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Regular Elements of $B_X(D)$ Defined by the Class $\Sigma_1(X,10)-I$

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Abstract

In order to obtain some results in the theory of semigroups, the concept of regularity, introduced by J. V. Neumann for elements of rings, is useful. In this work, all regular elements of semigroup defined by semilattices of the class $\Sigma_1(X,10)$ are studied. When *X* has finitely many elements, we have given the number of regular elements.

Keywords

Semilattice, Semigroup, Binary Relation

1. Introduction

Let *D* be a nonempty set of subsets of a given set *X*, closed under union. Such a set *D* is called a complete *X*-semilattice of unions. For any map *f* from *X* to *D*, we define a binary relation.

$$\alpha_f = \bigcup_{x\in X}\left(\{x\}\times f(x)\right).$$

The set of all α_f, denoted by $B_X(D)$, is a subsemigroup of B_X semigroup of all binary relations on *X*. (See [1]-[6].)

All notations, symbol and required definitions used in this work can be found in [7]. Recall the following results.

Lemma 1. [1], *Corollary* 1.18.1. *Let* $Y=\{y_1,y_2,\cdots,y_k\}$ *and* $D_j=\{T_1,\cdots,T_j\}$ *be two sets where* $k\ge 1$ *and* $j\ge 1$. *Then the number* $s(k,j)$ *of all possible mappings from* Y *to subsets* D_j' *of* D_j *such that*

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$T_j \in D_j'$ *is given by* $s(k, j) = j^k - (j-1)^k$.

Theorem 1. [1], *Theorem* 1.18.1. *Let* $D_j = \{T_1, T_2, \cdots, T_j\}$. *Let* $Y \subseteq X$ *be nonempty sets. Then the number of mappings from X to* D_j *such that* $f(y) = T_j$ *for some* $y \in Y$ *is equal to* $s = j^{|X \setminus Y|} \cdot \left(j^{|Y|} - (j-1)^{|Y|}\right)$.

2. Results

Let X be a nonempty set, D a X-semilattice of union with the conditions (see **Figure 1**);

$$\begin{aligned}
&Z_9 \subset Z_4 \subset Z_1 \subset \breve{D}, Z_9 \subset Z_5 \subset Z_1 \subset \breve{D},\\
&Z_9 \subset Z_6 \subset Z_1 \subset \breve{D}, Z_9 \subset Z_6 \subset Z_2 \subset \breve{D},\\
&Z_9 \subset Z_6 \subset Z_3 \subset \breve{D}, Z_9 \subset Z_7 \subset Z_3 \subset \breve{D},\\
&Z_9 \subset Z_8 \subset Z_3 \subset \breve{D};\\
&Z_i \setminus Z_j \neq \varnothing \left(1 \le i \ne j \le 3 \text{ or } 4 \le i \ne j \le 8\right);\\
&Z_1 \cup Z_2 = Z_1 \cup Z_3 = Z_2 \cup Z_3 = Z_4 \cup Z_2\\
&= Z_4 \cup Z_3 = Z_4 \cup Z_7 = Z_4 \cup Z_8 = Z_5 \cup Z_2\\
&= Z_5 \cup Z_3 = Z_5 \cup Z_7 = Z_5 \cup Z_8 = Z_7 \cup Z_1\\
&= Z_7 \cup Z_2 = Z_8 \cup Z_1 = Z_8 \cup Z_2 = \breve{D};\\
&Z_4 \cup Z_5 = Z_4 \cup Z_6 = Z_5 \cup Z_6 = Z_1;\\
&Z_6 \cup Z_7 = Z_6 \cup Z_8 = Z_7 \cup Z_8 = Z_3.
\end{aligned} \tag{1}$$

The class of X-semilattices where each element is isomorphic to D is denoted by $\Sigma_1(X, 10)$.

An element $\alpha \in B_X(D)$ is called regular if $\alpha \circ \beta \circ \alpha = \alpha$ for some $\beta \in B_X$. Our aim in this work is to identify all regular elements of $B_X(D)$ where D is given above.

Definition 1. *The complete X-semilattice of unions is called an XI-semilattice of unions if* $\Lambda(D, D_t) \in D$ *and* $Z = \bigcup_{t \in Z} \Lambda(D, D_t)$ *for any nonempty Z in D. Here* $\Lambda(D, D_t)$ *is an exact lower bound of* D_t *in D where* $D_t = \{Z' \in D \mid t \in Z'\}$.

The following Lemma is well known (see [7], Lemma 3).

Lemma 2. *All semilattices in the form of the diagrams in* **Figure 2** *are XI-semilattices.*

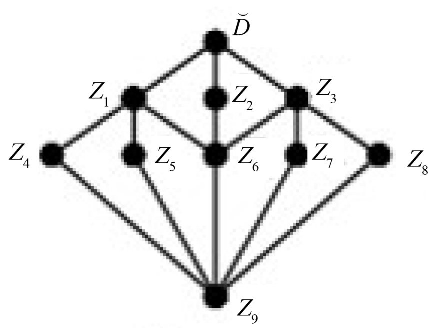

Figure 1. Diagram of semilattice of unions *D*.

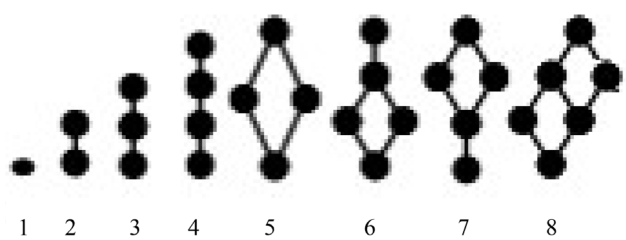

Figure 2. Diagram of all XI-subsemilattices of *D*.

Definition 2. *Let* D' *and* D'' *be two X-semilattices of unions. A one to one map from* D' *to* D'' *is said to be a complete isomorphism if*

$$\varphi(\cup D_1) = \bigcup_{T' \in D_1} \varphi(T')$$

for $D_1 \subseteq D'$.

Definition 3. [1], *Definition* 6.3.3. *Let* $\alpha \in B_X(D)$. *We say that a complete isomorphism* $\varphi : Q \to D'$ *is a complete* α*-isomorphism if*

a) $Q = V(D, \alpha)$;

b) $\varphi(\varnothing) = \varnothing$ for $\varnothing \in V(D,\alpha)$ and $\varphi(T)\alpha = T$ for any $T \in V(D,\alpha)$.

The following subsemilattices are all *XI*-semilattices of the *X*-semilattices of unions *D*.

a) $Q_1 = \{T\}$, where $T \in D$ (see diagram 1 of the **Figure 3**);

b) $Q_2 = \{T, T'\}$, where $T, T' \in D$ and $T \subset T'$ (see diagram 2 of the **Figure 3**);

c) $Q_3 = \{T, T', T''\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T''$ (see diagram 3 of the **Figure 3**);

d) $Q_4 = \{Z_9, T, T', \breve{D}\}$, where $T, T' \in D$ and $Z_9 \subset T \subset T' \subset \breve{D}$ (see diagram 4 of the **Figure 3**);

e) $Q_5 = \{T, T', T'', T' \cup T''\}$ where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$, (see diagram 5 of the **Figure 3**);

f) $Q_6 = \{Z_9, T, T', T \cup T', \breve{D}\}$, where $T, T' \in D$, $Z_9 \subset T$, $Z_9 \subset T'$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $T \cup T' \subset \breve{D}$ (see diagram 6 of the **Figure 3**);

g) $Q_7 = \{Z_9, Z_6, T, T', \breve{D}\}$, where $T, T' \in D$, $Z_6 \subset T$, $Z_6 \subset T'$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $T \cup T' = \breve{D}$ (see diagram 7 of the **Figure 3**);

h) $Q_8 = \{Z_9, T, T', T \cup T', Z, \breve{D}\}$, where $Z_9 \subset T$, $Z_9 \subset T' \subset Z$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $(T \cup T') \setminus Z \neq \varnothing$, $Z \setminus (T \cup T') \neq \varnothing$, $T \cup T' \cup Z = \breve{D}$ (see diagram 8 of the **Figure 3**);

For each $i = 1, 2, \cdots, 8$ we set $Q_i \vartheta_{XI} = \{D' \subset D \mid D' \text{ is isomorphic to } Q_i\}$.

One can see that

$$Q_1\vartheta_{XI} = \{\{Z_9\},\{Z_8\},\{Z_7\},\{Z_6\},\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}\};$$

$$\begin{aligned} Q_2\vartheta_{XI} = \{&\{Z_9,\breve{D}\},\{Z_9,Z_8\},\{Z_9,Z_7\},\{Z_9,Z_6\},\{Z_9,Z_5\},\{Z_9,Z_4\},\\ &\{Z_9,Z_3\},\{Z_9,Z_2\},\{Z_9,Z_1\},\{Z_8,Z_3\},\{Z_8,\breve{D}\},\{Z_7,Z_3\},\\ &\{Z_7,\breve{D}\},\{Z_6,Z_3\},\{Z_6,Z_2\},\{Z_6,Z_1\},\{Z_6,\breve{D}\},\{Z_5,Z_1\},\\ &\{Z_5,\breve{D}\},\{Z_4,Z_1\},\{Z_4,\breve{D}\},\{Z_3,\breve{D}\},\{Z_2,\breve{D}\},\{Z_1,\breve{D}\}\}; \end{aligned}$$

$\ldots$

$$\begin{aligned} Q_8\vartheta_{XI} = \{&\{Z_9,Z_8,Z_6,Z_3,Z_2,\breve{D}\},\{Z_9,Z_8,Z_6,Z_3,Z_1,\breve{D}\},\{Z_9,Z_7,Z_6,Z_3,Z_2,\breve{D}\},\\ &\{Z_9,Z_7,Z_6,Z_3,Z_1,\breve{D}\},\{Z_9,Z_6,Z_5,Z_3,Z_1,\breve{D}\},\{Z_9,Z_6,Z_5,Z_2,Z_1,\breve{D}\},\\ &\{Z_9,Z_6,Z_4,Z_3,Z_1,\breve{D}\},\{Z_9,Z_6,Z_4,Z_2,Z_1,\breve{D}\}\}. \end{aligned}$$

Assume that $D' \in Q_i\vartheta_{XI}$ and denote by the symbol $R(D')$ the set of all regular elements α of the semigroup

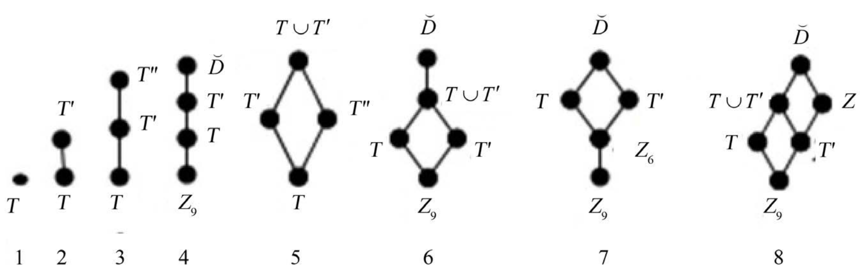

Figure 3. Diagram of all subsemilattices isomorphic to subsemilattices in **Figure 2**.

$B_X(D)$, for which the semilattices D' and Q_i are mutually α-isomorphic and $V(D,\alpha)=D'$ and

$$R^*(Q_i)=\bigcup_{D'\in Q_i\vartheta_{XI}} R(D')$$

(see [1], Definition 6.3.5).

The following results have the key role in this study.

Theorem 2. *Let* R_D *be the set of all regular elements of the semigroup* $B_X(D)$. *Then the following statements are true*:

a) $R(D')\cap R(D'')=\varnothing$ for any $D',D''\in\Sigma_{XI}(D)$ and $D'\neq D''$;

b) $R_D=\bigcup_{D'\in\Sigma_{XI}(D)}R(D')$;

c) if X is a finite set, then $|R_D|=\sum_{D'\in\Sigma_{XI}(D)}|R(D')|$ (see [1], Theorem 6.3.6).

Lemma 3. *Let* φ *be isomorphism between* Q_i *and* D_i' *semilattices,* $T\in Q_i$, $\bar{T}\in D_i'$ *and* $\varphi(T)=\bar{T}$. *If* X *is a finite set and* $|Q_i\vartheta_{XI}|=m_i$ $(i=1,2,\cdots,8)$, *then the following equalities are true*:

a) $|R(Q_1)|=1$;

b) $|R(Q_2)|=m_2\cdot\left(2^{|\bar{T}'\setminus\bar{T}|}-1\right)\cdot 2^{|X\setminus\bar{T}'|}$;

c) $|R(Q_3)|=m_3\cdot\left(2^{|\bar{T}'\setminus\bar{T}|}-1\right)\cdot\left(3^{|\bar{T}''\setminus\bar{T}'|}-2^{|\bar{T}''\setminus\bar{T}'|}\right)\cdot 3^{|X\setminus\bar{T}''|}$;

d) $|R(Q_4)|=m_4\cdot\left(2^{|\bar{T}\setminus Z_9|}-1\right)\cdot\left(3^{|\bar{T}'\setminus\bar{T}|}-2^{|\bar{T}'\setminus\bar{T}|}\right)\cdot\left(4^{|\breve{D}\setminus\bar{T}'|}-3^{|\breve{D}\setminus\bar{T}'|}\right)\cdot 4^{|X\setminus\breve{D}|}$;

e) $|R(Q_5)|=2\cdot m_5\cdot\left(2^{|\bar{T}'\setminus\bar{T}''|}-1\right)\cdot\left(2^{|\bar{T}''\setminus\bar{T}'|}-1\right)\cdot 4^{|X\setminus(\bar{T}'\cup\bar{T}'')|}$;

f) $|R(Q_6)|=2\cdot m_6\cdot\left(2^{|\bar{T}'\setminus\bar{T}''|}-1\right)\cdot\left(2^{|\bar{T}''\setminus\bar{T}'|}-1\right)\cdot\left(5^{|\breve{D}\setminus(\bar{T}'\cup\bar{T}'')|}-4^{|\breve{D}\setminus(\bar{T}'\cup\bar{T}'')|}\right)\cdot 5^{|X\setminus\breve{D}|}$;

g) $|R(Q_7)|=2\cdot m_7\cdot\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot 2^{|(\bar{T}\cap\bar{T}')\setminus Z_6|}\cdot\left(3^{|\bar{T}\setminus\bar{T}'|}-2^{|\bar{T}\setminus\bar{T}'|}\right)\cdot\left(3^{|\bar{T}'\setminus\bar{T}|}-2^{|\bar{T}'\setminus\bar{T}|}\right)\cdot 5^{|X\setminus\breve{D}|}$;

h) $|R(Q_8)|=2\cdot m_8\cdot\left(2^{|\bar{T}\setminus\bar{Z}|}-1\right)\cdot\left(2^{|\bar{T}'\setminus\bar{T}|}-1\right)\cdot\left(3^{|\bar{Z}\setminus(\bar{T}\cup\bar{T}')|}-2^{|\bar{Z}\setminus(\bar{T}\cup\bar{T}')|}\right)\cdot 6^{|X\setminus\breve{D}|}$.

Proof. The propositions a), b), c) and d) immediately follow from ([1], Theorem 6.3.5 and Theorem 13.1.2), while the equalities e), f), g) and h) follow from ([1], Theorem 6.3.5, Corolaries 13.3.4-5-6 and 13.7.3). □

3. Regular Elements of the Complete Semigroups of Binary Relations of the Class $\Sigma_1(X,10)$, When $\varnothing\notin D$ and $Z_9\neq\varnothing$

Theorem 3. *Let* $D=\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_1(X,10)$ *and* $Z_9\neq\varnothing$. *Then a binary relation* α *of the semigroup* $B_X(D)$ *whose quasinormal representation has the form* $\alpha=\bigcup_{T\in V(D,\alpha)}(Y_T^\alpha\times T)$, *will be a regular element of this semigroup iff there exist a complete* α*-isomorphism* φ *of the semilattice* $V(D,\alpha)$ *on some subsemilattice* D' *of the semilattice D which satisfies at least one of the following conditions*:

a) $\alpha=X\times T$, for some $T\in D$;

b) $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')$, for some $T,T'\in D$, $T\subset T'$ and $Y_T^\alpha,Y_{T'}^\alpha\notin\{\varnothing\}$ which satisfies the conditions: $Y_T^\alpha\supseteq\varphi(T)$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$;

c) $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T''}^\alpha\times T'')$, for some $T,T',T''\in D$, $T\subset T'\subset T''$, and $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$ which satisfies the conditions: $Y_T^\alpha\supseteq\varphi(T)$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq\varphi(T')$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$, $Y_{T''}^\alpha\cap\varphi(T'')\neq\varnothing$;

d) $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T''}^\alpha\times T'')\cup(Y_0^\alpha\times\breve{D})$, for some $T,T',T''\in D$, $T\subset T'\subset T''\subset\breve{D}$ and $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha,Y_0^\alpha\notin\{\varnothing\}$ which satisfies the conditions: $Y_T^\alpha\supseteq\varphi(T)$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq\varphi(T')$, $Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq\varphi(T'')$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$, $Y_{T''}^\alpha\cap\varphi(T'')\neq\varnothing$, $Y_0^\alpha\cap\varphi(\breve{D})\neq\varnothing$;

e) $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T''}^\alpha\times T'')\cup(Y_{T'\cup T''}^\alpha\times(T'\cup T''))$, where $T,T',T''\in D$, $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq\varphi(T')$, $Y_T^\alpha\cup Y_{T''}^\alpha\supseteq\varphi(T'')$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$, $Y_{T''}^\alpha\cap\varphi(T'')\neq\varnothing$;

f) $\alpha=\left(Y_9^\alpha\times Z_9\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where, $Y_9^\alpha,Y_6^\alpha,Y_T^\alpha,Y_{T'}^\alpha\notin\{\varnothing\}$, $T,T'\in\{Z_3,Z_2,Z_1\}$, $T\neq T'$, $T\cup T'=\breve{D}$ and satisfies the conditions: $Y_9^\alpha\supseteq\varphi(Z_9)$, $Y_9^\alpha\cup Y_6^\alpha\supseteq\varphi(Z_6)$, $Y_9^\alpha\cup Y_6^\alpha\cup Y_T^\alpha\supseteq\varphi(T)$, $Y_9^\alpha\cup Y_6^\alpha\cup Y_{T'}^\alpha\supseteq\varphi(T')$, $Y_6^\alpha\cap\varphi(Z_6)\neq\varnothing$, $Y_T^\alpha\cap\varphi(T)\neq\varnothing$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$;

g) $\alpha=\left(Y_9^\alpha\times Z_9\right)\cup\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T\cup T'}^\alpha\times(T\cup T')\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Y_T^\alpha,Y_{T'}^\alpha,Y_0^\alpha\notin\{\varnothing\}$, $T\in\{Z_8,Z_7,Z_6,Z_5\}$, $T'\in\{Z_7,Z_6,Z_5,Z_4\}$, $T\neq T'$, and satisfies the conditions: $Y_9^\alpha\cup Y_T^\alpha\supseteq\varphi(T)$, $Y_9^\alpha\cup Y_{T'}^\alpha\supseteq\varphi(T')$, $Y_T^\alpha\cap\varphi(T)\neq\varnothing$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$, $Y_0^\alpha\cap\varphi(\breve{D})\neq\varnothing$;

h) $\alpha=\left(Y_9^\alpha\times Z_9\right)\cup\left(Y_T^\alpha\times T\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_{T\cup Z_6}^\alpha\times(T\cup Z_6)\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Y_9^\alpha,Y_T^\alpha,Y_6^\alpha,Y_{T'}^\alpha\notin\{\varnothing\}$, $T\setminus Z_6\neq\varnothing$, $Z_6\setminus T\neq\varnothing$, $Z_9\subset Z_6\subset T'$ and satisfies the conditions: $Y_9^\alpha\supseteq\varphi(Z_9)$, $Y_9^\alpha\cup Y_6^\alpha\supseteq\varphi(Z_6)$, $Y_9^\alpha\cup Y_T^\alpha\supseteq\varphi(T)$, $Y_9^\alpha\cup Y_6^\alpha\cup Y_{T'}^\alpha\supseteq\varphi(T')$, $Y_T^\alpha\cap\varphi(Z_6)\neq\varnothing$, $Y_{T'}^\alpha\cap\varphi(T)\neq\varnothing$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$.

