Applied Mathematics, 2016, 7, 272-291
Published Online February 2016 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2016.73025

Interval Oscillation Criteria for Fractional Partial Differential Equations with Damping Term

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Received 28 January 2016; accepted 26 February 2016; published 29 February 2016

Abstract

In this article, we will establish sufficient conditions for the interval oscillation of fractional partial differential equations of the form

$$D_{+,t}^{\alpha}\left[r(t)D_{+,t}^{\alpha}\left(u(x,t)\right)\right]+q(x,t)D_{+,t}^{\alpha}u(x,t)+p(x,t)f\left(\int_{0}^{t}(t-s)^{-\alpha}u(x,s)\mathrm{d}s\right)g\left(D_{+,t}^{\alpha}u(x,t)\right)$$
$$=a(t)\Delta u(x,t)+F(x,t),(x,t)\in G=\Omega\times R_{+}.$$

It is based on the information only on a sequence of subintervals of the time space $[t_0,\infty)$ rather than whole half line. We consider *f* to be monotonous and non monotonous. By using a generalized Riccati technique, integral averaging method, Philos type kernals and new interval oscillation criteria are established. We also present some examples to illustrate our main results.

Keywords

Fractional, Parabolic, Oscillation, Fractional Differential Equation, Damping

1. Introduction

Fractional differential equations are now recognized as an excellent source of knowledge in modelling dynamical processes in self similar and porous structures, electrical networks, probability and statistics, visco elasticity, electro chemistry of corrosion, electro dynamics of complex medium, polymer rheology, industrial robotics,

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How to cite this paper: Sadhasivam, V. and Kavitha, J. (2016) Interval Oscillation Criteria for Fractional Partial Differential Equations with Damping Term. *Applied Mathematics*, **7**, 272-291. http://dx.doi.org/10.4236/am.2016.73025

economics, biotechnology, etc. For the theory and applications of fractional differential equations, we refer the monographs and journals in the literature [1]-[10]. The study of oscillation and other asymptotic properties of solutions of fractional order differential equations has attracted a good bit of attention in the past few years [11]-[13]. In the last few years, the fundamental theory of fractional partial differential equations with deviating arguments has undergone intensive development [14]-[22]. The qualitative theory of this class of equations is still in an initial stage of development.

In 1965, Wong and Burton [23] studied the differential equations of the form

$$u''+a(t)f(u)g(u')=0$$

In 1970, Burton and Grimer [24] has been investigated the qualitative properties of

$$(ru')'+af(u)g(u')=0$$

In 2009, Nandakumaran and Panigrahi [25] derived the oscillatory behavior of nonlinear homogeneous differential equations of the form

$$(r(t)y')'+q(t)y'+p(t)f(y)g(y')=0$$

Formulation of the Problems

In this article, we wish to study the interval oscillatory behavior of non linear fractional partial differential equations with damping term of the form

$$(E)\qquad D_{+,t}^{\alpha}\left[r(t)D_{+,t}^{\alpha}u(x,t)\right]+q(x,t)D_{+,t}^{\alpha}u(x,t)+p(x,t)f\left(\int_0^t(t-s)^{-\alpha}u(x,s)\mathrm{d}s\right)g\left(D_{+,t}^{\alpha}u(x,t)\right)$$
$$=a(t)\Delta u(x,t)+F(x,t),\quad (x,t)\in G=\Omega\times R_{+}$$

where Ω is a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega, \alpha\in(0,1)$ is a constant, $D_{+,t}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α of u with respect to t and Δ is the Laplacian operator in the Euclidean N-space R^N (ie) $\Delta u(x,t)=\sum_{r=1}^{N}\frac{\partial^2u(x,t)}{\partial x_r^2}$. Equation ($E$) is supplemented with the Neumann boundary condition

$$(B_1)\quad \frac{\partial u(x,t)}{\partial \nu}+\mu(x,t)u(x,t)=0,\quad (x,t)\in\partial\Omega\times R_{+},$$

where γ denotes the unit exterior normal vector to $\partial\Omega$ and $\mu(x,t)$ is a non negative continuous function on $\partial\Omega\times R_{+}$ and

$$(B_2)\quad u(x,t)=0,\quad (x,t)\in\partial\Omega\times R_{+}.$$

In what follows, we always assume without mentioning that

(A_1) $r\in C^{\alpha}(R_{+},R_{+}),\ a\in C(R_{+},R_{+})F\in C(\bar{G},R)$;

(A_2) $q\in C(\bar{G},R), q(t)=\min_{x\in\bar{\Omega}}q(x,t)$; $p\in C(\bar{G},R_{+})$, $p(t)=\min_{x\in\bar{\Omega}}p(x,t)$ with $p(t)\neq 0$ on any $[T_0,\infty)$ for some $T_0\geq 0$

(A_3) $f\in C(R,R)$ is convex with $uf(u)>0$ for $u\neq 0$.

