigroup.*

*Proof*. From ([1], Lemma 2), and by definition of the semilattice *D* it follows that the diagrams of *XI*-semilattices have the form of one of the diagrams given ([1], Figure 2). Now the given Theorem immediately follows from ([3], Theorem 2). □

## References


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