Proof. In this case from Lemma 2 it follows that diagrams 1-8 given in **Figure 2** exhaust all diagrams of *XI*-subsemilattices of the semilattice *D*. A quasinormal representation of regular elements of the semigroup $B_X(D)$, which are defined by these *XI*-semilattices, may have one of the form listed above. Then the validity of theorem immediately follows from ([1], Theorem 13.1.1, Theorem 13.3.1 and Theorem 13.7.1). □

Lemma 4. *Let* $D=\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_1(X,10)$ *and* $Z_9\neq\varnothing$. *Let* $R^*(Q_1)$ *be set of all regular elements of* $B_X(D)$ *such that each element satisfies the condition of* a) *of Theorem* 3. *Then* $|R^*(Q_1)|=10$.

Proof. Let binary relation α of the semigroup $B_X(D)$ satisfy the condition a) of Theorem 3. Then quasinormal representation of a binary relation α has a form $\alpha=X\times T$ for some $T\in D$. It is easy to see that $\alpha\circ\alpha=\alpha$ for all $T\in D$, *i.e.* binary relation α is a regular element of the semigroup $B_X(D)$. Therefore

$$|R^*(Q_1)|=10.$$ □

Now let binary relation α of the semigroup $B_X(D)$ satisfy the condition b) of Theorem 3 (see diagram 2 of the **Figure 3**). In this case we have $Q_2=\{T,T'\}$ where $T,T'\in D$ and $T\subset T'$. By definition of the semilattice *D* it follows that

$$\begin{aligned}Q_2\vartheta_{XI}=\{&\{Z_9,Z_8\},\{Z_9,Z_7\},\{Z_9,Z_6\},\{Z_9,Z_5\},\{Z_9,Z_4\},\{Z_9,Z_3\},\\&\{Z_9,Z_2\},\{Z_9,Z_1\},\{Z_9,\breve{D}\},\{Z_8,Z_3\},\{Z_8,\breve{D}\},\{Z_7,Z_3\},\\&\{Z_7,\breve{D}\},\{Z_6,Z_3\},\{Z_6,Z_2\},\{Z_6,Z_1\},\{Z_6,\breve{D}\},\{Z_5,Z_1\},\\&\{Z_5,\breve{D}\},\{Z_4,Z_1\},\{Z_4,\breve{D}\},\{Z_3,\breve{D}\},\{Z_2,\breve{D}\},\{Z_1,\breve{D}\}\}.\end{aligned}$$

It is easy to see that there is only one isomorphism from Q_2 to itself. That is $|\Phi(Q_2,Q_2)|=1$ and $|\Omega(Q_2)|=24$. If

$$\begin{aligned}&D_1'=\{Z_9,\breve{D}\},D_2'=\{Z_5,\breve{D}\},D_3'=\{Z_2,\breve{D}\},D_4'=\{Z_9,Z_8\},D_5'=\{Z_9,Z_7\},\\&D_6'=\{Z_9,Z_6\},D_7'=\{Z_9,Z_5\},D_8'=\{Z_9,Z_4\},D_9'=\{Z_9,Z_3\},D_{10}'=\{Z_9,Z_2\},\\&D_{11}'=\{Z_9,Z_1\},D_{12}'=\{Z_8,Z_3\},D_{13}'=\{Z_8,\breve{D}\},D_{14}'=\{Z_7,Z_3\},D_{15}'=\{Z_7,\breve{D}\},\\&D_{16}'=\{Z_6,Z_3\},D_{17}'=\{Z_6,Z_2\},D_{18}'=\{Z_6,Z_1\},D_{19}'=\{Z_6,\breve{D}\},D_{20}'=\{Z_5,Z_1\},\\&D_{21}'=\{Z_4,Z_1\},D_{22}'=\{Z_4,\breve{D}\},D_{23}'=\{Z_3,\breve{D}\},D_{24}'=\{Z_1,\breve{D}\}.\end{aligned}$$

then

$$R^*(Q_2)=\bigcup_{i=1}^{24}R(D_i'). \tag{2}$$

Lemma 5. *Let X be a finite set*,

$$D=\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_1(X,10)$$

and $Z_9\neq\varnothing$. Let $R^*(Q_2)$ be the set of all regular elements of $B_X(D)$ such that each element satisfies the condition b) of Theorem 3. Then

$$\left|R^*\left(Q_2\right)\right| = 24\cdot\left(2^{\left|\breve{D}\setminus Z_9\right|}-1\right)\cdot 2^{\left|X\setminus\breve{D}\right|}.$$

Proof. Let $Z, Z' \in D$, $Z \subset Z'$, $D' = \{Z, Z'\}$ and $\alpha \in R(D')$. Then quasinormal representation of a binary relation α has a form $\alpha = \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right)$ for some $T, T' \in D$, $T \subset T'$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$ and by statement b) of theorem 3 satisfies the conditions $Y_T^\alpha \supseteq Z$ and $Y_{T'}^\alpha \cap Z' \neq \varnothing$. By definition of the semilattice D we have $Z \supseteq Z_9$ and $\breve{D} \supseteq Z'$, *i.e.*, $Y_T^\alpha \supseteq Z_9$ and $Y_{T'}^\alpha \cap \breve{D} \neq \varnothing$. It follows that $\alpha \in R(D_1')$. Therefore we have

$$R^*\left(Q_2\right) = R\left(D_1'\right). \tag{3}$$

From this equality and by statement b) of Lemma 3 it immediately follows that

$$\left|R^*\left(Q_2\right)\right| = 24\cdot\left(2^{\left|\breve{D}\setminus Z_9\right|}-1\right)\cdot 2^{\left|X\setminus\breve{D}\right|}. \qquad \square$$

Let binary relation α of the semigroup $B_X(D)$ satisfy the condition c) of Theorem 3 (see diagram 3 of the **Figure 3**). In this case we have $Q_2 = \{T, T', T''\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T''$. By definition of the semilattice D it follows that

$$\begin{aligned} Q_3\vartheta_{XI} = {} & \{Z_9, Z_8, \breve{D}\}, \{Z_9, Z_7, \breve{D}\}, \{Z_9, Z_6, \breve{D}\}, \{Z_9, Z_5, \breve{D}\}, \{Z_9, Z_4, \breve{D}\}, \\ & \{Z_9, Z_3, \breve{D}\}, \{Z_9, Z_2, \breve{D}\}, \{Z_9, Z_1, \breve{D}\}, \{Z_9, Z_8, Z_3\}, \{Z_9, Z_7, Z_3\}, \\ & \{Z_9, Z_6, Z_3\}, \{Z_9, Z_6, Z_2\}, \{Z_9, Z_6, Z_1\}, \{Z_9, Z_5, Z_1\}, \{Z_9, Z_4, Z_1\}, \\ & \{Z_8, Z_3, \breve{D}\}, \{Z_7, Z_3, \breve{D}\}, \{Z_6, Z_3, \breve{D}\}, \{Z_6, Z_2, \breve{D}\}, \{Z_6, Z_1, \breve{D}\}, \\ & \{Z_5, Z_1, \breve{D}\}, \{Z_4, Z_1, \breve{D}\}. \end{aligned}$$

It is easy to see $\left|\Phi\left(Q_3, Q_3\right)\right| = 1$ and $\left|\Omega\left(Q_3\right)\right| = 22$. If

$$\begin{aligned} & D_1' = \{Z_9, Z_8, \breve{D}\}, D_2' = \{Z_9, Z_7, \breve{D}\}, D_3' = \{Z_9, Z_6, \breve{D}\}, D_4' = \{Z_9, Z_5, \breve{D}\}, \\ & D_5' = \{Z_9, Z_4, \breve{D}\}, D_6' = \{Z_9, Z_3, \breve{D}\}, D_7' = \{Z_9, Z_2, \breve{D}\}, D_8' = \{Z_9, Z_1, \breve{D}\}, \\ & D_9' = \{Z_9, Z_8, Z_3\}, D_{10}' = \{Z_9, Z_7, Z_3\}, D_{11}' = \{Z_9, Z_6, Z_3\}, D_{12}' = \{Z_9, Z_6, Z_2\}, \\ & D_{13}' = \{Z_9, Z_6, Z_1\}, D_{14}' = \{Z_9, Z_5, Z_1\}, D_{15}' = \{Z_9, Z_4, Z_1\}, D_{16}' = \{Z_8, Z_3, \breve{D}\}, \\ & D_{17}' = \{Z_7, Z_3, \breve{D}\}, D_{18}' = \{Z_6, Z_3, \breve{D}\}, D_{19}' = \{Z_6, Z_2, \breve{D}\}, D_{20}' = \{Z_6, Z_1, \breve{D}\}, \\ & D_{21}' = \{Z_5, Z_1, \breve{D}\}, D_{22}' = \{Z_4, Z_1, \breve{D}\}; \end{aligned}$$

then

$$R^*\left(Q_3\right) = \bigcup_{i=1}^{22} R\left(D_i'\right). \tag{4}$$

Lemma 6. *Let X be a finite set,*

$$D = \{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_1(X, 10)$$

and $Z_9 \neq \varnothing$. Let $R^*\left(Q_3\right)$ be the set of all regular elements of $B_X(D)$ such that each element satisfies the condition c) of Theorem 3. Then

$$\left|R^*\left(Q_3\right)\right| = \sum_{i=1}^{8}\left|R\left(D_i'\right)\right| - \sum_{(j,k)\in M_5(Q_3)}\left|R\left(D_j'\right)\cap R\left(D_k'\right)\right|$$

where

$$M_5\left(Q_3\right) = \{(2,6), (3,6), (3,7), (3,8), (4,8), (5,8)\}.$$

Proof. Let $D_i' = \{Y_i, Y_i', Y_i''\}$ $\left(Y_i \subset Y_i' \subset Y_i''\right)$ be arbitrary element of the set $Q_3\vartheta_{XI}$ and $\alpha \in R\left(D_i'\right)$. Then quasinormal representation of a binary relation α has a form $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)$ for some $T,T',T''\in D$, $T\subset T'\subset T''$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$ and by statement c) of Theorem 3 satisfies the conditions $Y_T^\alpha\supseteq Y_i$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_i'$, $Y_{T'}^\alpha\cap Y_i'\neq\varnothing$ and $Y_{T''}^\alpha\cap Y_i''\neq\varnothing$. By definition of the semilattice D we have $Y_i\supseteq Z_9$, $\breve{D}\supseteq Y_i''$. From this and by the condition $Y_T^\alpha\supseteq Y_i$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_i'$, $Y_{T'}^\alpha\cap Y_i'\neq\varnothing$, $Y_{T''}^\alpha\cap Y_i''\neq\varnothing$ we have

$$Y_T^\alpha\supseteq Z_9, Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_i', Y_{T'}^\alpha\cap Y_i'\neq\varnothing, Y_{T''}^\alpha\cap\breve{D}\neq\varnothing,$$

i.e. $\alpha\in R\left(D_j'\right)$, where $D_j'=\left\{Z_9,Y_i',\breve{D}\right\}$. It follows that $R\left(D_i'\right)\subseteq R\left(D_j'\right)$, From the last inclusion and by definition of the semilattice D we have $R\left(D_i'\right)\subseteq R\left(D_j'\right)$ for all $(i,j)\in M_1\left(Q_3\right)$, where

$$M_1\left(Q_3\right)=\{(9,1),(10,2),(11,3),(12,3),(13,3),(14,4),(15,5),$$
$$(16,6),(17,6),(18,6),(19,7),(20,8),(21,8),(22,8)\}.$$

Therefore the following equality

$$R^*\left(Q_3\right)=\bigcup_{i=1}^{8}R\left(D_i'\right) \tag{5}$$

holds. Now, let $D_i'=\left\{Z_9,Y_i,\breve{D}\right\},D_j'=\left\{Z_9,Y_j,\breve{D}\right\}\in\left\{D_1',D_2',\cdots,D_8'\right\}$, $D_i'\neq D_j'$ and $\alpha\in R\left(D_i'\right)\cap R\left(D_j'\right)$. Then for the binary relation α we have

$$Y_T^\alpha\supseteq Z_9, Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_i, Y_{T'}^\alpha\cap Y_i\neq\varnothing, Y_{T''}^\alpha\cap\breve{D}\neq\varnothing,$$
$$Y_T^\alpha\supseteq Z_9, Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_j, Y_{T'}^\alpha\cap Y_j\neq\varnothing, Y_{T''}^\alpha\cap\breve{D}\neq\varnothing.$$

From the last condition it follows that $Y_T^\alpha\supseteq Z_9, Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_i\cup Y_j$.

1) $Y_i\cup Y_j=\breve{D}$. Then we have that $\left(Y_T^\alpha\cup Y_{T'}^\alpha\right)\cap Y_{T''}^\alpha\supseteq\left(Y_i\cup Y_j\right)\cap Y_{T''}^\alpha=\breve{D}\cap Y_{T''}^\alpha\neq\varnothing$. But the inequality $\left(Y_T^\alpha\cup Y_{T'}^\alpha\right)\cap Y_{T''}^\alpha\neq\varnothing$ contradicts the condition that representation of binary relation α is quasinormal. So, the equality $R\left(D_i'\right)\cap R\left(D_j'\right)=\varnothing$ is true. From the last equality and by definition of the semilattice D we have $R\left(D_i'\right)\cap R\left(D_j'\right)=\varnothing$ for all $(i,j)\in M_2\left(Q_3\right)$, where

$$M_2\left(Q_3\right)=\{(1,4),(1,5),(1,7),(1,8),(2,4),(2,5),(2,7),(2,8),$$
$$(4,6),(4,7),(5,6),(5,7),(6,7),(6,8),(7,8)\}.$$

2) $D_i'=\left\{Z_9,Y_i,\breve{D}\right\},D_j'=\left\{Z_9,Y_j,\breve{D}\right\},D_k'=\left\{Z_9,Y_i\cup Y_j,\breve{D}\right\}\in\left\{D_1',D_2',\cdots,D_8'\right\}$, $D_i'\neq D_j'$, $D_i'\neq D_k'$, $D_j'\neq D_k'$, $\alpha\in R\left(D_i'\right)\cap R\left(D_j'\right)$ and $\alpha\in R\left(D_i'\right)\cap R\left(D_j'\right)\cap R\left(D_k'\right)$ are true. Then we have

$$Y_T^\alpha\supseteq Z_9, Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_i, Y_{T'}^\alpha\cap Y_i\neq\varnothing, Y_{T''}^\alpha\cap\breve{D}\neq\varnothing,$$
$$Y_T^\alpha\supseteq Z_9, Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_j, Y_{T'}^\alpha\cap Y_j\neq\varnothing, Y_{T''}^\alpha\cap\breve{D}\neq\varnothing,$$

and

$$Y_T^\alpha\supseteq Z_9, Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_i, Y_{T'}^\alpha\cap Y_i\neq\varnothing, Y_{T''}^\alpha\cap\breve{D}\neq\varnothing,$$
$$Y_T^\alpha\supseteq Z_9, Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_j, Y_{T'}^\alpha\cap Y_j\neq\varnothing, Y_{T''}^\alpha\cap\breve{D}\neq\varnothing,$$
$$Y_T^\alpha\supseteq Z_9, Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_i\cup Y_j, Y_{T'}^\alpha\cap\left(Y_i\cup Y_j\right)\neq\varnothing, Y_{T''}^\alpha\cap\breve{D}\neq\varnothing$$

respectively, *i.e.*, $\alpha\in R\left(D_i'\right)\cap R\left(D_j'\right)$ or $\alpha\in R\left(D_i'\right)\cap R\left(D_j'\right)\cap R\left(D_k'\right)$ if and only if

$$Y_T^\alpha\supseteq Z_9, Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_i\cup Y_j, Y_{T'}^\alpha\cap Y_i\neq\varnothing, Y_{T'}^\alpha\cap Y_j\neq\varnothing, Y_{T''}^\alpha\cap\breve{D}\neq\varnothing.$$

Therefore the equality $R\left(D_i'\right)\cap R\left(D_j'\right)=R\left(D_i'\right)\cap R\left(D_j'\right)\cap R\left(D_k'\right)$ is true. From the last equality and by definition of the semilattice D we have $R\left(D_i'\right)\cap R\left(D_j'\right)=R\left(D_i'\right)\cap R\left(D_j'\right)\cap R\left(D_k'\right)$ for all $(i,j,k)\in M_3\left(Q_3\right)$, where

$$M_3\left(Q_3\right)=\{(1,2,6),(1,3,6),(2,3,6),(3,4,8),(3,5,8),(4,5,8)\}.$$

3) $D_i'=\left\{Z_9,Y_i,\breve{D}\right\},D_j'=\left\{Z_9,Y_j,\breve{D}\right\},D_k'=\left\{Z_9,Y_k,\breve{D}\right\}$, $D_t'=\left\{Z_9,Y_i\cup Y_j\cup Y_k,\breve{D}\right\}\in\left\{D_1',D_2',\cdots,D_8'\right\}$,