(A_4) $g:R\to[L,\infty)$ is continuous where $L>0$.

By a solution of (E), (B_1) and (B_2) we mean a non trivial function $u(x,t)\in C^{2\alpha}(\bar{G},R)$ with $\int_0^t(t-s)^{-\alpha}u(x,s)\mathrm{d}s\in C'(\bar{G},R)$, $r(t)D_{+,t}^{\alpha}u(x,t)\in C^{\alpha}(\bar{G},R)$ and satisfies $\bar{G}$ and the boundary conditions (B_1) and (B_2). A solution $u(x,t)$ of (E), (B_1) or (E), (B_2) is said to be oscillatory in G if it has arbitrary large zeros; otherwise, it is nonoscillatory. An Equation (E) is called oscillatory if all its solutions are oscillatory. To the best of our knowledge, nothing is known regarding the interval oscillation criteria of (E), (B_1) and (E), (B_2) upto now. Motivativated by [22]-[25], we will establish new interval oscillation criteria for (E), (B_1) and (E), (B_2). Our results are essentially new.

Definition 1.1. A function $H = H(t,s)$ belongs to a function class *P* denoted by $H \in P$ if $H \in C(D, R_+)$ where $D = \{(t,s): -\infty < s \le t < \infty\}$ which satisfies $H(t,t) = 0$, $H(t,s) > 0$ for $t > s$ and has partial derivatives $\frac{\partial H}{\partial t}$ and $\frac{\partial H}{\partial s}$ on *D* such that

$$\frac{\partial H}{\partial t} = h_1(t,s)\sqrt{H(t,s)} \text{ and } \frac{\partial H}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}$$

where $h_1, h_2 \in L^1_{loc}(D, R)$.

2. Preliminaries

In this section, we will see the definitions of fractional derivatives and integrals. In this paper, we use the Riemann-Liouville left sided definition on the half axis R_+. The following notations will be used for the convenience.

$$U(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)\,\mathrm{d}x, \text{ where } |\Omega| = \int_\Omega \mathrm{d}x, \tag{1}$$

$$\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}, \xi_0 = \frac{t_0^\alpha}{\Gamma(1+\alpha)}, \tilde{q}(\xi) = q(t), \tilde{p}(\xi) = p(t), \xi_{c_i} = \frac{c_i^\alpha}{\Gamma(1+\alpha)}, \xi_{d_i} = \frac{d_i^\alpha}{\Gamma(1+\alpha)}$$

For $s, \xi \in [\xi_0, \infty)$ denote

$$Q_1(s,\xi) = h_1(s,\xi) - \frac{\tilde{q}(s)}{\tilde{r}(s)}\sqrt{H(s,\xi)}$$

$$Q_2(\xi,s) = h_2(\xi,s) + \frac{\tilde{q}(s)}{\tilde{r}(s)}\sqrt{H(\xi,s)}.$$

Definition 2.1 [2] The Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to *t* of a function $u(x,t)$ is given by

$$\left(D^\alpha_{+,t}u\right)(x,t) := \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-v)^{-\alpha} u(x,v)\,\mathrm{d}v$$

provided the right hand side is pointwise defined on R_+ where Γ is the gamma function.

Definition 2.2 [2] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y: R_+ \to R$ on the half-axis R_+ is given by

$$\left(I^\alpha_+ y\right)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1} y(v)\,\mathrm{d}v \quad \text{for} \quad t > 0$$

provided the right hand side is pointwise defined on R_+.

Definition 2.3 [2] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y: R_+ \to R$ on the half-axis R_+ is given by

$$\left(D^\alpha_+ y\right)(t) := \frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}}\left(I_+^{\lceil\alpha\rceil-\alpha} y\right)(t) \quad \text{for} \quad t > 0$$

provided the right hand side is pointwise defined on R_+ where $\lceil\alpha\rceil$ is the ceiling function of α.

Lemma 2.1 Let *y* be solution of (E) and

$$K(t) := \int_0^t (t-s)^{-\alpha} y(v)\,\mathrm{d}v \quad \text{for} \quad \alpha \in (0,1) \quad \text{and} \quad t > 0. \tag{2}$$

Then

$$K'(t) = \Gamma(1-\alpha)\left(D^\alpha_+ y\right)(t) \quad \text{for} \quad \alpha \in (0,1) \quad \text{and} \quad t > 0. \tag{3}$$

3. Oscillation with Monotonicity of $f(x)$ of (E) and (B_1)

In this section, we assume that (A_5) f is monotonous and satisfies the condition $f'(u) \ge M > 0$ where M is a constant.