$D'_p \neq D'_q$, $p,q \in \{i,j,k,t\}$, $p \neq q$, $\alpha \in R(D'_i) \cap R(D'_j) \cap R(D'_k)$ and $\alpha \in R(D'_i) \cap R(D'_j) \cap R(D'_k) \cap R(D'_t)$ are true. Then we have

$$Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_i, Y_{T'}^\alpha \cap Y_i \neq \varnothing, Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_j, Y_{T'}^\alpha \cap Y_j \neq \varnothing, Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_k, Y_{T'}^\alpha \cap Y_k \neq \varnothing, Y_{T''}^\alpha \cap \breve{D} \neq \varnothing$$

and

$$Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_i, Y_{T'}^\alpha \cap Y_i \neq \varnothing, Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_j, Y_{T'}^\alpha \cap Y_j \neq \varnothing, Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_k, Y_{T'}^\alpha \cap Y_k \neq \varnothing, Y_{T''}^\alpha \cap \breve{D} \neq \varnothing$$
$$Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_i \cup Y_j \cup Y_k, Y_{T'}^\alpha \cap (Y_i \cup Y_j \cup Y_k) \neq \varnothing, Y_{T''}^\alpha \cap \breve{D} \neq \varnothing$$

respectively, *i.e.*, $\alpha \in R(D'_i) \cap R(D'_j) \cap R(D'_k)$ and $\alpha \in R(D'_i) \cap R(D'_j) \cap R(D'_k) \cap R(D'_t)$ if and only if

$$Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_i \cup Y_j \cup Y_k, Y_{T'}^\alpha \cap Y_i \neq \varnothing, Y_{T'}^\alpha \cap Y_j \neq \varnothing, Y_{T'}^\alpha \cap Y_k \neq \varnothing, Y_{T''}^\alpha \cap \breve{D} \neq \varnothing.$$

Therefore the equality $R(D'_i) \cap R(D'_j) \cap R(D'_k) = R(D'_i) \cap R(D'_j) \cap R(D'_k) \cap R(D'_t)$ is true. From the last equality and by definition of the semilattice D we have

$$R(D'_i) \cap R(D'_j) \cap R(D'_k) = R(D'_i) \cap R(D'_j) \cap R(D'_k) \cap R(D'_t)$$

for all $(i,j,k,t) \in M_4(Q_3)$, where

$$M_4(Q_3) = \{(1,2,3,6),(3,4,5,8)\}.$$

Now, by equality (4) and conditions 1), 2) and 3) it follows that the following equality is true

$$\left|R^*(Q_3)\right| = \sum_{i=1}^{8} \left|R(D'_i)\right| - \sum_{(j,k)\in M_5(Q_3)} \left|R(D'_j) \cap R(D'_k)\right|$$

where

$$M_5(Q_3) = \{(2,6),(3,6),(3,7),(3,8),(4,8),(5,8)\}.$$ □

Lemma 7. *Let* $D' = \{Z_9, Y, \breve{D}\}$, $D'' = \{Z_9, Y', \breve{D}\}$ *where* $Y, Y' \in D$ *and* $Y' \supseteq Y$. *If quasinormal representation of binary relation* α *of the semigroup* $B_X(D)$ *has a form* $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_0^\alpha \times \breve{D})$ *for some* $T, T' \in D$, $T \subset T' \subset \breve{D}$ *and* $Y_T^\alpha, Y_{T'}^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* $\alpha \in R(D') \cap R(D'')$ *iff*

$$Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y', Y_{T'}^\alpha \cap Y \neq \varnothing, Y_0^\alpha \cap \breve{D} \neq \varnothing.$$

Proof. If $\alpha \in R(D') \cap R(D'')$, then by statement c) of Theorem 3 we have

$$\begin{aligned} &Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y, Y_{T'}^\alpha \cap Y \neq \varnothing, Y_0^\alpha \cap \breve{D} \neq \varnothing; \\ &Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y', Y_{T'}^\alpha \cap Y' \neq \varnothing, Y_0^\alpha \cap \breve{D} \neq \varnothing. \end{aligned} \tag{6}$$

From the last condition we have

$$Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y', Y_{T'}^\alpha \cap Y \neq \varnothing, Y_0^\alpha \cap \breve{D} \neq \varnothing, \tag{7}$$

since $Y' \supseteq Y$ by assumption.

On the other hand, if the conditions of (7) holds, then (6) immediately follows, *i.e.* $\alpha \in R(D') \cap R(D'')$. Lemma is proved. □

Lemma 8. *Let* $D' = \{Z_9, Y, \breve{D}\}, D'' = \{Z_9, Y', \breve{D}\} \in \{D'_1, D'_2, \cdots, D'_8\}$, $Y' \supseteq Y$ *and X be a finite set. Then the following equality holds*

$$\left|R(D') \cap R(D'')\right| = 22 \cdot 2^{|Y' \setminus Y|} \cdot \left(2^{|Y \setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Y'|} - 2^{|\breve{D} \setminus Y'|}\right) \cdot 3^{|X \setminus \breve{D}|}.$$

Proof. Let $D' = \{Z_9, Y, \breve{D}\}, D'' = \{Z_9, Y', \breve{D}\} \in \{D'_1, D'_2, \cdots, D'_8\}$, where $Y' \supseteq Y$. Assume that

$\alpha \in R(D') \cap R(D'')$ and a quasinormal representation of a regular binary relation α has a form $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_0^\alpha \times \breve{D})$ for some $T, T' \in D$, $T \subset T' \subset \breve{D}$ and $Y_T^\alpha, Y_{T'}^\alpha, Y_0^\alpha \notin \{\varnothing\}$. Then according to Lemma 7, we have

$$Y_T^\alpha \supseteq Z_9, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y', Y_{T'}^\alpha \cap Y \neq \varnothing, Y_0^\alpha \cap \breve{D} \neq \varnothing \tag{8}$$

Further, let f_α be a mapping of the set X in the semilattice D satisfying the conditions $f_\alpha(t) = t\alpha$ for all $t \in X$. $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$ are the restrictions of the mapping f_α on the sets Z_9, $Y' \setminus Z_9$, $\breve{D} \setminus Y'$, $X \setminus \breve{D}$ respectively. It is clear that the intersection of elements of the set $\{Z_9, Y' \setminus Z_9, \breve{D} \setminus Y', X \setminus \breve{D}\}$ is an empty set, and $Z_9 \cup (Y' \setminus Z_9) \cup (\breve{D} \setminus Y') \cup (X \setminus \breve{D}) = X$. We are going to find properties of the maps $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$.

1) $t \in Z_9$. Then by the properties (1) we have $Z_9 \subseteq Y_T^\alpha$, *i.e.*, $t \in Y_T^\alpha$ and $t\alpha = T$ by definition of the set Y_T^α. Therefore $f_{0\alpha}(t) = T$ for all $t \in Z_9$.

2) $t \in Y' \setminus Z_9$. Then by the properties (1) we have $Y' \setminus Z_9 \subseteq Y' \subseteq Y_T^\alpha \cup Y_{T'}^\alpha$, *i.e.*, $t \in Y_T^\alpha \cup Y_{T'}^\alpha$ and $t\alpha \in \{T, T'\}$ by definition of the sets Y_T^α and $Y_{T'}^\alpha$. Therefore $f_{1\alpha}(t) \in \{T, T'\}$ for all $t \in Y' \setminus Z_9$.

By suppose we have that $Y_{T'}^\alpha \cap Y \neq \varnothing$, *i.e.* $t'\alpha = T'$ for some $t' \in Y$. If $t' \in Z_9$, then $t' \in Z_9 \subseteq Y_T^\alpha$. Therefore $t'\alpha = T$. That is contradiction to the equality $t'\alpha = T'$, while $T \neq T'$ by definition of the semilattice D.

Therefore $f_{1\alpha}(t') = T'$ for some $t' \in Y \setminus Z_9$.

3) $t \in \breve{D} \setminus Y'$. Then by properties (1) we have $\breve{D} \setminus Y' \subseteq \breve{D} \subseteq X = Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_0^\alpha$, *i.e.*, $t \in Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_0^\alpha$ and $t\alpha \in \{T, T', \breve{D}\}$ by definition of the sets Y_T^α, $Y_{T'}^\alpha$ and Y_0^α. Therefore $f_{3\alpha}(t) \in \{T, T', \breve{D}\}$ for all $t \in \breve{D} \setminus Y'$.

By suppose we have that $Y_0^\alpha \cap \breve{D} \neq \varnothing$, *i.e.* $t''\alpha = \breve{D}$ for some $t'' \in \breve{D}$. If $t'' \in Y'$, then $t'' \in Y' \subseteq Y_T^\alpha \cup Y_{T'}^\alpha$. Therefore $t''\alpha \in \{T, T'\}$ by definition of the set Y_T^α and $Y_{T'}^\alpha$. We have contradiction to the equality $t''\alpha = \breve{D}$.

Therefore $f_{3\alpha}(t'') = \breve{D}$ for some $t'' \in \breve{D} \setminus Y'$.

4) $t \in X \setminus \breve{D}$. Then by definition of a quasinormal representation of a binary relation α and by property (1) we have $t \in X \setminus \breve{D} \subseteq X = Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_0^\alpha$, *i.e.* $t\alpha \in \{T, T', \breve{D}\}$ by definition of the sets $Y_T^\alpha, Y_{T'}^\alpha$ and Y_0^α. Therefore $f_{4\alpha}(t) \in \{T, T', \breve{D}\}$ for all $t \in X \setminus \breve{D}$.

We have seen that for every binary relation $\alpha \in R(D') \cap R(D'')$ there exists ordered system $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha})$. It is obvious that for disjoint binary relations there exist disjoint ordered systems.

Further, let

$$f_0 : Z_9 \to \{T\}, \; f_1 : Y' \setminus Z_9 \to \{T, T'\},$$

$$f_2 : \breve{D} \setminus Y' \to \{T, T', \breve{D}\}, \; f_3 : X \setminus \breve{D} \to \{T, T', \breve{D}\}$$

be such mappings that satisfy the conditions:

$f_0(t) = T$ for all $t \in Z_9$;

$f_1(t) \in \{T, T'\}$ for all $t \in Y' \setminus Z_9$ and $f_1(t') = T'$ for some $t' \in Y \setminus Z_9$;

$f_2(t) \in \{T, T', \breve{D}\}$ for all $t \in \breve{D} \setminus Y'$ and $f_2(t'') = \breve{D}$ for some $t'' \in \breve{D} \setminus Y'$;

$f_3(t) \in \{T, T', \breve{D}\}$ for all $t \in X \setminus \breve{D}$.

Now we define a map *f* from *X* to the semilattice *D*, which satisfies the condition:

$$f(t) = \begin{cases} f_0(t), \text{if } t \in Z_9, \\ f_1(t), \text{if } t \in Y' \setminus Z_9, \\ f_2(t), \text{if } t \in \breve{D} \setminus Y', \\ f_3(t), \text{if } t \in X \setminus \breve{D}. \end{cases}$$

Further, let $\beta = \bigcup_{x \in X} (\{x\} \times f(x))$, $Y_T^\beta = \{t \mid t\beta = T\}$, $Y_{T'}^\beta = \{t \mid t\beta = T'\}$ and $Y_0^\beta = \{t \mid t\beta = \breve{D}\}$. Then binary relation β may be represented by

$$\beta = (Y_T^\beta \times T) \cup (Y_{T'}^\beta \times T') \cup (Y_0^\beta \times \breve{D})$$

and satisfies the conditions

$$Y_T^{\beta} \supseteq Z_9, Y_T^{\beta} \cup Y_{T'}^{\beta} \supseteq Y', Y_{T'}^{\beta} \cap Y \neq \varnothing, Y_0^{\beta} \cap \breve{D} \neq \varnothing.$$

(By suppose $f_1(t') = T'$ for some $t' \in Y \setminus Z_9$ and $f_2(t'') = \breve{D}$ for some $t'' \in \breve{D} \setminus Y'$), *i.e.*, by lemma 7 we have that $\beta \in R(D') \cap R(D'')$.

Therefore for every binary relation $\alpha \in R(D') \cap R(D'')$ and ordered system $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha})$ there exists one to one mapping.

By Lemma 1 and by Theorem 1 the number of the mappings $f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha}, f_{4\alpha}$ are respectively

$$1, 2^{|(Y'\setminus Z_9)\setminus(Y\setminus Z_9)|} \cdot \left(2^{|Y\setminus Z_9|} - 1\right), 3^{|\breve{D}\setminus Y'|} - 2^{|\breve{D}\setminus Y'|}, 3^{|X\setminus\breve{D}|}.$$

Note that the number $2^{|Y'\setminus(Y\cup Z_9)|} \cdot \left(2^{|Y\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Y'|} - 2^{|\breve{D}\setminus Y'|}\right) \cdot 3^{|X\setminus\breve{D}|}$ does not depend on choice of chains $T \subset T' \subset T''$ $(T, T', T'' \in D)$ of the semilattice D. Since the number of such different chains of the semilattice D is equal to 22, for arbitrary $T, T', T'' \in D$ where $T \subset T' \subset T''$, the number of regular elements of the set $R(D') \cap R(D'')$ is equal to

$$|R(D') \cap R(D'')| = 22 \cdot 2^{|Y'\setminus Y|} \cdot \left(2^{|Y\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Y'|} - 2^{|\breve{D}\setminus Y'|}\right) \cdot 3^{|X\setminus\breve{D}|}. \qquad \square$$

Therefore we obtain

$$\begin{aligned}
|R(D_1') \cap R(D_6')| &= 22 \cdot 2^{|Z_3\setminus Z_8|} \cdot \left(2^{|Z_8\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_3|} - 2^{|\breve{D}\setminus Z_3|}\right) \cdot 3^{|X\setminus\breve{D}|}, \\
|R(D_2') \cap R(D_6')| &= 22 \cdot 2^{|Z_3\setminus Z_7|} \cdot \left(2^{|Z_7\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_3|} - 2^{|\breve{D}\setminus Z_3|}\right) \cdot 3^{|X\setminus\breve{D}|}, \\
|R(D_3') \cap R(D_6')| &= 22 \cdot 2^{|Z_3\setminus Z_6|} \cdot \left(2^{|Z_6\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_3|} - 2^{|\breve{D}\setminus Z_3|}\right) \cdot 3^{|X\setminus\breve{D}|}, \\
|R(D_3') \cap R(D_7')| &= 22 \cdot 2^{|Z_2\setminus Z_6|} \cdot \left(2^{|Z_6\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_2|} - 2^{|\breve{D}\setminus Z_2|}\right) \cdot 3^{|X\setminus\breve{D}|}, \\
|R(D_3') \cap R(D_8')| &= 22 \cdot 2^{|Z_1\setminus Z_6|} \cdot \left(2^{|Z_6\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_1|} - 2^{|\breve{D}\setminus Z_1|}\right) \cdot 3^{|X\setminus\breve{D}|}, \\
|R(D_4') \cap R(D_8')| &= 22 \cdot 2^{|Z_1\setminus Z_5|} \cdot \left(2^{|Z_5\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_1|} - 2^{|\breve{D}\setminus Z_1|}\right) \cdot 3^{|X\setminus\breve{D}|}, \\
|R(D_5') \cap R(D_8')| &= 22 \cdot 2^{|Z_1\setminus Z_4|} \cdot \left(2^{|Z_4\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_1|} - 2^{|\breve{D}\setminus Z_1|}\right) \cdot 3^{|X\setminus\breve{D}|}.
\end{aligned} \qquad (9)$$

Lemma 9. *Let* X *be a finite set,* $D = \{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_1(X, 10)$ *and* $Z_9 \neq \varnothing$. *Let* $R^*(Q_3)$ *be set of all regular elements of* $B_X(D)$ *such that each element satisfies the condition* c) *of Theorem* 3. *Then*

$$\begin{aligned}
|R^*(Q_3)| = {} & 22 \cdot \left(2^{|Z_8\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_8|} - 2^{|\breve{D}\setminus Z_8|}\right) \cdot 3^{|X\setminus\breve{D}|} + 22 \cdot \left(2^{|Z_7\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_7|} - 2^{|\breve{D}\setminus Z_7|}\right) \cdot 3^{|X\setminus\breve{D}|} \\
& + 22 \cdot \left(2^{|Z_6\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_6|} - 2^{|\breve{D}\setminus Z_6|}\right) \cdot 3^{|X\setminus\breve{D}|} + 22 \cdot \left(2^{|Z_5\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_5|} - 2^{|\breve{D}\setminus Z_5|}\right) \cdot 3^{|X\setminus\breve{D}|} \\
& + 22 \cdot \left(2^{|Z_4\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_4|} - 2^{|\breve{D}\setminus Z_4|}\right) \cdot 3^{|X\setminus\breve{D}|} + 22 \cdot \left(2^{|Z_3\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_3|} - 2^{|\breve{D}\setminus Z_3|}\right) . 3^{|X\setminus\breve{D}|} \\
& + 22 \cdot \left(2^{|Z_2\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_2|} - 2^{|\breve{D}\setminus Z_2|}\right) \cdot 3^{|X\setminus\breve{D}|} + 22 \cdot \left(2^{|Z_1\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_1|} - 2^{|\breve{D}\setminus Z_1|}\right) \cdot 3^{|X\setminus\breve{D}|} \\
& - 22 \cdot 2^{|Z_3\setminus Z_8|} \cdot \left(2^{|Z_8\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_3|} - 2^{|\breve{D}\setminus Z_3|}\right) \cdot 3^{|X\setminus\breve{D}|} - 22 \cdot 2^{|Z_3\setminus Z_7|} \cdot \left(2^{|Z_7\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_3|} - 2^{|\breve{D}\setminus Z_3|}\right) \cdot 3^{|X\setminus\breve{D}|} \\
& - 22 \cdot 2^{|Z_3\setminus Z_6|} \cdot \left(2^{|Z_6\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_3|} - 2^{|\breve{D}\setminus Z_3|}\right) \cdot 3^{|X\setminus\breve{D}|} - 22 \cdot 2^{|Z_2\setminus Z_6|} \cdot \left(2^{|Z_6\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_2|} - 2^{|\breve{D}\setminus Z_2|}\right) \cdot 3^{|X\setminus\breve{D}|} \\
& - 22 \cdot 2^{|Z_1\setminus Z_6|} \cdot \left(2^{|Z_6\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_1|} - 2^{|\breve{D}\setminus Z_1|}\right) \cdot 3^{|X\setminus\breve{D}|} - 22 \cdot 2^{|Z_1\setminus Z_5|} \cdot \left(2^{|Z_5\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_1|} - 2^{|\breve{D}\setminus Z_1|}\right) \cdot 3^{|X\setminus\breve{D}|} \\
& - 22 \cdot 2^{|Z_1\setminus Z_4|} \cdot \left(2^{|Z_4\setminus Z_9|} - 1\right) \cdot \left(3^{|\breve{D}\setminus Z_1|} - 2^{|\breve{D}\setminus Z_1|}\right) \cdot 3^{|X\setminus\breve{D}|}.
\end{aligned}$$

Proof. The given Lemma immediately follows from Lemma 6 and from the Equalities (5).

Now let a binary relation α of the semigroup $B_X(D)$ satisfy the condition (d) of Theorem 3 (see diagram 4 of the **Figure 3**). In this case we have $Q_4 = \{Z_9, T, T', \breve{D}\}$, where $T, T' \in D$ and $Z_9 \subset T \subset T' \subset \breve{D}$. By definition of the semilattice D it follows that

$$Q_4\vartheta_{XI} = \{\{Z_9, Z_8, Z_3, \breve{D}\}, \{Z_9, Z_7, Z_3, \breve{D}\}, \{Z_9, Z_6, Z_3, \breve{D}\}, \{Z_9, Z_6, Z_2, \breve{D}\}, \{Z_9, Z_6, Z_1, \breve{D}\}, \{Z_9, Z_5, Z_1, \breve{D}\}, \{Z_9, Z_4, Z_1, \breve{D}\}\}.$$

It is easy to see $|\Phi(Q_4, Q_4)| = 1$ and $|\Omega(Q_4)| = 7$. If

$$D_1' = \{Z_9, Z_8, Z_3, \breve{D}\},\ D_2' = \{Z_9, Z_7, Z_3, \breve{D}\},\ D_3' = \{Z_9, Z_6, Z_3, \breve{D}\},$$
$$D_4' = \{Z_9, Z_6, Z_2, \breve{D}\},\ D_5' = \{Z_9, Z_6, Z_1, \breve{D}\},\ D_6' = \{Z_9, Z_5, Z_1, \breve{D}\},$$
$$D_7' = \{Z_9, Z_4, Z_1, \breve{D}\};$$

then

$$R^*(Q_4) = \bigcup_{i=1}^{7} R(D_i'). \tag{10}$$

Lemma 10. *Let X be a finite set,*

$$D = \{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_1(X, 10)$$

and $Z_9 \neq \varnothing$. Let $R^*(Q_4)$ be set of all regular elements of $B_X(D)$ such that each element satisfies the condition d) of Theorem 3. Then

$$\begin{aligned}
|R^*(Q_4)| &= 7\cdot\left(2^{|Z_8\setminus Z_9|}-1\right)\cdot\left(3^{|Z_3\setminus Z_8|}-2^{|Z_3\setminus Z_8|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+7\cdot\left(2^{|Z_7\setminus Z_9|}-1\right)\cdot\left(3^{|Z_3\setminus Z_7|}-2^{|Z_3\setminus Z_7|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+7\cdot\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot\left(3^{|Z_3\setminus Z_6|}-2^{|Z_3\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+7\cdot\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+7\cdot\left(2^{|Z_6\setminus Z_9|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+7\cdot\left(2^{|Z_5\setminus Z_9|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+7\cdot\left(2^{|Z_4\setminus Z_9|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}.
\end{aligned}$$

Proof. Let $D_i' = \{Z_9, Y_i, Y_i', \breve{D}\}, D_j' = \{Z_9, Y_j, Y_j', \breve{D}\} \in \{D_1', D_2', \cdots, D_7'\}$, $D_i' \neq D_j'$ and $\alpha \in D_i' \cap D_j'$. Then $\alpha = (Y_9^\alpha \times Z_9) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_0^\alpha \times \breve{D})$, where $Z_9 \subset T \subset T' \subset \breve{D}$, $Y_9^\alpha, Y_T^\alpha, Y_{T'}^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and the following inclusions and inequalities are true

$$Y_9^\alpha \supseteq Z_9,\ Y_9^\alpha \cup Y_T^\alpha \supseteq Y_i,\ Y_9^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_i',$$
$$Y_T^\alpha \cap Y_i \neq \varnothing,\ Y_{T'}^\alpha \cap Y_i' \neq \varnothing,\ Y_0^\alpha \cap \breve{D} \neq \varnothing;$$
$$Y_9^\alpha \supseteq Z_9,\ Y_9^\alpha \cup Y_T^\alpha \supseteq Y_j,\ Y_9^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_j',$$
$$Y_T^\alpha \cap Y_j \neq \varnothing,\ Y_{T'}^\alpha \cap Y_j' \neq \varnothing,\ Y_0^\alpha \cap \breve{D} \neq \varnothing.$$

From this it follows that

$$Y_9^\alpha \supseteq Z_9,\ Y_9^\alpha \cup Y_T^\alpha \supseteq Y_i \cup Y_j,\ Y_9^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_i' \cup Y_j'.$$

We consider the following cases.

1) $Y_i \cup Y_j = \breve{D}$ or $Y_i' \cup Y_j' = \breve{D}$. Then we have $(Y_9^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha) \cap Y_0^\alpha \supseteq \breve{D} \cap Y_0^\alpha \neq \varnothing$. But the inequality

$\left(Y_9^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha\right) \cap Y_0^\alpha \neq \varnothing$ contradicts the condition that representation of binary relation α is quasinormal. So, the equality $R\left(D_i'\right) \cap R\left(D_j'\right) = \varnothing$ holds. From the last equality and by definition of the semilattice D we have $R\left(D_i'\right) \cap R\left(D_j'\right) = \varnothing$ for all $(i, j) \in M_1\left(Q_4\right)$, where

$$M_1\left(Q_4\right) = \{(1,4),(1,5),(1,6),(1,7),(2,4),(2,5),(2,6),(2,7),(3,4),(3,5),(3,6),(3,7),(4,5),(4,6),(4,7)\}. \quad (10a)$$

2) $Y_i \cup Y_j = Z_3 \in \{Y_i', Y_j'\}$ or $Y_i \cup Y_j = Z_1 \in \{Y_i', Y_j'\}$ Then we have $\left(Y_9^\alpha \cup Y_T^\alpha\right) \cap Y_{T'}^\alpha \supseteq Z_3 \cap Y_{T'}^\alpha \neq \varnothing$ or $\left(Y_9^\alpha \cup Y_T^\alpha\right) \cap Y_{T'}^\alpha \supseteq Z_1 \cap Y_{T'}^\alpha \neq \varnothing$. But the inequality $\left(Y_9^\alpha \cup Y_T^\alpha\right) \cap Y_{T'}^\alpha \neq \varnothing$ or $\left(Y_9^\alpha \cup Y_T^\alpha\right) \cap Y_{T'}^\alpha \neq \varnothing$ contradicts the condition that representation of binary relation α is quasinormal. So, the equality $R\left(D_i'\right) \cap R\left(D_j'\right) = \varnothing$ holds. From the last equality and by definition of the semilattice D we have $R\left(D_i'\right) \cap R\left(D_j'\right) = \varnothing$ for all $(i, j) \in M_2\left(Q_4\right)$, where

$$M_2\left(Q_4\right) = \{(1,2),(1,3),(2,3),(5,6),(5,7),(6,7)\} \quad (10b)$$

By conditions (10a) and (10b) it follows that

$$\left|R^*\left(Q_4\right)\right| = \sum_{i=1}^{7}\left|R\left(D_i'\right)\right|.$$

From the last equality we have that the given Lemma is true. □

Now let a binary relation α of the semigroup $B_X(D)$ satisfy the condition e) of Theorem 3 (see diagram 5 of the **Figure 3**). In this case we have $Q_5 = \{Z_9, T, T', T \cup T'\}$, where $T, T' \in D$ and $T \setminus T' \neq \varnothing$ and $T' \setminus T \neq \varnothing$. By definition of the semilattice D it follows that

$$\begin{aligned} Q_5\vartheta_{XI} = {} & \{Z_9,Z_5,Z_4,Z_1\},\{Z_9,Z_6,Z_4,Z_1\},\{Z_9,Z_6,Z_5,Z_1\},\{Z_9,Z_7,Z_6,Z_3\},\{Z_9,Z_8,Z_6,Z_3\},\\ & \{Z_9,Z_8,Z_7,Z_3\},\{Z_9,Z_8,Z_4,\breve{D}\},\{Z_9,Z_8,Z_5,\breve{D}\},\{Z_9,Z_7,Z_2,\breve{D}\},\{Z_9,Z_7,Z_4,\breve{D}\},\\ & \{Z_9,Z_7,Z_5,\breve{D}\},\{Z_9,Z_8,Z_1,\breve{D}\},\{Z_9,Z_8,Z_2,\breve{D}\},\{Z_9,Z_4,Z_2,\breve{D}\},\{Z_9,Z_4,Z_3,\breve{D}\},\\ & \{Z_9,Z_5,Z_2,\breve{D}\},\{Z_9,Z_5,Z_3,\breve{D}\},\{Z_9,Z_7,Z_1,\breve{D}\},\{Z_9,Z_2,Z_1,\breve{D}\},\{Z_9,Z_3,Z_1,\breve{D}\},\\ & \{Z_9,Z_3,Z_2,\breve{D}\},\{Z_6,Z_2,Z_1,\breve{D}\},\{Z_6,Z_3,Z_1,\breve{D}\},\{Z_6,Z_3,Z_2,\breve{D}\}. \end{aligned}$$

It is easy to see $\left|\Phi\left(Q_5, Q_5\right)\right| = 2$ and $\left|\Omega\left(Q_5\right)\right| = 24$. If

$$\begin{aligned} & D_1' = \{Z_9,Z_8,Z_7,Z_3\},\ D_2' = \{Z_9,Z_7,Z_8,Z_3\},\ D_3' = \{Z_9,Z_8,Z_6,Z_3\},\\ & D_4' = \{Z_9,Z_6,Z_8,Z_3\}, D_5' = \{Z_9,Z_8,Z_5,\breve{D}\},\ D_6' = \{Z_9,Z_5,Z_8,\breve{D}\}, D_7' = \{Z_9,Z_8,Z_4,\breve{D}\},\\ & D_8' = \{Z_9,Z_4,Z_8,\breve{D}\}, D_9' = \{Z_9,Z_8,Z_2,\breve{D}\}, D_{10}' = \{Z_9,Z_2,Z_8,\breve{D}\}, D_{11}' = \{Z_9,Z_8,Z_1,\breve{D}\},\\ & D_{12}' = \{Z_9,Z_1,Z_8,\breve{D}\}, D_{13}' = \{Z_9,Z_7,Z_6,Z_3\},\ D_{14}' = \{Z_9,Z_6,Z_7,Z_3\},\ D_{15}' = \{Z_9,Z_7,Z_5,\breve{D}\},\\ & D_{16}' = \{Z_9,Z_5,Z_7,\breve{D}\},\ D_{17}' = \{Z_9,Z_7,Z_4,\breve{D}\},\ D_{18}' = \{Z_9,Z_4,Z_7,\breve{D}\}, D_{19}' = \{Z_9,Z_7,Z_2,\breve{D}\},\\ & D_{20}' = \{Z_9,Z_2,Z_7,\breve{D}\}, D_{21}' = \{Z_9,Z_7,Z_1,\breve{D}\}, D_{22}' = \{Z_9,Z_1,Z_7,\breve{D}\},\ D_{23}' = \{Z_9,Z_6,Z_5,Z_1\},\\ & D_{24}' = \{Z_9,Z_5,Z_6,Z_1\}, D_{25}' = \{Z_9,Z_6,Z_4,Z_1\},\ D_{26}' = \{Z_9,Z_4,Z_6,Z_1\},\ D_{27}' = \{Z_9,Z_5,Z_4,Z_1\},\\ & D_{28}' = \{Z_9,Z_4,Z_5,Z_1\}, D_{29}' = \{Z_9,Z_5,Z_3,\breve{D}\},\ D_{30}' = \{Z_9,Z_3,Z_5,\breve{D}\}, D_{31}' = \{Z_9,Z_5,Z_2,\breve{D}\},\\ & D_{32}' = \{Z_9,Z_2,Z_5,\breve{D}\},\ D_{33}' = \{Z_9,Z_4,Z_3,\breve{D}\}, D_{34}' = \{Z_9,Z_3,Z_4,\breve{D}\},\ D_{35}' = \{Z_9,Z_4,Z_2,\breve{D}\},\\ & D_{36}' = \{Z_9,Z_2,Z_4,\breve{D}\},\ D_{37}' = \{Z_9,Z_3,Z_2,\breve{D}\},\ D_{38}' = \{Z_9,Z_2,Z_3,\breve{D}\},\ D_{39}' = \{Z_9,Z_3,Z_1,\breve{D}\},\\ & D_{40}' = \{Z_9,Z_1,Z_3,\breve{D}\}, D_{41}' = \{Z_9,Z_2,Z_1,\breve{D}\},\ D_{42}' = \{Z_9,Z_1,Z_2,\breve{D}\}, D_{43}' = \{Z_6,Z_2,Z_1,\breve{D}\},\\ & D_{44}' = \{Z_6,Z_1,Z_2,\breve{D}\}, D_{45}' = \{Z_6,Z_3,Z_1,\breve{D}\}, D_{46}' = \{Z_6,Z_1,Z_3,\breve{D}\},\ D_{47}' = \{Z_6,Z_3,Z_2,\breve{D}\},\\ & D_{48}' = \{Z_6,Z_2,Z_3,\breve{D}\}. \end{aligned}$$

then

$$R^*(Q_5)=\bigcup_{i=1}^{48}R(D_i'). \quad (11)$$

Lemma 11. *Let X be a finite set,*

$$D=\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_1(X,10)$$

and $Z_9\neq\varnothing$. Let $R^*(Q_5)$ be set of all regular elements of $B_X(D)$ such that each element satisfies the condition e) of Theorem 3. Then

$$\left|R^*(Q_5)\right|=\sum_{i=1}^{42}\left|R(D_i')\right|-\sum_{(k,j)\in M_4'}\left|R(D_k')\cap R(D_j')\right|,$$

where

$$\begin{aligned}M_4'(Q_5)=\{&(3,9),(3,11),(4,10),(4,12),(5,11),(5,30),(5,39),(6,12),(6,29),\\&(6,40),(7,11),(7,34),(7,39),(8,12),(8,33),(8,40),(9,37),\\&(10,38),(11,39),(12,40),(13,19),(13,21),(14,20),(14,22),\\&(15,21),(15,30),(15,39),(16,22),(16,29),(16,40),(17,21),\\&(17,34),(17,39),(18,22),(18,33),(18,40),(19,37),(20,38),\\&(21,39),(22,40),(23,30),(23,32),(23,39),(23,41),(24,29),\\&(24,31),(25,34),(25,36),(26,33),(26,35),(29,40),(30,39),\\&(31,42),(32,41),(33,40),(34,39),(35,42),(36,41)\}.\end{aligned}$$

Proof. Let $\tilde{D}=\{Z,Z',Z'',Z'\cup Z''\}$ be arbitrary element of the set $\{D_1',D_2',\cdots,D_{48}'\}$ and $\alpha\in R(D')$. Then quasinormal representation of a binary relation α of the semigroup $B_X(D)$ has a form

$$\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times(T'\cup T'')\right),$$

where $T,T',T''\in D$, $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$ and by statement e) of Theorem 3 satisfies the following conditions

$$Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z',Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Z'',Y_{T'}^\alpha\cap Z'\neq\varnothing,Y_{T''}^\alpha\cap Z''\neq\varnothing.$$

From this we have that the inclusions

$$\begin{aligned}&R(D_{43}')\subseteq R(D_{41}'),R(D_{44}')\subseteq R(D_{42}'),R(D_{45}')\subseteq R(D_{39}'),\\&R(D_{46}')\subseteq R(D_{40}'),R(D_{47}')\subseteq R(D_{41}'),R(D_{37}')\subseteq R(D_{38}').\end{aligned}$$

are fulfilled. Therefore from the Equality (1) it follows that

$$R^*(Q_5)=\bigcup_{i=1}^{42}R(D_i'). \quad (12)$$

Let $D_i'=\{Y_i,Y_i',Y_i''Y_i'\cup Y_i''\}$ and $D_j'=\{Y_j,Y_j',Y_j'',Y_j'\cup Y_j''\}$ be such elements of the set $\{D_1',D_2',\cdots,D_{42}'\}$ that $D_i'\neq D_j'$ and $\alpha\in R(D_i')\cap R(D_j')$. Then quasinormal representation of a binary relation α of the semigroup $B_X(D)$ has a form

$$\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times(T'\cup T'')\right),$$

where $T,T',T''\in D$, $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$ and by statement e) of Theorem 3 satisfies the following conditions

$$Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z',\quad Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Z'',\quad Y_{T'}^\alpha\cap Z'\neq\varnothing \text{ and } Y_{T''}^\alpha\cap Z''\neq\varnothing.$$

Then by statement e) of Theorem 3 we have

$$\begin{aligned}&Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_i',Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Y_i'',Y_{T'}^\alpha\cap Y_i'\neq\varnothing,Y_{T''}^\alpha\cap Y_i''\neq\varnothing,\\&Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_j',Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Y_j'',Y_{T'}^\alpha\cap Y_j'\neq\varnothing,Y_{T''}^\alpha\cap Y_j''\neq\varnothing.\end{aligned}$$

From this conditions it follows that

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_i' \cup Y_j',\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y_i'' \cup Y_j''.$$

For D_i' and D_j' we consider the following cases.