Theorem 3.1 If the fractional differential inequality

$$D_+^\alpha\left[r(t)D_+^\alpha U(t)\right] + q(t)D_+^\alpha U(t) + p(t)f(K(t))L - F(t) \le 0 \quad (4)$$

has no eventually positive solution, then every solution of (E) and (B_1) is oscillatory in $G_{t_1} = \Omega\times[t_1,\infty)$, where $t_1 \ge 0$.

Proof. Suppose to the contrary that there is a non oscillatory solution $u(x,t)$ of the problem (E) and (B_1) which has no zero in $\Omega\times[t_0,\infty)$ for some $t_0 > 0$. Without loss of generality, we may assume that $u(x,t) > 0$ in $\Omega\times[t_1,\infty)$, $t_1 \ge t_0$. Integrating (E) with respect to x over Ω, we have

$$\begin{aligned}&\int_\Omega D_{+,t}^\alpha\left[r(t)D_{+,t}^\alpha u(x,t)\right]\mathrm{d}x + \int_\Omega q(x,t)D_{+,t}^\alpha u(x,t)\mathrm{d}x + \int_\Omega p(x,t)f\left(\int_0^t (t-s)^{-\alpha}u(x,s)\mathrm{d}s\right)g\left(D_{+,t}^\alpha u(x,t)\right)\mathrm{d}x\\ &= a(t)\int_\Omega \Delta u(x,t)\mathrm{d}x + \int_\Omega F(x,t)\mathrm{d}x.\end{aligned} \quad (5)$$

Using Green's formula and boundary condition (B_1), it follows that

$$\int_\Omega \Delta u(x,t)\mathrm{d}x = \int_{\partial\Omega}\frac{\partial u(x,t)}{\partial\gamma}\mathrm{d}S = -\int_{\partial\Omega}\mu(x,t)u(x,t)\mathrm{d}S \le 0, \quad t \ge t_1. \quad (6)$$

$$\begin{aligned}\int_\Omega q(x,t)D_{+,t}^\alpha u(x,t)\mathrm{d}x &\ge q(t)D_{+,t}^\alpha\left(\int_\Omega u(x,t)\mathrm{d}x\right)\\ &\ge q(t)|\Omega|D_{+,t}^\alpha\left(\frac{1}{|\Omega|}\int_\Omega u(x,t)\mathrm{d}x\right)\\ &\ge q(t)|\Omega|D_+^\alpha U(t), \quad t \ge t_1.\end{aligned} \quad (7)$$

By Jensen's inequality and (A_2) we get

$$\begin{aligned}&\int_\Omega p(x,t)f\left(\int_0^t (t-s)^{-\alpha}u(x,s)\mathrm{d}s\right)g\left(D_{+,t}^\alpha u(x,t)\right)\mathrm{d}x\\ &\ge p(t)\int_\Omega f\left(\int_0^t (t-s)^{-\alpha}u(x,s)\mathrm{d}s\right)g\left(D_{+,t}^\alpha u(x,t)\right)\mathrm{d}x\\ &\ge p(t)|\Omega| f\left(\int_0^t (t-s)^{-\alpha}\left(\frac{1}{|\Omega|}\int_\Omega u(x,s)\mathrm{d}x\right)\mathrm{d}s\right)g\left(D_{+,t}^\alpha u(x,t)\right)\\ &\ge p(t)|\Omega| f\left(\int_0^t (t-s)^{-\alpha}U(s)\mathrm{d}s\right)g\left(D_{+,t}^\alpha u(x,t)\right).\end{aligned}$$

By using $g\left(D_{+,t}^\alpha u(x,t)\right) \ge L > 0$ we have

$$\int_\Omega p(x,t)f\left(\int_0^t (t-s)^{-\alpha}u(x,s)\mathrm{d}s\right)g\left(D_{+,t}^\alpha u(x,t)\right)\mathrm{d}x \ge p(t)|\Omega| f(K(t))L, \quad t \ge t_1. \quad (8)$$

In view of (1), (6)-(8), (5) yield

$$D_+^\alpha\left[r(t)D_+^\alpha U(t)|\Omega|\right] + q(t)|\Omega|U(t) + Lp(t)|\Omega| f(K(t)) \le \int_\Omega F(x,t)\mathrm{d}x.$$

Take $F(t) = \frac{1}{|\Omega|}\int_\Omega F(x,t)\mathrm{d}x$, therefore

$$D_+^\alpha\left[r(t)D_+^\alpha U(t)\right] + q(t)D_+^\alpha U(t) + Lp(t)f(K(t)) - F(t) \le 0, t \ge t_1$$

Therefore $U(t)$ is eventually positive solution of (4). This contradicts the hypothesis and completes the proof.