1) $Y_i' \cup Y_j' = \breve{D}$ or $Y_i'' \cup Y_j'' = \breve{D}$. Then $\left(Y_T^\alpha \cup Y_{T'}^\alpha\right) \cap Y_{T''}^\alpha \supseteq \breve{D} \cap Y_{T''}^\alpha \supseteq Y_i'' \cap Y_{T''}^\alpha \neq \varnothing$ or $\left(Y_T^\alpha \cup Y_{T''}^\alpha\right) \cap Y_{T'}^\alpha \supseteq \breve{D} \cap Y_{T'}^\alpha \supseteq Y_i' \cap Y_{T'}^\alpha \neq \varnothing$ respectively. But the inequalities $\left(Y_T^\alpha \cup Y_{T'}^\alpha\right) \cap Y_{T''}^\alpha \neq \varnothing$ and $\left(Y_T^\alpha \cup Y_{T''}^\alpha\right) \cap Y_{T'}^\alpha \neq \varnothing$ contradict the condition that representation of binary relation α is quasinormal. So, the equality $R\left(D_i'\right) \cap R\left(D_j'\right) = \varnothing$ holds. From the last equality it follows that $R\left(D_i'\right) \cap R\left(D_j'\right) = \varnothing$ for all $(i, j) \in \bigcup_{i=1}^{41} M_i$, where

$$M_1(Q_5) = \{(1,5),(1,6),(1,7),(1,8),(1,9),(1,10),(1,11),(1,12),(1,15),(1,16),(1,17),$$
$$(1,18),(1,19),(1,20),(1,21),(1,22),(1,23),(1,24),(1,25),(1,26),(1,27),$$
$$(1,28),(1,29),(1,30),(1,31),(1,32),(1,33),(1,34),(1,35),(1,36),(1,37),$$
$$(1,38),(1,39),(1,40),(1,41),(1,42)\};$$

$$M_2(Q_5) = \{(2,5),(2,6),(2,7),(2,8),(2,9),(2,10),(2,11),(2,12),(2,15),(2,16),$$
$$(2,17),(2,18),(2,19),(2,20),(2,21),(2,22),(2,23),(2,24),(2,25),$$
$$(2,26),(2,27),(2,28),(2,29),(2,30),(2,31),(2,32),(2,33),(2,34),$$
$$(2,35),(2,36),(2,37),(2,38),(2,39),(2,40)(2,41),(2,42)\};$$

$$M_3(Q_5) = \{(3,6),(3,8),(3,10),(3,12),(3,16),(3,18),(3,20),(3,22),(3,24),(3,26),(3,27),$$
$$(3,28),(3,29),(3,31),(3,32),(3,33),(3,35),(3,36),(3,38),(3,40),(3,41),(3,42)\};$$

$$M_4(Q_5) = \varnothing;$$

$$M_5(Q_5) = \{(5,6),(5,8),(5,9),(5,10),(5,12),(5,14),(5,16),(5,18),(5,19),(5,20),$$
$$(5,22),(5,24),(5,26),(5,27),(5,28),(5,29),(5,31),(5,32),(5,33),$$
$$(5,35),(5,36),(5,37),(5,38),(5,40),(5,41),(5,42)\};$$

$$M_6(Q_5) = \{(6,7),(6,9),(6,10),(6,11),(6,13),(6,15),(6,17),(6,19),(6,20),(6,21),$$
$$(6,23),(6,25),(6,27),(6,28),(6,30),(6,31),(6,32),(6,34),(6,35),$$
$$(6,36),(6,37),(6,38),(6,39),(6,41),(6,42)\};$$

$$M_7(Q_5) = \{(7,8),(7,9),(7,10),(7,12),(7,14),(7,16),(7,18),(7,19),(7,20),(7,22),$$
$$(7,24),(7,26),(7,27),(7,28),(7,29),(7,31),(7,32),(7,33),(7,35),$$
$$(7,36),(7,37),(7,38),(7,40),(7,41),(7,42)\};$$

$$M_8(Q_5) = \{(8,9),(8,10),(8,11),(8,13),(8,15),(8,17),(8,19),(8,20),(8,21),$$
$$(8,23),(8,25),(8,27),(8,28),(8,30),(8,31),(8,30),(8,32),$$
$$(8,34),(8,35),(8,36),(8,37),(8,38),(8,39),(8,41),(8,42)\};$$

$$M_9(Q_5) = \{(9,10),(9,11),(9,12),(9,14),(9,15),(9,16),(9,17),(9,18),(9,20),(9,21),$$
$$(9,22),(9,23),(9,24),(9,25),(9,26),(9,27),(9,28),(9,29),(9,30),(9,31),$$
$$(9,32),(9,33),(9,34),(9,35),(9,36),(9,38),(9,39),(9,40),(9,41),(9,42)\};$$

$$M_{10}(Q_5) = \{(10,11),(10,12),(10,13),(10,15),(10,16),(10,17),(10,18),(10,19),$$
$$(10,21),(10,22),(10,23),(10,24),(10,25),(10,26),(10,27),(10,28),$$
$$(10,29),(10,30),(10,31),(10,32),(10,33),(10,34),(10,35),(10,36),$$
$$(10,37),(10,39),(10,40),(10,41),(10,42)\};$$

$$M_{11}(Q_5)=\{(11,12),(11,14),(11,16),(11,18),(11,19),(11,20),(11,22),(11,24),(11,26),(11,27),(11,28),(11,29),(11,31),(11,32),(11,33),(11,35),(11,36),(11,37),(11,38),(11,40),(11,41),(11,42)\};$$

$$M_{12}(Q_5)=\{(12,13),(12,15),(12,17),(12,19),(12,20),(12,21),(12,23),(12,25),(12,27),(12,28),(12,30),(12,31),(12,32),(12,34),(12,35),(12,36),(12,37),(12,38),(12,39),(12,41),(12,42)\};$$

$$M_{13}(Q_5)=\{(13,16),(13,18),(13,20),(13,22),(13,24),(13,26),(13,27),(13,28),(13,29),(13,31),(13,32),(13,33),(13,35),(13,36),(13,38),(13,40),(13,41),(13,42)\};$$

$$M_{14}(Q_5)=\{(14,15),(14,17),(14,19),(14,21),(14,23),(14,25),(14,27),(14,28),(14,30),(14,31),(14,32),(14,34),(14,35),(14,36),(14,37),(14,39),(14,41),(14,42)\};$$

$$M_{15}(Q_5)=\{(15,16),(15,18),(15,19),(15,20),(15,22),(15,24),(15,26),(15,27),(15,28),(15,29),(15,31),(15,32),(15,33),(15,35)(15,36),(15,37),(15,38),(15,40),(15,41),(15,42)\};$$

$$M_{16}(Q_5)=\{(16,17),(16,19),(16,20),(16,21),(16,23),(16,25),(16,27),(16,28),(16,30),(16,31),(16,32),(16,34),(16,35),(16,36),(16,37),(16,38),(16,39),(16,41),(16,42)\};$$

$$M_{17}(Q_5)=\{(17,18),(17,19),(17,20),(17,22),(17,24),(17,26),(17,27),(17,28),(17,29),(17,31),(17,32),(17,33),(17,35),(17,36),(17,37),(17,38),(17,40),(17,41),(17,42)\};$$

$$M_{18}(Q_5)=\{(18,19),(18,20),(18,21),(18,23),(18,25),(18,27),(18,28),(18,30),(18,31),(18,32),(18,34),(18,35),(18,36),(18,37),(18,38),(18,39),(18,41),(18,42)\};$$

$$M_{19}(Q_5)=\{(19,20),(19,21),(19,22),(19,23),(19,24),(19,25),(19,26),(19,27),(19,28),(19,29),(19,30),(19,31),(19,32),(19,33),(19,34),(19,35),(19,36),(19,38),(19,39),(19,40),(19,41),(19,42)\};$$

$$M_{20}(Q_5)=\{(20,21),(20,22),(20,23),(20,24),(20,25),(20,26),(20,27),(20,28),(20,29),(20,30),(20,31),(20,32),(20,33),(20,34),(20,35),(20,36),(20,37),(20,39),(20,40),(20,41),(20,42)\};$$

$$M_{21}(Q_5)=\{(21,22),(21,24),(21,26),(21,27),(21,28),(21,29),(21,31),(21,32),(21,33),(21,35),(21,36),(21,37),(21,38),(21,40),(21,41),(21,42)\};$$

$$M_{22}(Q_5)=\{(22,23),(22,25),(22,27),(22,28),(22,30),(22,31),(22,32),(22,34),(22,35),(22,36),(22,37),(22,38),(22,39),(22,41),(22,42)\};$$

$$M_{23}(Q_5)=\{(23,29),(23,31),(23,33),(23,35),(23,37),(23,38),(23,40),(23,42)\};$$

$$M_{24}(Q_5)=\{(24,30),(24,32),(24,34),(24,36),(24,37),(24,38),(24,39),(24,41)\};$$

$$M_{25}(Q_5)=\{(25,29),(25,31),(25,33),(25,35),(25,37),(25,38),(25,40),(25,42)\};$$

$$M_{26}(Q_5)=\{(26,30),(26,32),(26,34),(26,36),(26,37),(26,38),(26,39),(26,41)\};$$

$$M_{27}(Q_5)=\{(27,29),(27,30),(27,31),(27,32),(27,33),(27,34),(27,35),(27,36),\\ (27,37),(27,38),(27,39),(27,40),(27,41),(27,42)\};$$

$$M_{28}(Q_5)=\{(28,29),(28,30),(28,31),(28,32),(28,33),(28,34),(28,35),(28,36),\\ (28,37),(28,38),(28,39),(28,40),(28,41),(28,42)\};$$

$$M_{29}(Q_5)=\{(29,30),(29,31),(29,32),(29,34),(29,35),(29,36),(29,37),(29,38),\\ (29,39),(29,41),(29,42)\};$$

$$M_{30}(Q_5)=\{(30,31),(30,32),(30,33),(30,35),(30,36),(30,37),(30,38),(30,40),\\ (30,41),(30,42)\};$$

$$M_{31}(Q_5)=\{(31,32),(31,33),(31,34),(31,36),(31,37),(31,38),(31,39),(31,40),\\ (31,41)\};$$

$$M_{32}(Q_5)=\{(32,33),(32,34),(32,35),(32,37),(32,38),(32,39),(32,40),(32,42)\};$$

$$M_{33}(Q_5)=\{(33,34),(33,35),(33,36),(33,37),(33,38),(33,39),(33,41),(33,42)\};$$

$$M_{34}(Q_5)=\{(34,35),(34,36),(34,37),(34,38),(34,40),(34,41),(34,42)\};$$

$$M_{35}(Q_5)=\{(35,36),(35,37),(35,38),(35,39),(35,40),(35,41)\};$$

$$M_{36}(Q_5)=\{(36,37),(36,38),(36,39),(36,40),(36,42)\};$$

$$M_{37}(Q_5)=\{(37,38),(37,39),(37,40),(37,41),(37,42)\};$$

$$M_{38}(Q_5)=\{(38,39),(38,40),(38,41),(38,42)\};$$

$$M_{39}(Q_5)=\{(39,40),(39,41),(39,42)\};$$

$$M_{40}(Q_5)=\{(40,41),(40,42)\};M_{41}(Q_5)=\{(41,42)\}.$$

2) $Y_i', Y_j', Y_i'', Y_j'' \in \{Z_6, Z_5, Z_4, Z_1\}$ or $Y_i', Y_j', Y_i'', Y_j'' \in \{Z_8, Z_7, Z_6, Z_3\}$. Then by definition of the semilattice D it follows that the inequalities $(Y_T^\alpha \cup Y_{T'}^\alpha) \cap Y_{T''}^\alpha \supseteq Z_1 \cap Y_{T''}^\alpha \supseteq Y_i'' \cap Y_{T''}^\alpha \neq \varnothing$, $(Y_T^\alpha \cup Y_{T''}^\alpha) \cap Y_{T'}^\alpha \supseteq Z_1 \cap Y_{T'}^\alpha \supseteq Y_i' \cap Y_{T'}^\alpha \neq \varnothing$ or $(Y_T^\alpha \cup Y_{T'}^\alpha) \cap Y_{T''}^\alpha \supseteq Z_3 \cap Y_{T''}^\alpha \supseteq Y_i'' \cap Y_{T''}^\alpha \neq \varnothing$, $(Y_T^\alpha \cup Y_{T''}^\alpha) \cap Y_{T'}^\alpha \supseteq Z_3 \cap Y_{T'}^\alpha \supseteq Y_i' \cap Y_{T'}^\alpha \neq \varnothing$ are true respectively. But the inequalities $(Y_T^\alpha \cup Y_{T'}^\alpha) \cap Y_{T''}^\alpha \neq \varnothing$ and $(Y_T^\alpha \cup Y_{T''}^\alpha) \cap Y_{T'}^\alpha \neq \varnothing$ contradict the condition that representation of binary relation α is quasinormal. So, the equality $R(D_i') \cap R(D_j') = \varnothing$ holds. From the last equality, by definition of the semilattice D it follows that $R(D_i') \cap R(D_j') = \varnothing$ for all $(i, j) \in M_1'$, where

$$M_1'(Q_5)=\{(1,2),(1,3),(1,4),(1,13),(1,14),(2,3),(2,4),(2,13),(2,14),(3,4),\\ (3,13),(3,14),(4,13),(4,14),(13,14),(23,24),(23,25),(23,26),\\ (23,27),(23,28),(23,39),(23,41),(24,25),(24,26),(24,27),\\ (24,28),(25,26),(25,27),(25,28),(26,27),(26,28),(27,28)\}.$$

3) If $\alpha \in R(D'_3) \cap R(D'_{15})$, then

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_8, Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_6, Y_{T'}^\alpha \cap Z_8 \neq \varnothing, Y_{T''}^\alpha \cap Z_6 \neq \varnothing,$$
$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_7, Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_5, Y_{T'}^\alpha \cap Z_7 \neq \varnothing, Y_{T''}^\alpha \cap Z_5 \neq \varnothing.$$

Then by definition of the semilattice D it follows that the inequalities

$$\left(Y_T^\alpha \cup Y_{T'}^\alpha\right) \cap Y_{T''}^\alpha \supseteq \left(Z_8 \cup Z_7\right) \cap Y_{T''}^\alpha = Z_3 \cap Y_{T''}^\alpha \supseteq Z_6 \cap Y_{T''}^\alpha \neq \varnothing$$

are true. But the inequalities $\left(Y_T^\alpha \cup Y_{T'}^\alpha\right) \cap Y_{T''}^\alpha \neq \varnothing$ contradict the condition that representation of binary relation α is quasinormal. So, the equality $R(D'_3) \cap R(D'_{15}) = \varnothing$ holds. From the last equality it follows that $(3,15) \in M'_2$, where

$$\begin{aligned} M'_2(Q_5) = \{&(3,15),(3,17),(3,19),(3,21),(3,23),(3,25),(3,30),(3,34),(3,37),\\ &(3,39),(4,16),(4,18),(4,22),(4,24),(4,26),(4,29),(4,33),(4,38),\\ &(4,40),(5,13),(5,25),(6,14),(6,24),(6,26),(7,13),(7,23),(8,14),\\ &(8,24),(9,13),(10,14),(11,13),(12,14),(12,26),(13,23),(13,25),\\ &(13,30),(13,34),(13,37),(13,39),(14,24),(14,26),(14,29),(14,33),\\ &(14,38),(14,40),(15,25),(16,26),(17,23),(18,24),(21,23),(21,25),\\ &(22,24),(22,26),(23,34),(23,36),(24,33),(24,35),(24,40),(24,42),\\ &(25,30),(25,32),(25,39),(25,41),(26,29),(26,31),(26,40),(26,42)\}. \end{aligned}$$

By similar way one can prove that $R(D'_i) \cap R(D'_j) = \varnothing$ for any $(i,j) \in M'_2 \setminus \{(3,5)\}$.

4) $D'_i = \{Y_i, Y'_i, Y''_i, Y'_i \cup Y''_i\}$, $D'_j = \{Y_j, Y'_j, Y''_j, Y'_j \cup Y''_j\}$ and $D'_j = \{Y_k, Y'_i \cup Y'_j, Y''_i \cup Y''_j, Y'_i \cup Y''_i \cup Y'_j \cup Y''_j\}$ are such elements of the set $\{D'_1, D'_2, \cdots, D'_{42}\}$ that $D'_i \neq D'_j$, $D'_i \neq D'_k$, $D'_j \neq D'_k$, $\alpha \in R(D'_i) \cap R(D'_j)$ and $\alpha \in R(D'_i) \cap R(D'_j) \cap R(D'_k)$, then by statement e) of theorem 3 satisfies the following conditions:

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y'_i, Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y''_i, Y_{T'}^\alpha \cap Y'_i \neq \varnothing, Y_{T''}^\alpha \cap Y''_i \neq \varnothing,$$
$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y'_j, Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y''_j, Y_{T'}^\alpha \cap Y'_j \neq \varnothing, Y_{T''}^\alpha \cap Y''_j \neq \varnothing$$

and

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y'_i, Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y''_i, Y_{T'}^\alpha \cap Y'_i \neq \varnothing, Y_{T''}^\alpha \cap Y''_i \neq \varnothing, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y'_j,$$
$$Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y''_j, Y_{T'}^\alpha \cap Y'_j \neq \varnothing, Y_{T''}^\alpha \cap Y''_j \neq \varnothing, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y'_i \cup Y'_j,$$
$$Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y''_i \cup Y''_j, Y_{T'}^\alpha \cap Z_8 \neq \left(Y'_i \cup Y'_j\right), Y_{T''}^\alpha \cap \left(Y''_i \cup Y''_j\right) \neq \varnothing$$

respectively, *i.e.*, $\alpha \in R(D'_3) \cap R(D'_5)$ or $\alpha \in R(D'_3) \cap R(D'_5) \cap R(D'_{11})$ if and only if

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y'_i \cup Y'_j, Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y''_i \cup Y''_j,$$
$$Y_{T'}^\alpha \cap Y'_i \neq \varnothing, Y_{T''}^\alpha \cap Y''_i \neq \varnothing, Y_{T'}^\alpha \cap Y'_j \neq \varnothing, Y_{T''}^\alpha \cap Y''_j \neq \varnothing.$$

Therefore, the equality $R(D'_i) \cap R(D'_j) = R(D'_i) \cap R(D'_j) \cap R(D'_k)$ is true. From the last equality by definition of the semilattice D it follows that $R(D'_i) \cap R(D'_j) = R(D'_i) \cap R(D'_j) \cap R(D'_k)$ for all $(i,j,k) \in M'_3$, where

$$\begin{aligned} M'_3(Q_5) = \{&(3,5,11),(3,7,11),(4,6,12),(4,8,12),(4,20,38),(5,7,11),(5,15,30),(5,17,39),\\ &(5,21,39),(5,23,30),(5,34,39),(6,8,12),(6,16,29),(6,18,40),(6,22,40),\\ &(6,33,40),(7,15,39),(7,17,34),(7,21,39),(7,25,34),(7,30,39),(8,16,40),\\ &(8,18,33),(8,22,40),(8,26,33),(8,29,40),(9,19,37),(10,20,38),(11,15,39),\\ &(11,17,39),(11,21,39),(11,23,39),(11,25,39),(11,30,39),(12,16,40),(12,18,40),\\ &(12,22,40),(12,24,40),(12,29,40),(12,33,40),(13,15,21),(13,17,21),(14,16,22),\\ &(14,18,22),(15,17,21),(15,23,30),(15,34,39),(16,18,22),(16,24,29),(16,33,40),\\ &(17,25,34),(17,30,39),(18,26,33),(18,29,40),(21,30,39),(21,34,39),(22,29,40),\\ &(22,33,40),(29,33,40),(30,34,39),(31,35,42),(32,36,41)\}. \end{aligned}$$

From the equalities $R(D_i') \cap R(D_j') = \varnothing \quad ((i, j) \in M_1' \cup M_2')$ and $R(D_i') \cap R(D_j') = R(D_i') \cap R(D_j') \cap R(D_k') \quad ((i, j, k) \in M_3')$ given above it follows that

$$\left|R^*(Q_5)\right| = \sum_{i=1}^{42}\left|R(D_i')\right| - \sum_{(k,j)\in M_4'(Q_5)}\left|R(D_k') \cap R(D_j')\right|,$$

where

$$\begin{aligned} M_4'(Q_5) = \{&(3,9),(3,11),(4,10),(4,12),(5,11),(5,30),(5,39),(6,12),(6,29),\\ &(6,40),(7,11),(7,34),(7,39),(8,12),(8,33),(8,40),(9,37),\\ &(10,38),(11,39),(12,40),(13,19),(13,21),(14,20),(14,22),\\ &(15,21),(15,30),(15,39),(16,22),(16,29),(16,40),(17,21),\\ &(17,34),(17,39),(18,22),(18,33),(18,40),(19,37),(20,38),\\ &(21,39),(22,40),(23,30),(23,32),(24,29),(24,31),(25,34),\\ &(25,36),(26,33),(26,35),(29,40),(30,39),(31,42),(32,41),\\ &(33,40),(34,39),(35,42),(36,41)\}. \end{aligned}$$

□

Lemma 12. *Let* $D_i' = \{Z_9, Y_i, Y_i', Y_i \cup Y_i'\}$ *and* $D_j' = \{Z_9, Y_j, Y_j', Y_j \cup Y_j'\}$ *be arbitrary elements of the set* $\{D_1', D_2', \cdots, D_{42}'\}$*, where* $D_i' \neq D_j'$*,* $Y_j \supseteq Y_i$ *and* $Y_j' \supseteq Y_i'$*. If quasinormal representation of binary relation* α *of the semigroup* $B_X(D)$ *has a form* $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T'\cup T''}^\alpha \times (T' \cup T''))$*, for some* $T, T', T'' \in D$*,* $T \subset T'$*,* $T \subset T''$ *and* $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$*, then* $\alpha \in R(D_i') \cap R(D_j')$ *iff*

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_j,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y_j',\ Y_{T'}^\alpha \cap Y_i \neq \varnothing,\ Y_{T''}^\alpha \cap Y_i' \neq \varnothing.$$

Proof. If $\alpha \in R(D_i') \cap R(D_j')$, then by statement e) of Theorem 3 we have

$$\begin{aligned} &Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_i,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y_i',\ Y_{T'}^\alpha \cap Y_i \neq \varnothing,\ Y_{T''}^\alpha \cap Y_i' \neq \varnothing;\\ &Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_j,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y_j',\ Y_{T'}^\alpha \cap Y_j \neq \varnothing,\ Y_{T''}^\alpha \cap Y_j' \neq \varnothing. \end{aligned} \tag{13}$$

From the last condition we have

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_j,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y_j',\ Y_{T'}^\alpha \cap Y_i \neq \varnothing,\ Y_{T''}^\alpha \cap Y_i' \neq \varnothing, \tag{14}$$

since $Y_j \supseteq Y_i$ and $Y_j' \supseteq Y_i'$ by supposition.

On the other hand, if the conditions of (14) hold, then the conditions of (13) follow, *i.e.* $\alpha \in R(D_i') \cap R(D_j')$.

□

Lemma 13. *Let* $D_i' = \{Z_9, Y_i, Y_i', Y_i \cup Y_i'\}$ *and* $D_j' = \{Z_9, Y_j, Y_j', Y_j \cup Y_j'\}$ *be arbitrary elements of the set* $\{D_1', D_2', \cdots, D_{42}'\}$*, where* $D_i' \neq D_j'$*,* $Y_j \supseteq Y_i$ *and* $Y_j' \supseteq Y_i'$*. Then the following equality holds*:

$$\begin{aligned} &\left|R(D_i') \cap R(D_j')\right|\\ &= 24 \cdot 2^{\left|(Y_j \setminus (Y_i \cup Y_i'))\right|} \cdot \left(2^{\left|Y_i \setminus Y_j'\right|} - 1\right) \cdot 2^{\left|Y_i' \setminus (Y_i' \cup Y_j)\right|} \cdot \left(2^{\left|Y_i' \setminus Y_j\right|} - 1\right) \cdot 4^{\left|X \setminus (Y_j \setminus Y_j')\right|} \end{aligned}$$

Proof. Let $D_i' = \{Z_9, Y_i, Y_i', Y_i \cup Y_i'\}$ and $D_j' = \{Z_9, Y_j, Y_j', Y_j \cup Y_j'\}$ be arbitrary elements of the set $\{D_1', D_2', \cdots, D_{42}'\}$, where $D_i' \neq D_j'$, $Y_j \supseteq Y_i$ and $Y_j' \supseteq Y_i'$. If $\alpha \in R(D_i') \cap R(D_j')$. Then quasinormal representation of a binary relation α of semigroup $B_X(D)$ has a form

$$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T'\cup T''}^\alpha \times (T' \cup T''))$$

for some $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$ and by the lemma 12 satisfies the conditions

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_j,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y_j',\ Y_{T'}^\alpha \cap Y_i \neq \varnothing,\ Y_{T''}^\alpha \cap Y_i' \neq \varnothing. \tag{15}$$

Now, let f_α be a mapping from X to the semilattice D satisfying the conditions $f_\alpha(t) = t\alpha$ for all $t \in X$. $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$ are the restrictions of the mapping f_α on the sets

$Y_j \cap Y'_j, Y_j \setminus Y'_j, Y'_j \setminus Y_j, X \setminus (Y_j \cup Y'_j)$ respectively. It is clear that the intersection of elements of the set $\{Y_j \cap Y'_j, Y_j \setminus Y'_j, Y'_j \setminus Y_j, X \setminus (Y_j \cup Y'_j)\}$ is an empty set and

$$(Y_j \cap Y'_j) \cup (Y_j \setminus Y'_j) \cup (Y'_j \setminus Y_j) \cup (X \setminus (Y_j \cup Y'_j)) = X.$$

We are going to find properties of the maps $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$.

(1) $t \in Y_j \cap Y'_j$. Then by the properties (1) we have

$$t \in Y_j \cap Y'_j \subseteq (Y_T^\alpha \cup Y_{T'}^\alpha) \cap (Y_T^\alpha \cup Y_{T''}^\alpha) = Y_T^\alpha,$$

since $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_j, Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y'_j$, *i.e.*, $t \in Y_T^\alpha$ and $t\alpha = T$ by definition of the set Y_T^α. Therefore $f_{0\alpha}(t) = T$ for all $t \in Y_j \cap Y'_j$.

(2) $t \in Y_j \setminus Y'_j$. Then by the properties (1) we have $Y_j \setminus Y'_j \subseteq Y_j \subseteq Y_T^\alpha \cup Y_{T'}^\alpha$, *i.e.*, $t \in Y_T^\alpha \cup Y_{T'}^\alpha$ and $t\alpha \in \{T, T'\}$ by definition of the set Y_T^α and $Y_{T'}^\alpha$. Therefore $f_{1\alpha}(t) \in \{T, T'\}$ for all $t \in Y_j \setminus Y'_j$.

By suppose we have that $Y_{T'}^\alpha \cap Y_i \neq \varnothing$, *i.e.* $t'\alpha = T'$ for some $t' \in Y_i$. Then $t' \in Y'_j$ since $Y'_i \subseteq Y'_j$. If $t' \in Y'_j$, then $t' \in Y'_j \subseteq Y_T^\alpha \cup Y_{T''}^\alpha$. Therefore $t'\alpha \in \{T, T''\}$. That contradicts the equality $t'\alpha = T'$, while $T' \neq T$ and $T' \neq T''$ by definition of the semilattice *D*.

Therefore $f_{1\alpha}(t') = T'$ for some $t' \in Y_i \setminus Y'_j$.

(3) $t \in Y'_j \setminus Y_j$. Then by the properties (1) we have $Y' \setminus Y_j \subseteq Y' \subseteq Y_T^\alpha \cup Y_{T''}^\alpha$, *i.e.*, $t \in Y_T^\alpha \cup Y_{T''}^\alpha$ and $t\alpha \in \{T, T''\}$ by definition of the set Y_T^α and $Y_{T''}^\alpha$. Therefore $f_{2\alpha}(t) \in \{T, T''\}$ for all $t \in Y'_j \setminus Y_j$.

By suppose we have that $Y_{T''}^\alpha \cap Y'_i \neq \varnothing$, *i.e.* $t''\alpha = T''$ for some $t'' \in Y'_i$. Then $t'' \in Y'_j$ since $Y'_i \subseteq Y'_j$. If $t'' \in Y_j$ then $t'' \in Y_j \subseteq Y_T^\alpha \cup Y_{T'}^\alpha$. Therefore $t''\alpha \in \{T, T'\}$. That contradicts the equality $t''\alpha = T''$, while $T'' \neq T$ and $T'' \neq T'$ by definition of the semilattice *D*.

Therefore $f_{2\alpha}(t'') = T''$ for some $t'' \in Y'_i \setminus Y_j$.

(4) $t \in X \setminus (Y_j \cup Y'_j)$. Then by definition quasinormal representation of a binary relation α and by property (1) we have $X \setminus (Y_j \cup Y'_j) \subseteq X = Y_T^\alpha \cup Y_T^\alpha \cup Y_{T''}^\alpha \cup Y_{T' \cup T''}^\alpha$, *i.e.* $t\alpha \in \{T, T', T'', T' \cup T''\}$ by definition of the sets $Y_T^\alpha, Y_{T'}^\alpha$, $Y_{T''}^\alpha$ and $Y_{T' \cup T''}^\alpha$. Therefore $f_{3\alpha}(t) \in \{T, T', T'', T' \cup T''\}$ for all $t \in X \setminus (Y_j \cup Y'_j)$.

Therefore for every binary relation $\alpha \in R(D'_i) \cap R(D'_j)$ there exists ordered system $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha})$. It is obvious that for disjoint binary relations there exists disjoint ordered systems.

Further, let

$$f_0 : Y_j \cap Y'_j \to \{T\},$$

$$f_1 : Y_j \setminus Y'_j \to \{T, T'\},$$

$$f_2 : Y'_j \setminus Y_j \to \{T, T''\},$$

$$f_3 : X \setminus (Y_j \cup Y'_j) \to \{T, T', T'', T' \cup T''\}$$

be such mappings, which satisfy the conditions

$f_0(t) = T$ for all $t \in Y_j \cap Y'_j$;

$f_1(t) \in \{T, T'\}$ for all $t \in Y_j \setminus Y'_j$ and $f_1(t'_1) = T'$ for some $t'_1 \in Y_i \setminus Y'_j$;

$f_2(t) \in \{T, T''\}$ for all $t \in Y'_j \setminus Y_j$ and $f_2(t'_2) = T''$ for some $t'_2 \in Y'_i \setminus Y_j$;

$f_{3\alpha}(t) \in \{T, T', T'', T' \cup T''\}$ for all $t \in X \setminus (Y_j \cup Y'_j)$.

Now we define a map *f* from *X* to the semilattice *D*, which satisfies the condition

$$f(t) = \begin{cases} f_0(t), \text{if } t \in Y_j \cap Y'_j, \\ f_1(t), \text{if } t \in Y_j \setminus Y'_j, \\ f_2(t), \text{if } t \in Y'_j \setminus Y_j, \\ f_3(t), \text{if } t \in X \setminus (Y_j \cup Y'_j). \end{cases}$$

Further, let $\beta = \bigcup_{x \in X} (\{x\} \times f(x))$, $Y_T^\beta = \{t \mid t\beta = T\}$, $Y_{T'}^\beta = \{t \mid t\beta = T'\}$, $Y_{T''}^\beta = \{t \mid t\beta = T''\}$ and $Y_{T' \cup T''}^\beta = \{t \mid t\beta = T' \cup T''\}$. Then binary relation β may be represented by

$$\beta = (Y_T^\beta \times T) \cup (Y_{T'}^\beta \times T') \cup (Y_{T''}^\beta \times T'') \cup (Y_{T' \cup T''}^\beta \times (T' \cup T''))$$

and satisfies the conditions

$$Y_T^{\beta} \cup Y_{T'}^{\beta} \supseteq Y_j, Y_T^{\beta} \cup Y_{T''}^{\beta} \supseteq Y_j', Y_{T'}^{\beta} \cap Y_i \neq \varnothing, Y_{T''}^{\beta} \cap Y_i' \neq \varnothing.$$

(By suppose $f_1(t_1') = T'$ for some $t_1' \in Y_i \setminus Y_j'$ and $f_2(t_2') = T''$ for some $t_2' \in Y_i' \setminus Y_j$), *i.e.*, by lemma 12 we have that $\beta \in R(D_i') \cap R(D_j')$. Therefore for every binary relation $\alpha \in R(D_i') \cap R(D_j')$ and ordered system $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha})$ there exists one to one mapping.

The number of the mappings $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$ $(\alpha \in R(D_i') \cap R(D_j'))$ are respectively

$$1, 2^{\left|(Y_j \setminus Y_j') \setminus (Y_i \setminus Y_j')\right|} \cdot \left(2^{\left|Y_i \setminus Y_j'\right|} - 1\right), 2^{\left|(Y_j' \setminus Y_j) \setminus (Y_i' \setminus Y_j)\right|} \cdot \left(2^{\left|Y_i' \setminus Y_j\right|} - 1\right), 4^{\left|X \setminus (Y_j \cup Y_j')\right|}.$$

Note that the number $2^{\left|(Y_j \setminus (Y_i \cup Y_j'))\right|} \cdot \left(2^{\left|Y_i \setminus Y_j'\right|} - 1\right) \cdot 2^{\left|(Y_j' \setminus (Y_i' \cup Y_j))\right|} \cdot \left(2^{\left|Y_i' \setminus Y_j'\right|} - 1\right) \cdot 4^{\left|X \setminus (Y_j \cup Y_j')\right|}$ does not depend on choice of elements $T, T', T'' \in D$ of the semilattice *D*, where $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$ and $T'' \setminus T' \neq \varnothing$. Since the number of such different elements of the form (T', T'') of the semilattice *D* are equal to 24, the number of regular elements of the set $R(D_i') \cap R(D_j')$ is equal to

$$\begin{aligned} &\left|R(D_i') \cap R(D_j')\right| \\ &= 24 \cdot 2^{\left|(Y_j \setminus (Y_j \cup Y_j'))\right|} \cdot \left(2^{\left|Y_i \setminus Y_j'\right|} - 1\right) \cdot 2^{\left|(Y_j' \setminus (Y_i' \cup Y_j))\right|} \cdot \left(2^{\left|Y_i' \setminus Y_j\right|} - 1\right) \cdot 4^{\left|X \setminus (Y_j \cup Y_j')\right|}. \end{aligned}$$