Remark 3.1 Let

$$r(t)=\tilde{r}(\xi),\quad U(t)=\tilde{U}(\xi),\quad p(t)=\tilde{p}(\xi)$$
$$f(K(t))=f(\tilde{K}(\xi)),\quad F(t)=\tilde{F}(\xi).$$

Then $D_{+}^{\alpha}U(t)=\tilde{U}'(\xi)$ we use this transformation in (4). The inequality becomes

$$\left(\tilde{r}(\xi)\tilde{U}'(\xi)\right)'+\tilde{q}(\xi)U'(\xi)+L\tilde{p}(\xi)f\left(\tilde{K}(\xi)\right)-\tilde{F}(\xi)\le 0. \tag{9}$$

Theorem (3.1) can be stated as, if the differential inequality

$$\left[\tilde{r}(\xi)\tilde{U}'(\xi)\right]'+\tilde{q}(\xi)\tilde{U}'(\xi)+L\tilde{p}(\xi)f\left(\tilde{K}(\xi)\right)-\tilde{F}(\xi)\le 0,$$

has no eventually positive solution then every solution of (E) and (B_1) is oscillatory in $G_{\xi_1}=\Omega\times[\xi_1,\infty)$ where $\xi_1\ge 0$.

Theorem 3.2 Suppose that the conditions (A_1) - (A_5) hold. Assume that for any $T_0\ge t_0$ there exist c_i, δ_i, d_i for $i=1,2$ such that $T_0\le c_1<\delta_1<d_1\le c_2<\delta_2<d_2$, $t\in[c_2,d_2]\cup[c_2,d_2]$ satisfying

$$F(t)=\begin{cases}\le 0, & t\in[c_1,d_1]\\ \ge 0, & t\in[c_2,d_2]\end{cases} \tag{10}$$

If there exist $\xi_{\delta_i}\in\left(\xi_{c_i},\xi_{d_i}\right)$, $H\in P$ and $\rho\in C^{\alpha}\left([t_0,\infty),R_{+}\right)$ such that

$$\begin{aligned}&\frac{1}{H\left(\xi_{\delta_i},\xi_{c_i}\right)}\int_{\xi_{c_i}}^{\xi_{\delta_i}}H\left(s,\xi_{c_i}\right)\tilde{\phi}(s)\mathrm{d}s+\frac{1}{H\left(\xi_{d_i},\xi_{\delta_i}\right)}\int_{\xi_{\delta_i}}^{\xi_{d_i}}H\left(\xi_{d_i},s\right)\tilde{\phi}(s)\mathrm{d}s\\&>\frac{1}{4M\Gamma(1-\alpha)H\left(\xi_{\delta_i},\xi_{c_i}\right)}\int_{\xi_{c_i}}^{\xi_{\delta_i}}\tilde{r}(s)\tilde{v}(s)Q_1^2\left(s,\xi_{c_i}\right)\mathrm{d}s\\&+\frac{1}{4M\Gamma(1-\alpha)H\left(\xi_{d_i},\xi_{\delta_i}\right)}\int_{\xi_{\delta_i}}^{\xi_{d_i}}\tilde{r}(s)\tilde{v}(s)Q_2^2\left(\xi_{d_i},s\right)\mathrm{d}s,\quad\text{for } i=1,2,\end{aligned} \tag{11}$$

where $\tilde{v}$ and $\tilde{\phi}$ are defined as

$$\tilde{v}(\xi)=\exp\left(-2M\int_{\xi_0}^{\xi}\tilde{\rho}(s)\mathrm{d}s\right)$$
$$\tilde{\phi}(\xi)=\tilde{v}(\xi)\left\{L\tilde{p}(\xi)-\tilde{q}(\xi)\tilde{\rho}(\xi)+M\Gamma(1-\alpha)\tilde{r}(\xi)\tilde{\rho}^2(\xi)-\left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right\}.$$

Then every solution of (E), (B_1) is oscillatory in G.

Proof. Suppose to the contrary that $u(x,t)$ be a non oscillatory solution of the problem (E), (B_1) say $u(x,t)\ne 0$ in $\Omega\times[T_0,\infty)$ for some $T_0\ge t_0$. Define the following Riccati transformation function

$$w(t)=v(t)r(t)\left[\frac{D_{+}^{\alpha}U(t)}{f(K(t))}+\rho(t)\right],\quad t\ge T_0.$$

Then for $t\ge T_0$

$$\begin{aligned}D_{+}^{\alpha}w(t)&=D_{+}^{\alpha}v(t)\frac{w(t)}{v(t)}\\&+v(t)\left\{\frac{D_{+}^{\alpha}\left(r(t)D_{+}^{\alpha}U(t)\right)}{f(K(t))}-r(t)\frac{D_{+}^{\alpha}U(t)}{f^2(K(t))}f'(K(t))D_{+}^{\alpha}(K(t))+D_{+}^{\alpha}\left(\rho(t)r(t)\right)\right\}.\end{aligned}$$

By using $f'(K(t))\ge M>0$ and inequality (4) we get

$$D_+^\alpha w(t) \le D_+^\alpha v(t)\frac{w(t)}{v(t)} + v(t)\left[-\frac{q(t)D_+^\alpha U(t)}{f(K(t))} - Lp(t) - Mr(t)\frac{D_+^\alpha U(t)}{f^2(K(t))}D_+^\alpha(K(t)) + D_+^\alpha(\rho(t)r(t))\right] + \frac{F(t)v(t)}{f(K(t))}. \quad (12)$$

By assumption, if $u(x,t)>0$ then we can choose $c_1, d_1 \ge T_0$ with $c_1 < d_1$ such that $F(t) \le 0$ on the interval $[c_1, d_1]$. If $u(x,t)<0$ then we can choose $c_2, d_2 \ge T_0$ with $c_2 < d_2$ such that $F(t) \ge 0$ on the interval $[c_1, d_1]$ So

$$\frac{F(t)v(t)}{f(K(t))} \le 0, \quad t \in [c_i, d_i], \quad i = 1, 2,$$

therefore inequality (12) becomes

$$D_+^\alpha w(t) \le D_+^\alpha v(t)\frac{w(t)}{v(t)} + v(t)\left[-\frac{q(t)D_+^\alpha U(t)}{f(K(t))} - Lp(t) - Mr(t)\frac{D_+^\alpha U(t)}{f^2(K(t))}D_+^\alpha(K(t)) + D_+^\alpha(\rho(t)r(t))\right] \quad (13)$$

$$t \in [c_i, d_i], \quad i = 1, 2.$$

Let $w(t) = \tilde{w}(\xi)$, $v(t) = \tilde{v}(\xi)$, $q(t) = \tilde{q}(\xi)$, $U(t) = \tilde{U}(\xi)$, $p(t) = \tilde{p}(\xi)$, $K(t) = \tilde{K}(\xi)$, $\rho(t) = \tilde{\rho}(\xi)$. Then $D_+^\alpha w(t) = \tilde{w}'(\xi)$, $D_+^\alpha U(t) = \tilde{U}'(\xi)$, $D_+^\alpha K(t) = \tilde{K}'(\xi)$, so (13) is transformed into

$$\tilde{w}'(\xi) \le \tilde{v}'(\xi)\frac{\tilde{w}(\xi)}{\tilde{v}(\xi)} + \tilde{v}(\xi)\left[\frac{-\tilde{q}(\xi)\tilde{U}'(\xi)}{f(\tilde{K}(\xi))} - L\tilde{p}(\xi) - M\tilde{r}(\xi)\frac{\tilde{U}'(\xi)}{f^2(\tilde{K}(\xi))}\tilde{K}'(\xi) + \left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right]$$

$$\le -2M\tilde{v}(\xi)\tilde{\rho}(\xi)\frac{\tilde{w}(\xi)}{\tilde{v}(\xi)} + \tilde{v}(\xi)\left[-\tilde{q}(\xi)\left(\frac{\tilde{w}(\xi)}{\tilde{v}(\xi)\tilde{r}(\xi)} - \tilde{\rho}(\xi)\right) - L\tilde{p}(\xi) - M\tilde{r}(\xi)\frac{\tilde{U}'(\xi)}{f^2(\tilde{K}(\xi))}\Gamma(1-\alpha)\tilde{U}'(\xi) + \left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right]$$

$$\le -2M\tilde{w}(\xi)\tilde{\rho}(\xi) - \frac{\tilde{q}(\xi)\tilde{w}(\xi)}{\tilde{r}(\xi)} + \tilde{v}(\xi)\left[\tilde{q}(\xi)\tilde{\rho}(\xi) - L\tilde{p}(\xi) - M\tilde{r}(\xi)\Gamma(1-\alpha)\left(\frac{\tilde{w}(\xi)}{\tilde{v}(\xi)\tilde{r}(\xi)} - \tilde{\rho}(\xi)\right)^2 + \left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right]$$

$$\le -2M\tilde{\rho}(\xi)\tilde{w}(\xi)\left[1 - M\Gamma(1-\alpha)\right] - \frac{\tilde{q}(\xi)\tilde{w}(\xi)}{\tilde{r}(\xi)} - \tilde{v}(\xi)\left[-\tilde{q}(\xi)\tilde{\rho}(\xi) + L\tilde{p}(\xi) + M\Gamma(1-\alpha)\tilde{\rho}^2(\xi) - \left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right] - M\Gamma(1-\alpha)\frac{\tilde{w}^2(\xi)}{\tilde{r}(\xi)\tilde{v}(\xi)}$$

$$\tilde{w}'(\xi) \le -\tilde{q}(\xi)\frac{\tilde{w}(\xi)}{\tilde{r}(\xi)} - \tilde{\phi}(\xi) - M\Gamma(1-\alpha)\frac{\tilde{w}^2(\xi)}{\tilde{r}(\xi)\tilde{v}(\xi)}$$