□

Lemma 14. *Let X be a finite set,*

$$D = \left\{Z_9, Z_8, Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\} \in \Sigma_1(X, 10)$$

and $Z_9 \neq \varnothing$. *Let* $R^*(Q_5)$ *be set of all regular elements of* $B_X(D)$ *such that each element satisfies the condition e) of Theorem 3. Then* $\left|R^*(Q_5)\right| = \sum_{i=1}^{42} \left|R(D_i')\right| - \sum_{(k,j) \in M_4'} \left|R(D_k') \cap R(D_j')\right|$, *where*

$$\begin{aligned} \sum_{i=1}^{42} \left|R(D_i')\right| = {} & 48 \cdot \left(2^{|Z_8 \setminus Z_7|} - 1\right) \cdot \left(2^{|Z_7 \setminus Z_8|} - 1\right) \cdot 4^{|X \setminus Z_3|} + 48 \cdot \left(2^{|Z_8 \setminus Z_6|} - 1\right) \cdot \left(2^{|Z_6 \setminus Z_8|} - 1\right) \cdot 4^{|X \setminus Z_3|} \\ & + 48 \cdot \left(2^{|Z_8 \setminus Z_5|} - 1\right) \cdot \left(2^{|Z_5 \setminus Z_8|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} + 48 \cdot \left(2^{|Z_8 \setminus Z_4|} - 1\right) \cdot \left(2^{|Z_4 \setminus Z_8|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} \\ & + 48 \cdot \left(2^{|Z_8 \setminus Z_2|} - 1\right) \cdot \left(2^{|Z_2 \setminus Z_8|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} + 48 \cdot \left(2^{|Z_8 \setminus Z_1|} - 1\right) \cdot \left(2^{|Z_1 \setminus Z_8|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} \\ & + 48 \cdot \left(2^{|Z_7 \setminus Z_6|} - 1\right) \cdot \left(2^{|Z_6 \setminus Z_7|} - 1\right) \cdot 4^{|X \setminus Z_3|} + 48 \cdot \left(2^{|Z_7 \setminus Z_5|} - 1\right) \cdot \left(2^{|Z_5 \setminus Z_7|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} \\ & + 48 \cdot \left(2^{|Z_7 \setminus Z_4|} - 1\right) \cdot \left(2^{|Z_4 \setminus Z_7|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} + 48 \cdot \left(2^{|Z_7 \setminus Z_2|} - 1\right) \cdot \left(2^{|Z_2 \setminus Z_7|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} \\ & + 48 \cdot \left(2^{|Z_7 \setminus Z_1|} - 1\right) \cdot \left(2^{|Z_1 \setminus Z_7|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} + 48 \cdot \left(2^{|Z_6 \setminus Z_5|} - 1\right) \cdot \left(2^{|Z_5 \setminus Z_6|} - 1\right) \cdot 4^{|X \setminus Z_1|} \\ & + 48 \cdot \left(2^{|Z_6 \setminus Z_4|} - 1\right) \cdot \left(2^{|Z_4 \setminus Z_6|} - 1\right) \cdot 4^{|X \setminus Z_1|} + 48 \cdot \left(2^{|Z_5 \setminus Z_4|} - 1\right) \cdot \left(2^{|Z_4 \setminus Z_5|} - 1\right) \cdot 4^{|X \setminus Z_1|} \\ & + 48 \cdot \left(2^{|Z_5 \setminus Z_3|} - 1\right) \cdot \left(2^{|Z_3 \setminus Z_5|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} + 48 \cdot \left(2^{|Z_5 \setminus Z_2|} - 1\right) \cdot \left(2^{|Z_2 \setminus Z_5|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} \\ & + 48 \cdot \left(2^{|Z_4 \setminus Z_3|} - 1\right) \cdot \left(2^{|Z_3 \setminus Z_4|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} + 48 \cdot \left(2^{|Z_4 \setminus Z_2|} - 1\right) \cdot \left(2^{|Z_2 \setminus Z_4|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} \\ & + 48 \cdot \left(2^{|Z_3 \setminus Z_2|} - 1\right) \cdot \left(2^{|Z_2 \setminus Z_3|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} + 48 \cdot \left(2^{|Z_3 \setminus Z_1|} - 1\right) \cdot \left(2^{|Z_1 \setminus Z_3|} - 1\right) \cdot 4^{|X \setminus \breve{D}|} \\ & + 48 \cdot \left(2^{|Z_2 \setminus Z_1|} - 1\right) \cdot \left(2^{|Z_1 \setminus Z_2|} - 1\right) \cdot 4^{|X \setminus \breve{D}|}. \end{aligned}$$

and

$$
\begin{aligned}
&\sum_{(k,j)\in M_4'}\left|R\left(D_k'\right)\cap R\left(D_j'\right)\right| \\
&= 48\cdot\left(2^{|Z_8\setminus Z_2|}-1\right)\cdot 2^{|Z_2\setminus Z_3|}\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot 2^{|Z_1\setminus Z_3|}\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_5\setminus Z_8|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot\left(2^{|Z_8\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_4\setminus Z_8|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot\left(2^{|Z_8\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot\left(2^{|Z_8\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot\left(2^{|Z_7\setminus Z_2|}-1\right)\cdot 2^{|Z_2\setminus Z_3|}\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot 2^{|Z_1\setminus Z_3|}\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot\left(2^{|Z_7\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot\left(2^{|Z_7\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot\left(2^{|Z_7\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot 2^{|Z_3\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot 2^{|Z_2\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot 2^{|Z_3\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot 2^{|Z_2\setminus Z_1|}\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot\left(2^{|Z_5\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|} \\
&\quad+48\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+48\cdot\left(2^{|Z_4\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}.
\end{aligned}
$$

Proof. The given Lemma immediately follows from Lemma 11 and Lemma 13. □

Let binary relation α of the semigroup $B_X(D)$ satisfy the condition g) of Theorem 3 (see diagram 7 of the **Figure 3**). In this case we have $Q_6=\{Z_9,Z_6,T,T',\breve{D}\}$, where $T,T'\in\{Z_3,Z_2,Z_1\}$, $T\setminus T'\neq\varnothing$ and $T'\setminus T\neq\varnothing$. By definition of the semilattice D it follows that

$$Q_6\vartheta_{XI}=\left\{\{Z_9,Z_6,Z_3,Z_2,\breve{D}\},\{Z_9,Z_6,Z_3,Z_1,\breve{D}\},\{Z_9,Z_6,Z_2,Z_1,\breve{D}\}\right\}.$$

It is easy to see $\left|\Phi(Q_6,Q_6)\right|=2$ and $\left|\Omega(Q_6)\right|=3$. If

$$
\begin{aligned}
&D_1'=\{Z_9,Z_6,Z_3,Z_2,\breve{D}\},\ D_2'=\{Z_9,Z_6,Z_2,Z_3,\breve{D}\},\ D_3'=\{Z_9,Z_6,Z_3,Z_1,\breve{D}\},\\
&D_4'=\{Z_9,Z_6,Z_1,Z_3,\breve{D}\},\ D_5'=\{Z_9,Z_6,Z_2,Z_1,\breve{D}\},\ D_6'=\{Z_9,Z_6,Z_1,Z_2,\breve{D}\}.
\end{aligned}
$$

(see **Figure 4**).

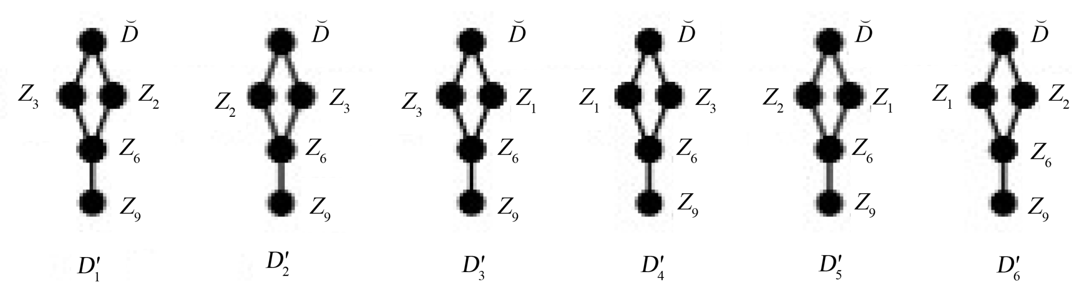

Figure 4. Diagram of all subsemilattices isomorphic to 7 in **Figure 2**.

Then

$$R^*(Q_6)=\bigcup_{i=1}^{6}R(D'_i). \tag{16}$$

Lemma 15. *Let* X *be a finite set,* $D=\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_1(X,10)$ *and* $Z_9\neq\varnothing$. *Let* $R^*(Q_6)$ *be set of all regular elements of* $B_X(D)$ *such that each element satisfies the condition f) of Theorem* 3. *Then*

$$\begin{aligned}\left|R^*(Q_6)\right|&=6\left(2^{|Z_6\setminus Z_9|}-1\right)2^{|(Z_3\cap Z_2)\setminus Z_6|}\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)5^{|X\setminus\breve{D}|}\\&+6\left(2^{|Z_6\setminus Z_9|}-1\right)2^{|(Z_3\cap Z_1)\setminus Z_6|}\left(3^{|Z_3\setminus Z_1|}-2^{|Z_3\setminus Z_1|}\right)\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)5^{|X\setminus\breve{D}|}\\&+6\left(2^{|Z_6\setminus Z_9|}-1\right)2^{|(Z_2\cap Z_1)\setminus Z_6|}\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)5^{|X\setminus\breve{D}|}.\end{aligned}$$

Proof. Let $D'_i=\{Z_9,Z_6,Y_i,Y'_i,Y_i\cup Y'_i\}$, $D'_j=\{Z_9,Z_6,Y_j,Y'_j,Y_j\cup Y'_j\}\in\{D'_1,D'_2,\cdots,D'_6\}$, $D'_i\neq D'_j$ and $\alpha\in R(D'_i)\cap R(D'_j)$. Then quasinormal representation of a binary relation α of the semigroup $B_X(D)$ has a form

$$\alpha=\left(Y_9^\alpha\times Z_9\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T\cup T'}^\alpha\times(T\cup T')\right),$$

where $T,T'\in\{Z_3,Z_2,Z_1\}$, $T\setminus T'\neq\varnothing$ and $T'\setminus T\neq\varnothing$ and by statement f) of theorem 3 satisfies the following conditions

$$\begin{aligned}&Y_9^\alpha\supseteq Z_9,\ Y_9^\alpha\cup Y_6^\alpha\supseteq Z_6,\ Y_9^\alpha\cup Y_6^\alpha\cup Y_T^\alpha\supseteq Y_i,\ Y_9^\alpha\cup Y_6^\alpha\cup Y_{T'}^\alpha\supseteq Y'_i,\\&Y_6^\alpha\cap Z_6\neq\varnothing,\ Y_T^\alpha\cap Y_i\neq\varnothing,\ Y_{T'}^\alpha\cap Y'_i\neq\varnothing;\\&Y_9^\alpha\supseteq Z_9,\ Y_9^\alpha\cup Y_6^\alpha\supseteq Z_6,\ Y_9^\alpha\cup Y_6^\alpha\cup Y_T^\alpha\supseteq Y_j,\ Y_9^\alpha\cup Y_6^\alpha\cup Y_{T'}^\alpha\supseteq Y'_j,\\&Y_6^\alpha\cap Z_6\neq\varnothing,\ Y_T^\alpha\cap Y_j\neq\varnothing,\ Y_{T'}^\alpha\cap Y'_j\neq\varnothing.\end{aligned}$$

From this conditions it follows that

$$Y_9^\alpha\supseteq Z_9,\ Y_9^\alpha\cup Y_6^\alpha\supseteq Z_6,\ Y_9^\alpha\cup Y_6^\alpha\cup Y_T^\alpha\supseteq Y_i\cup Y_j,\ Y_9^\alpha\cup Y_6^\alpha\cup Y_{T'}^\alpha\supseteq Y'_i\cup Y'_j.$$

For D'_i and D'_j we consider the following case.

$Y_i\cup Y_j=\breve{D}$ or $Y'_i\cup Y'_j=\breve{D}$. Then

$$\left(Y_9^\alpha\cup Y_6^\alpha\cup Y_T^\alpha\right)\cap Y_{T'}^\alpha\supseteq\breve{D}\cap Y_{T'}^\alpha\supseteq Y'_i\cap Y_{T'}^\alpha\neq\varnothing$$

or

$$\left(Y_9^\alpha\cup Y_6^\alpha\cup Y_{T'}^\alpha\right)\cap Y_T^\alpha\supseteq\breve{D}\cap Y_T^\alpha\supseteq Y_i\cap Y_T^\alpha\neq\varnothing.$$

But the inequality $\left(Y_9^\alpha\cup Y_6^\alpha\cup Y_T^\alpha\right)\cap Y_{T'}^\alpha\neq\varnothing$ and $\left(Y_9^\alpha\cup Y_6^\alpha\cup Y_{T'}^\alpha\right)\cap Y_T^\alpha\neq\varnothing$ contradict the condition that representation of binary relation α is quasinormal. So, the equality $R(D'_i)\cap R(D'_j)=\varnothing$ holds. From the last equality by definition of the semilattice D it follows that $R(D'_i)\cap R(D'_j)=\varnothing$ for all $(i,j)\in M_1(Q_6)$, where

$$\begin{aligned}M_1(Q_6)=\{&(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),\\&(2,6),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}.\end{aligned} \tag{17}$$

Now by Equalities (16) and by condition (17) it follows that

$$\left|R^*(Q_6)\right|=\sum_{i=1}^{6}\left|R(D'_i)\right|.$$

By statement f) of Lemma 3 the given Lemma is true. □

Now let binary relation α of the semigroup $B_X(D)$ satisfy the condition f) of Theorem 3 (see diagram 6 of the **Figure 3**). In this case we have $Q_7=\{Z_9,T,T',T\cup T',\breve{D}\}$, where $T,T'\in\{Z_8,Z_7,Z_6,Z_5,Z_4\}$, $T\setminus T'\neq\varnothing$ and $T'\setminus T\neq\varnothing$. By definition of the semilattice D it follows that

$$Q_7\vartheta_{XI}=\left\{\left\{Z_9,Z_8,Z_7,Z_3,\breve{D}\right\},\left\{Z_9,Z_8,Z_6,Z_3,\breve{D}\right\},\left\{Z_9,Z_7,Z_6,Z_3,\breve{D}\right\},\right.$$
$$\left.\left\{Z_9,Z_6,Z_5,Z_1,\breve{D}\right\},\left\{Z_9,Z_6,Z_4,Z_1,\breve{D}\right\},\left\{Z_9,Z_5,Z_4,Z_1,\breve{D}\right\}\right\};$$

It is easy to see $\left|\Phi\left(Q_7,Q_7\right)\right|=2$ and $\left|\Omega\left(Q_7\right)\right|=6$. If

$$D_1'=\left\{Z_9,Z_8,Z_7,Z_3,\breve{D}\right\},\ D_2'=\left\{Z_9,Z_7,Z_8,Z_3,\breve{D}\right\},\ D_3'=\left\{Z_9,Z_8,Z_6,Z_3,\breve{D}\right\},$$
$$D_4'=\left\{Z_9,Z_6,Z_8,Z_3,\breve{D}\right\},\ D_5'=\left\{Z_9,Z_7,Z_6,Z_3,\breve{D}\right\},\ D_6'=\left\{Z_9,Z_6,Z_7,Z_3,\breve{D}\right\},$$
$$D_7'=\left\{Z_9,Z_6,Z_5,Z_1,\breve{D}\right\},\ D_8'=\left\{Z_9,Z_5,Z_6,Z_1,\breve{D}\right\},\ D_6'=\left\{Z_9,Z_6,Z_4,Z_1,\breve{D}\right\},$$
$$D_{10}'=\left\{Z_9,Z_4,Z_6,Z_1,\breve{D}\right\},D_{11}'=\left\{Z_9,Z_5,Z_4,Z_1,\breve{D}\right\},D_{12}'=\left\{Z_9,Z_4,Z_5,Z_1,\breve{D}\right\}$$

(see **Figure 5**).

Then

$$R^*\left(Q_7\right)=\bigcup_{i=1}^{12}R\left(D_i'\right).$$

Lemma 16. *Let* X *be a finite set,* $D=\left\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\right\}\in\Sigma_1\left(X,10\right)$ *and* $Z_9\neq\varnothing$. *Let* $R^*\left(Q_7\right)$ *be set of all regular elements of* $B_X\left(D\right)$ *such that each element satisfies the condition g) of Theorem* 3. *Then*

$$\begin{aligned}\left|R^*\left(Q_7\right)\right|&=12\cdot\left(2^{\left|Z_8\setminus Z_7\right|}-1\right)\cdot\left(2^{\left|Z_7\setminus Z_8\right|}-1\right)\cdot\left(5^{\left|\breve{D}\setminus Z_3\right|}-4^{\left|\breve{D}\setminus Z_3\right|}\right)\cdot5^{\left|X\setminus\breve{D}\right|}\\&+12\cdot\left(2^{\left|Z_8\setminus Z_6\right|}-1\right)\cdot\left(2^{\left|Z_6\setminus Z_8\right|}-1\right)\cdot\left(5^{\left|\breve{D}\setminus Z_3\right|}-4^{\left|\breve{D}\setminus Z_3\right|}\right)\cdot5^{\left|X\setminus\breve{D}\right|}\\&+12\cdot\left(2^{\left|Z_7\setminus Z_6\right|}-1\right)\cdot\left(2^{\left|Z_6\setminus Z_7\right|}-1\right)\cdot\left(5^{\left|\breve{D}\setminus Z_3\right|}-4^{\left|\breve{D}\setminus Z_3\right|}\right)\cdot5^{\left|X\setminus\breve{D}\right|}\\&+12\cdot\left(2^{\left|Z_6\setminus Z_5\right|}-1\right)\cdot\left(2^{\left|Z_5\setminus Z_6\right|}-1\right)\cdot\left(5^{\left|\breve{D}\setminus Z_1\right|}-4^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot5^{\left|X\setminus\breve{D}\right|}\\&+12\cdot\left(2^{\left|Z_6\setminus Z_4\right|}-1\right)\cdot\left(2^{\left|Z_4\setminus Z_6\right|}-1\right)\cdot\left(5^{\left|\breve{D}\setminus Z_1\right|}-4^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot5^{\left|X\setminus\breve{D}\right|}\\&+12\cdot\left(2^{\left|Z_5\setminus Z_4\right|}-1\right)\cdot\left(2^{\left|Z_4\setminus Z_5\right|}-1\right)\cdot\left(5^{\left|\breve{D}\setminus Z_1\right|}-4^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot5^{\left|X\setminus\breve{D}\right|}.\end{aligned}$$

Proof. Let $D_i'=\left\{Z_9,Y_i,Y_i',Y_i\cup Y_i',\breve{D}\right\},D_j'=\left\{Z_9,Y_j,Y_j',Y_j\cup Y_j',\breve{D}\right\}\in\left\{D_1',D_2',\cdots,D_{12}'\right\}$, $D_i'\neq D_j'$ and $\alpha\in R\left(D_i'\right)\cap R\left(D_j'\right)$. Then quasinormal representation of a binary relation α of the semigroup $B_X\left(D\right)$ has a form

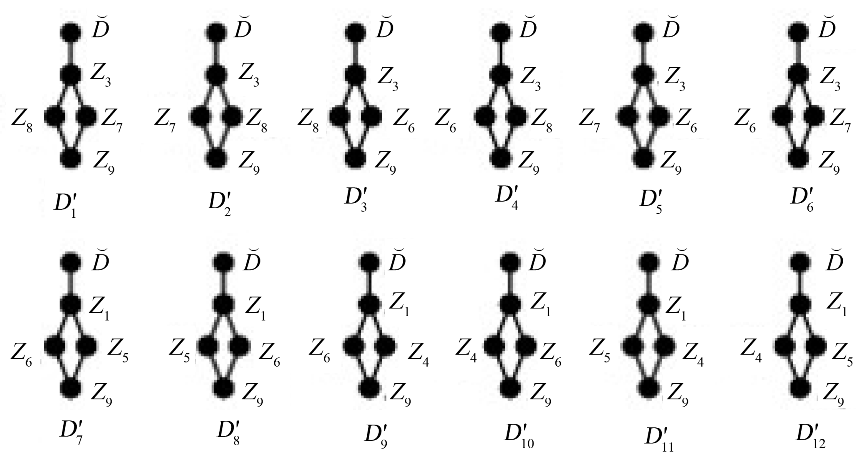

Figure 5. Diagram of all subsemilattices isomorphic to 6 in **Figure 2**.

$$\alpha = \left(Y_9^\alpha \times Z_9\right) \cup \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_{T\cup T'}^\alpha \times \left(T \cup T'\right)\right) \cup \left(Y_0^\alpha \times \breve{D}\right),$$

where $T, T' \in \{Z_8, Z_7, Z_6, Z_5, Z_4\}$, $T \setminus T' \neq \varnothing$ and $T' \setminus T \neq \varnothing$ and by statement g) of Theorem 3 satisfies the following conditions

$$Y_9^\alpha \cup Y_T^\alpha \supseteq Y_i, Y_9^\alpha \cup Y_{T'}^\alpha \supseteq Y_i', Y_T^\alpha \cap Y_i \neq \varnothing, Y_{T'}^\alpha \cap Y_i' \neq \varnothing, Y_0^\alpha \cap \breve{D} \neq \varnothing;$$
$$Y_9^\alpha \cup Y_T^\alpha \supseteq Y_j, Y_9^\alpha \cup Y_{T'}^\alpha \supseteq Y_j', Y_T^\alpha \cap Y_j \neq \varnothing, Y_{T'}^\alpha \cap Y_j' \neq \varnothing, Y_0^\alpha \cap \breve{D} \neq \varnothing.$$

From this conditions it follows that

$$Y_9^\alpha \cup Y_T^\alpha \supseteq Y_i \cup Y_j, Y_9^\alpha \cup Y_{T'}^\alpha \supseteq Y_i' \cup Y_j'.$$

For D_i' and D_j' we consider the following cases.