That is

$$\tilde{\phi}(\xi) \le -\tilde{w}'(\xi) - \tilde{q}(\xi)\frac{\tilde{w}(\xi)}{\tilde{r}(\xi)} - M\Gamma(1-\alpha)\frac{\tilde{w}^2(\xi)}{\tilde{r}(\xi)\tilde{v}(\xi)}, \quad \xi \in \left[\xi_{c_i}, \xi_{d_i}\right], \ i = 1, 2. \quad (14)$$

Let ξ_{δ_i} be an arbitrary point in $\left(\xi_{c_i}, \xi_{d_i}\right)$ substituting ξ with s multiplying both sides of (14) by $H(\xi, s)$

and integrating it over $\left[\xi_{\delta_i},\xi\right)$ for $\xi\in\left[\xi_{\delta_i},\xi_{d_i}\right)$ $i=1,2$ we obtain

$$\begin{aligned}
&\int_{\xi_{\delta_i}}^{\xi} H(\xi,s)\tilde{\phi}(s)\mathrm{d}s \\
&\le -\int_{\xi_{\delta_i}}^{\xi} H(\xi,s)\tilde{w}'(s)\mathrm{d}s-\int_{\xi_{\delta_i}}^{\xi} H(\xi,s)\tilde{q}(s)\frac{\tilde{w}(s)}{\tilde{r}(s)}\mathrm{d}s-\int_{\xi_{\delta_i}}^{\xi}\frac{M\Gamma(1-\alpha)}{\tilde{r}(s)\tilde{v}(s)}\tilde{w}^2(s)H(\xi,s)\mathrm{d}s \\
&=H\left(\xi,\xi_{\delta_i}\right)\tilde{w}\left(\xi_{\delta_i}\right)-\int_{\xi_{\delta_i}}^{\xi}\left[h_2(\xi,s)\sqrt{H(\xi,s)}\tilde{w}(s)+H(\xi,s)\frac{\tilde{q}(s)}{\tilde{r}(s)}\tilde{w}(s)+\frac{M\Gamma(1-\alpha)}{\tilde{r}(s)\tilde{v}(s)}\tilde{w}^2(s)H(\xi,s)\right]\mathrm{d}s \\
&=H\left(\xi,\xi_{\delta_i}\right)\tilde{w}\left(\xi_{\delta_i}\right)-\int_{\xi_{\delta_i}}^{\xi}\left[\sqrt{\frac{M\Gamma(1-\alpha)H(\xi,s)}{\tilde{r}(s)\tilde{v}(s)}}\tilde{w}(s)+\frac{1}{2}\sqrt{\frac{\tilde{r}(s)\tilde{v}(s)}{M\Gamma(1-\alpha)}}\left(h_2(\xi,s)+\frac{\tilde{q}(s)}{\tilde{r}(s)}\sqrt{H(\xi,s)}\right)\right]^2\mathrm{d}s \\
&\quad+\frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{\delta_i}}^{\xi}\left(h_2(\xi,s)+\frac{\tilde{q}(s)}{\tilde{r}(s)}\sqrt{H(\xi,s)}\right)^2\tilde{r}(s)\tilde{v}(s)\mathrm{d}s \\
&=H\left(\xi,\xi_{\delta_i}\right)\tilde{w}\left(\xi_{\delta_i}\right)-\int_{\xi_{\delta_i}}^{\xi}\left[\sqrt{\frac{M\Gamma(1-\alpha)H(\xi,s)}{\tilde{r}(s)\tilde{v}(s)}}\tilde{w}(s)+\frac{1}{2}\sqrt{\frac{\tilde{r}(s)\tilde{v}(s)}{M\Gamma(1-\alpha)}}Q_2(\xi,s)\right]^2\mathrm{d}s \\
&\quad+\frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{\delta_i}}^{\xi}Q_2^2(\xi,s)\tilde{r}(s)\tilde{v}(s)\mathrm{d}s\le H\left(\xi,\xi_{\delta_i}\right)\tilde{w}\left(\xi_{\delta_i}\right)+\frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{\delta_i}}^{\xi}Q_2^2(\xi,s)\tilde{r}(s)\tilde{v}(s)\mathrm{d}s
\end{aligned}$$

Letting $\xi\to\xi_{d_i}-$ and dividing both sides by $H\left(\xi_{d_i},\xi_{\delta_i}\right)$

$$\frac{1}{H\left(\xi_{d_i},\xi_{\delta_i}\right)}\int_{\xi_{\delta_i}}^{\xi_{d_i}}H\left(\xi_{d_i},s\right)\tilde{\phi}(s)\mathrm{d}s\le\tilde{w}\left(\xi_{\delta_i}\right)+\frac{1}{4M\Gamma(1-\alpha)H\left(\xi_{d_i},\xi_{\delta_i}\right)}\int_{\xi_{\delta_i}}^{\xi_{d_i}}Q_2^2\left(\xi_{d_i},s\right)\tilde{r}(s)\tilde{v}(s)\mathrm{d}s \tag{15}$$

On the other hand, substituting ξ by s multiply both sides of (14) by $H(s,\xi)$ and integrating it over $\left(\xi,\xi_{\delta_i}\right)$ for $\xi\in\left[\xi_{c_i},\xi_{\delta_i}\right)$ we obtain

$$\begin{aligned}
\int_{\xi}^{\xi_{\delta_i}}H(s,\xi)\tilde{\phi}(s)\mathrm{d}s&\le-\int_{\xi}^{\xi_{\delta_i}}H(s,\xi)\tilde{w}'(s)\mathrm{d}s-\int_{\xi}^{\xi_{\delta_i}}H(s,\xi)\tilde{q}(s)\frac{\tilde{w}(s)}{\tilde{r}(s)}\mathrm{d}s-\int_{\xi}^{\xi_{\delta_i}}\frac{M\Gamma(1-\alpha)}{\tilde{r}(s)\tilde{v}(s)}\tilde{w}^2(s)H(s,\xi)\mathrm{d}s \\
&=-H\left(\xi_{\delta_i},\xi\right)\tilde{w}\left(\xi_{\delta_i}\right) \\
&\quad+\int_{\xi}^{\xi_{\delta_i}}\left[h_1(s,\xi)\sqrt{H(s,\xi)}\tilde{w}(s)-H(s,\xi)\frac{\tilde{q}(s)}{\tilde{r}(s)}\tilde{w}(s)-\frac{M\Gamma(1-\alpha)}{\tilde{r}(s)\tilde{v}(s)}H(s,\xi)\right]\mathrm{d}s \\
&\le-H\left(\xi_{\delta_i},\xi\right)\tilde{w}\left(\xi_{\delta_i}\right)+\frac{1}{4M\Gamma(1-\alpha)}\int_{\xi}^{\xi_{\delta_i}}Q_1^2(s,\xi)\tilde{r}(s)\tilde{v}(s)\mathrm{d}s
\end{aligned}$$

Letting $\xi\to\xi_{c_i}+$ and dividing both sides by $H\left(\xi_{\delta_i},\xi_{c_i}\right)$

$$\frac{1}{H\left(\xi_{\delta_i},\xi_{c_i}\right)}\int_{\xi_{c_i}}^{\xi_{\delta_i}}H\left(s,\xi_{c_i}\right)\tilde{\phi}(s)\mathrm{d}s\le-\tilde{w}\left(\xi_{\delta_i}\right)+\frac{1}{4M\Gamma(1-\alpha)H\left(\xi_{\delta_i},\xi_{c_i}\right)}\int_{\xi_{c_i}}^{\xi_{\delta_i}}Q_1^2\left(s,\xi_{c_i}\right)\tilde{r}(s)\tilde{v}(s)\mathrm{d}s \tag{16}$$

Now we claim that every non trivial solution of differential inequality (9) has atleast one zero in $\left(\xi_{c_i},\xi_{\delta_i}\right)$.

Suppose the contrary. By remark, without loss of generality, we may assume that there is a solution of (9) such that $\tilde{U}(\xi)>0$ for $\xi\in\left(\xi_{c_i},\xi_{\delta_i}\right)$. Adding (15) and (16) we get the inequality

$$\begin{aligned}
&\frac{1}{H\left(\xi_{\delta_i},\xi_{c_i}\right)}\int_{\xi_{c_i}}^{\xi_{\delta_i}}H\left(s,\xi_{c_i}\right)\tilde{\phi}(s)\mathrm{d}s+\frac{1}{H\left(\xi_{d_i},\xi_{\delta_i}\right)}\int_{\xi_{\delta_i}}^{\xi_{d_i}}H\left(\xi_{d_i},s\right)\tilde{\phi}(s)\mathrm{d}s \\
&\le\frac{1}{4M\Gamma(1-\alpha)H\left(\xi_{\delta_i},\xi_{c_i}\right)}\int_{\xi_{c_i}}^{\xi_{\delta_i}}Q_1^2\left(s,\xi_{c_i}\right)\tilde{r}(s)\tilde{v}(s)\mathrm{d}s+\frac{1}{4M\Gamma(1-\alpha)H\left(\xi_{d_i},\xi_{\delta_i}\right)}\int_{\xi_{\delta_i}}^{\xi_{d_i}}Q_2^2\left(\xi_{d_i},s\right)\tilde{r}(s)\tilde{v}(s)\mathrm{d}s
\end{aligned}$$

which contradicts the assumption (11). Thus the claim holds.