1) $Y_i \cup Y_j = \breve{D}$ or $Y_i' \cup Y_j' = \breve{D}$. Then

$$\left(Y_9^\alpha \cup Y_T^\alpha\right) \cap Y_{T'}^\alpha \supseteq \breve{D} \cap Y_{T'}^\alpha \supseteq Y_i' \cap Y_{T'}^\alpha \neq \varnothing$$

or

$$\left(Y_9^\alpha \cup Y_{T'}^\alpha\right) \cap Y_T^\alpha \supseteq \breve{D} \cap Y_T^\alpha \supseteq Y_i \cap Y_T^\alpha \neq \varnothing.$$

But the inequalities $\left(Y_9^\alpha \cup Y_T^\alpha\right) \cap Y_{T'}^\alpha \neq \varnothing$ and $\left(Y_9^\alpha \cup Y_{T'}^\alpha\right) \cap Y_T^\alpha \neq \varnothing$ contradicts the condition that representation of a binary relation α is quasinormal. So, the equality $R\left(D_i'\right) \cap R\left(D_j'\right) = \varnothing$ holds. From the last equality by definition of the semilattice D it follows that $R\left(D_i'\right) \cap R\left(D_j'\right) = \varnothing$ for all $(i, j) \in M_1\left(Q_7\right)$, where

$$\begin{aligned} M_1\left(Q_7\right) = \{&(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,9),(1,10),(1,11),(1,12),\\ &(2,3),(2,4),(2,5),(2,6),(2,7),(2,8),(2,9),(2,10),(2,11),\\ &(2,12),(3,4),(3,4),(3,5),(3,6),(3,8),(3,10),(3,11),(3,12),\\ &(4,5),(4,6),(4,7),(4,9),(4,11),(4,12),(5,6),(5,8),(5,10),\\ &(5,11),(5,12),(6,7),(6,9),(6,11),(6,12)\}. \end{aligned}$$

2) $Y_i \cup Y_i' = Y_j \cup Y_j' = Z_3$ or $Y_i \cup Y_i' = Y_j \cup Y_j' = Z_1$. Then

$$\left(Y_9^\alpha \cup Y_T^\alpha\right) \cap Y_{T'}^\alpha \supseteq Z_3 \cap Y_{T'}^\alpha \supseteq Y_i' \cap Y_{T'}^\alpha \neq \varnothing$$

or

$$\left(Y_9^\alpha \cup Y_{T'}^\alpha\right) \cap Y_T^\alpha \supseteq Z_1 \cap Y_T^\alpha \supseteq Y_i \cap Y_T^\alpha \neq \varnothing.$$

But the inequalities $\left(Y_9^\alpha \cup Y_T^\alpha\right) \cap Y_{T'}^\alpha \neq \varnothing$ and $\left(Y_9^\alpha \cup Y_{T'}^\alpha\right) \cap Y_T^\alpha \neq \varnothing$ contradicts the condition that representation of a binary relation α is quasinormal. So, the equality $R\left(D_i'\right) \cap R\left(D_j'\right) = \varnothing$ holds. From the last equality by definition of the semilattice D it follows that $R\left(D_i'\right) \cap R\left(D_j'\right) = \varnothing$ for all $(i, j) \in M_2\left(Q_7\right)$, where

$$\begin{aligned} M_2\left(Q_7\right) = \{&(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),\\ &(3,4),(3,5),(3,6),(4,5),(4,6),(5,6),(7,8),(7,9),(7,10),\\ &(7,11),(7,12),(8,9),(8,10),(8,11),(8,12),(9,10),(9,11),\\ &(9,12),(10,11),(10,12),(11,12)\}. \end{aligned}$$

3) $Y_i \cup Y_i' = Z_3$, $Y_j \cup Y_j' = Z_1$ or $Y_i \cup Y_i' = Z_1$, $Y_j \cup Y_j' = Z_3$. Then

$$\left(Y_9^\alpha \cup Y_T^\alpha\right) \cap Y_{T'}^\alpha \supseteq Z_3 \cap Y_{T'}^\alpha \supseteq Z_6 \cap Y_{T'}^\alpha \neq \varnothing$$

or

$$\left(Y_9^\alpha \cup Y_{T'}^\alpha\right) \cap Y_T^\alpha \supseteq Z_3 \cap Y_T^\alpha \supseteq Z_6 \cap Y_T^\alpha \neq \varnothing.$$

But the inequalities $\left(Y_9^\alpha \cup Y_T^\alpha\right) \cap Y_{T'}^\alpha \neq \varnothing$ and $\left(Y_9^\alpha \cup Y_{T'}^\alpha\right) \cap Y_T^\alpha \neq \varnothing$ contradicts the condition that representation of a binary relation α is quasinormal. So, the equality $R\left(D_i'\right) \cap R\left(D_j'\right) = \varnothing$ holds. From the last equality by definition of the semilattice D it follows that $R\left(D_i'\right) \cap R\left(D_j'\right) = \varnothing$ for all $(i, j) \in M_3\left(Q_7\right)$, where

$$M_3(Q_7)=\{(3,7),(3,9),(4,8),(4,10),(5,7),(5,9),(6,8),(6,10)\}$$

Now by conditions 1), 2) and 3) it follows that

$$\left|R^*(Q_7)\right|=\sum_{i=1}^{6}\left|R(D_i')\right|.$$

By statement (g) of Lemma 3 the given Lemma is true. □

Let binary relation α of the semigroup $B_X(D)$ satisfy the condition h) of Theorem 3 (see diagram 8 of the **Figure 3**). In this case we have $Q_8=\{Z_9,T,Z_6,T\cup Z_6,T',\breve{D}\}$, where $T\in\{Z_8,Z_7,Z_5,Z_4\}$, $T'\in\{Z_3,Z_2,Z_1\}$. By definition of the semilattice D it follows that

$$\begin{aligned}Q_8\vartheta_{XI}=&\{Z_9,Z_8,Z_6,Z_3,Z_2,\breve{D}\},\{Z_9,Z_8,Z_6,Z_3,Z_1,\breve{D}\},\\&\{Z_9,Z_7,Z_6,Z_3,Z_2,\breve{D}\},\{Z_9,Z_7,Z_6,Z_3,Z_1,\breve{D}\},\\&\{Z_9,Z_6,Z_5,Z_3,Z_1,\breve{D}\},\{Z_9,Z_6,Z_5,Z_2,Z_1,\breve{D}\},\\&\{Z_9,Z_6,Z_4,Z_3,Z_1,\breve{D}\},\{Z_9,Z_6,Z_4,Z_2,Z_1,\breve{D}\}.\end{aligned}$$

It is easy to see $\left|\Phi(Q_8,Q_8)\right|=1$ and $\left|\Omega(Q_8)\right|=8$. If

$$\begin{aligned}&D_1'=\{Z_9,Z_8,Z_6,Z_3,Z_2,\breve{D}\},D_2'=\{Z_9,Z_8,Z_6,Z_3,Z_1,\breve{D}\},\\&D_3'=\{Z_9,Z_7,Z_6,Z_3,Z_2,\breve{D}\},D_4'=\{Z_9,Z_7,Z_6,Z_3,Z_1,\breve{D}\},\\&D_5'=\{Z_9,Z_6,Z_5,Z_3,Z_1,\breve{D}\},D_6'=\{Z_9,Z_6,Z_5,Z_2,Z_1,\breve{D}\},\\&D_7'=\{Z_9,Z_6,Z_4,Z_3,Z_1,\breve{D}\},D_8'=\{Z_9,Z_6,Z_4,Z_2,Z_1,\breve{D}\}\end{aligned}$$

(see **Figure 6**).

Then

$$R^*(Q_8)=\bigcup_{i=1}^{8}R(D_i').$$

Lemma 17. *Let X be a finite set, $D=\{Z_9,Z_8,Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_1(X,10)$ and $Z_9\neq\varnothing$. Let $R^*(Q_8)$ be set of all regular elements of $B_X(D)$ such that each element satisfies the condition h) of Theorem* 3. *Then*

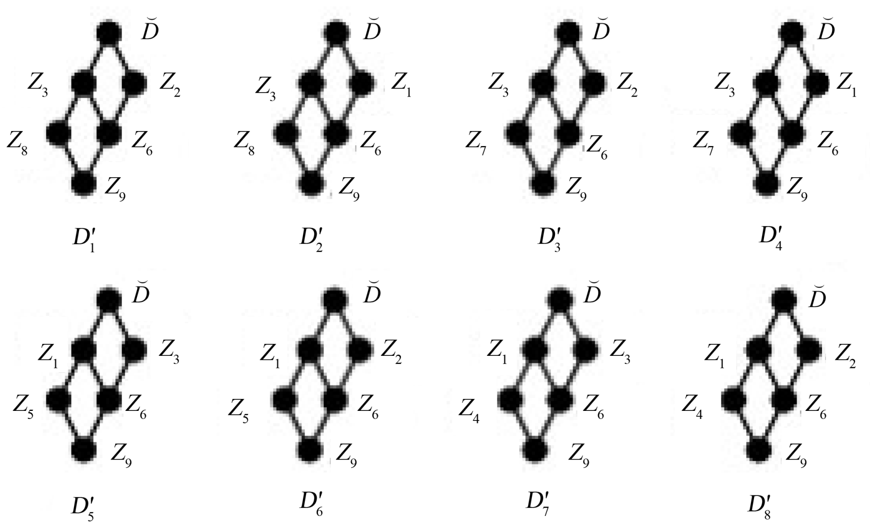

Figure 6. Diagram of all subsemilattices isomorphic to 8 in **Figure 2**.

$$\begin{aligned}\left|R^*\left(Q_8\right)\right| &= 8\cdot\left(2^{|Z_8\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 6^{\left|X\setminus\breve{D}\right|}\\ &+8\cdot\left(2^{|Z_8\setminus Z_1|}-1\right)\cdot\left(2^{|Z_6\setminus Z_8|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 6^{\left|X\setminus\breve{D}\right|}\\ &+8\cdot\left(2^{|Z_7\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 6^{\left|X\setminus\breve{D}\right|}\\ &+8\cdot\left(2^{|Z_7\setminus Z_1|}-1\right)\cdot\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 6^{\left|X\setminus\breve{D}\right|}\\ &+8\cdot\left(2^{|Z_5\setminus Z_3|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(3^{|Z_3\setminus Z_1|}-2^{|Z_3\setminus Z_1|}\right)\cdot 6^{\left|X\setminus\breve{D}\right|}\\ &+8\cdot\left(2^{|Z_5\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{\left|X\setminus\breve{D}\right|}\\ &+8\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(3^{|Z_3\setminus Z_1|}-2^{|Z_3\setminus Z_1|}\right)\cdot 6^{\left|X\setminus\breve{D}\right|}\\ &+8\cdot\left(2^{|Z_4\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_4|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{\left|X\setminus\breve{D}\right|}.\end{aligned}$$

Proof. Let $D'_i=\left\{Z_9,Y_i,Z_6,Y_i\cup Z_6,Y'_i,\breve{D}\right\}, D'_j=\left\{Z_9,Y_j,Z_6,Y_j\cup Z_6,Y'_j,\breve{D}\right\}\in\left\{D'_1,D'_2,\cdots,D'_8\right\}$, where $D'_i\neq D'_j$, $Y_i,Y_j\in\left\{Z_8,Z_7,Z_5,Z_4\right\}$, $Y'_i,Y'_j\in\left\{Z_3,Z_2,Z_1\right\}$ and $\alpha\in R\left(D'_i\right)\cap R\left(D'_j\right)$. Then quasinormal representation of a binary relation α of the semigroup $B_X\left(D\right)$ has a form

$$\alpha=\left(Y_9^\alpha\times Z_9\right)\cup\left(Y_T^\alpha\times T\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_{T\cup Z_6}^\alpha\times\left(T\cup Z_6\right)\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$$

where $Y_9^\alpha,Y_T^\alpha,Y_6^\alpha,Y_{T'}^\alpha\in\{\varnothing\}$, $T\in\left\{Z_8,Z_7,Z_5,Z_4\right\}$, $T'\in\left\{Z_3,Z_2,Z_1\right\}$ and by statement g) of Theorem 3 satisfies the following conditions

$$\begin{aligned}&Y_9^\alpha\supseteq Z_9, Y_9^\alpha\cup Y_T^\alpha\supseteq Y_i, Y_9^\alpha\cup Y_6^\alpha\supseteq Z_6, Y_9^\alpha\cup Y_6^\alpha\cup Y_{T'}^\alpha\supseteq Y'_i,\\ &Y_T^\alpha\cap Y_i\neq\varnothing, Y_6^\alpha\cap Z_6\neq\varnothing, Y_{T'}^\alpha\cap Y'_i\neq\varnothing;\\ &Y_9^\alpha\supseteq Z_9, Y_9^\alpha\cup Y_T^\alpha\supseteq Y_j, Y_9^\alpha\cup Y_6^\alpha\supseteq Z_6, Y_9^\alpha\cup Y_6^\alpha\cup Y_{T'}^\alpha\supseteq Y'_j,\\ &Y_T^\alpha\cap Y_j\neq\varnothing, Y_6^\alpha\cap Z_6\neq\varnothing, Y_{T'}^\alpha\cap Y'_j\neq\varnothing,\end{aligned}$$

From this conditions it follows that

$$Y_9^\alpha\cup Y_T^\alpha\supseteq Y_i\cup Y_j, Y_9^\alpha\cup Y_6^\alpha\cup Y_{T'}^\alpha\supseteq Y'_i\cup Y'_j.$$

For D'_i and D'_j we consider the following case.

$Y'_i\cup Y'_j=\breve{D}$. Then $\left(Y_9^\alpha\cup Y_6^\alpha\cup Y_{T'}^\alpha\right)\cap Y_T^\alpha\supseteq\breve{D}\cap Y_T^\alpha\supseteq Y_i\cap Y_T^\alpha\neq\varnothing$. But the inequality $\left(Y_9^\alpha\cup Y_6^\alpha\cup Y_{T'}^\alpha\right)\cap Y_T^\alpha\neq\varnothing$ contradicts the condition that representation of binary relation α is quasinormal. So, the equality $R\left(D'_i\right)\cap R\left(D'_j\right)=\varnothing$ holds. From the last equality by definition of the semilattice D it follows that $R\left(D'_i\right)\cap R\left(D'_j\right)=\varnothing$ for all $\left(i,j\right)\in M_1\left(Q_8\right)$, where

$$\begin{aligned}M_1\left(Q_8\right)=\{&(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(2,3),(2,4),(2,5),\\ &(2,6),(2,7),(2,8),(3,4),(3,5),(3,6),(3,7),(3,8),(4,5),(4,6),\\ &(4,7),(4,8),(5,6),(5,7),(5,8),(6,7),(6,8),(7,8)\}.\end{aligned}$$

Therefore we have

$$\left|R^*\left(Q_8\right)\right|=\sum_{i=1}^{8}\left|R\left(D'_i\right)\right|.$$

By statement h) of Lemma 3 the given Lemma is true. □

Let us assume that

$$s_1=\sum_{i=1}^{8}\left|R^*\left(Q_i\right)\right|.$$

Theorem 4. *Let* $D\in\Sigma_1\left(X,10\right)$, $Z_9\neq\varnothing$. *If X is a finite set and* R_D *is a set of all regular elements of the* *semigroup* $B_X(D)$ *then* $|R_D| = s_1$.

Proof. This Theorem immediately follows from Theorem 2 and Theorem 3. □

Example 1. *Let* $X = \{1,2,3,4,5,6,7,8\}$,

$$P_0 = \{6\}, P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\}, P_4 = \{4\},$$

$$P_5 = \{5\}, P_7 = \{7\}, P_8 = \{8\}, P_9 = P_6 = \varnothing.$$

Then $\breve{D} = \{1,2,3,4,5,6,7,8\}$, $Z_1 = \{2,3,4,5,6,7,8\}$, $Z_2 = \{1,3,4,5,6,7,8\}$, $Z_3 = \{1,2,4,5,6,7,8\}$, $Z_4 = \{2,3,5,6,7,8\}$, $Z_5 = \{2,3,4,6,7,8\}$, $Z_6 = \{4,5,6,7,8\}$, $Z_7 = \{1,2,4,5,6,8\}$, $Z_8 = \{1,2,4,5,6,7\}$ and $Z_9 = \{6\}$.

$$D = \{\{1,2,3,4,5,6,7,8\},\{2,3,4,5,6,7,8\},\{1,3,4,5,6,7,8\},\{1,2,4,5,6,7,8\},$$
$$\{2,3,5,6,7,8\},\{2,3,4,6,7,8\},\{4,5,6,7,8\},\{1,2,4,5,6,8\},\{1,2,4,5,6,7\},\{6\}\}$$

We have $Z_9 \neq \varnothing$, $|R^*(Q_1)| = 10$, $|R^*(Q_2)| = 3048$, $|R^*(Q_3)| = 14652$, $|R^*(Q_4)| = 2443$, $|R^*(Q_5)| = 3600$, $|R^*(Q_6)| = 540$, $|R^*(Q_7)| = 168$, $|R(Q_8)| = 64$, $R_D = 24525$.

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