We consider a sequence $\{T_j\} \subset [t_0, \infty)$ such that $T_j \to \infty$ as $j \to \infty$. By the assumptions of the theorem for each $j \in N$ there exist $c_{ij}, \delta_{ij}, d_{ij} \in R$ such that $T_j \le c_{ij} \le \delta_{ij} \le d_{ij}$ and (11) holds with $\xi_{c_i}, \xi_{\delta_i}, \xi_{d_i}$, replaced by $\xi_{c_{ij}}, \xi_{\delta_{ij}}, \xi_{d_{ij}}$ respectively for $i = 1,2$ $j \in N$. From that, every non trivial solution $\tilde{U}(\xi)$ of (9) has at least one zero in $\xi_j \in (\xi_{c_{ij}}, \xi_{d_{ij}})$. Noting that $\xi_j \ge \xi_{c_{ij}} \ge \xi_{T_j}$ $j \in N$ we see that every solution $\tilde{U}(\xi)$ has arbitrary large zero. This contradicts the fact that $\tilde{U}(\xi)$ is non oscillatory by (9) and the assumption $u(x,t) \ne 0$ in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Hence every solution of the problem (E), (B_1) is oscillatory in *G*.

Theorem 3.3 Assume that the conditions (A_1) - (A_5) hold. Assume that there exist $H \in P$ $\rho \in C'([\xi_0, \infty), (0, \infty))$ such that for any $\xi_i \ge \xi_0$ $i = 1,2$,

$$\limsup_{\xi \to \infty} \int_{\xi_i}^{\xi} \left[H(s, \xi_i) \tilde{\phi}(s) - \frac{1}{4M\Gamma(1-\alpha)} \tilde{r}(s) \tilde{v}(s) Q_1^2(s, \xi_i) \right] \mathrm{d}s > 0 \tag{17}$$

and

$$\limsup_{\xi \to \infty} \int_{\xi_i}^{\xi} \left[H(\xi, s) \tilde{\phi}(s) - \frac{1}{4M\Gamma(1-\alpha)} \tilde{r}(s) \tilde{v}(s) Q_2^2(\xi, s) \right] \mathrm{d}s > 0, \tag{18}$$

where $\tilde{v}(\xi)$ and $\tilde{\phi}(\xi)$ are defined as in Theorem 3.2. Then every solution of $(E), (B_1)$ is oscillatory in *G*.

Proof. For any $\xi_{T_i} \ge \xi_0$, $i = 1,2$ that is, $\xi_{T_2} \ge \xi_{T_1} \ge \xi_0$, let $\xi_{c_i} = \xi_{T_i}$, $i = 1,2$. In (17) take $\xi_i = \xi_{c_i}$. Then there exists $\xi_{\delta_i} > \xi_{c_i}$ $i = 1,2$ such that

$$\int_{\xi_{c_i}}^{\xi_{\delta_i}} \left[H(s, \xi_{c_i}) \tilde{\phi}(s) - \frac{1}{4M\Gamma(1-\alpha)} \tilde{r}(s) \tilde{v}(s) Q_1^2(s, \xi_{c_i}) \right] \mathrm{d}s > 0 \tag{19}$$

In (18) take $\xi_i = \xi_{\delta_i}$. Then there exist $\xi_{d_i} > \xi_{\delta_i}$ $i = 1,2$ such that

$$\int_{\xi_{\delta_i}}^{\xi_{d_i}} \left[H(\xi_{d_i}, s) \tilde{\phi}(s) - \frac{1}{4M\Gamma(1-\alpha)} \tilde{r}(s) \tilde{v}(s) Q_2^2(\xi_{d_i}, s) \right] \mathrm{d}s > 0 \tag{20}$$

Dividing Equations (19) and (20) by $H(\xi_{\delta_i}, \xi_{c_i})$ and $H(\xi_{d_i}, \xi_{\delta_i})$ respectively and adding we get

$$\frac{1}{H(\xi_{\delta_i}, \xi_{c_i})} \int_{\xi_{c_i}}^{\xi_{\delta_i}} H(s, \xi_{c_i}) \tilde{\phi}(s) \mathrm{d}s + \frac{1}{H(\xi_{d_i}, \xi_{\delta_i})} \int_{\xi_{\delta_i}}^{\xi_{d_i}} H(\xi_{d_i}, s) \tilde{\phi}(s) \mathrm{d}s$$

$$> \frac{1}{4M\Gamma(1-\alpha) H(\xi_{\delta_i}, \xi_{c_i})} \int_{\xi_{c_i}}^{\xi_{\delta_i}} Q_1^2(s, \xi_{c_i}) \tilde{r}(s) \tilde{v}(s) \mathrm{d}s + \frac{1}{4M\Gamma(1-\alpha) H(\xi_{d_i}, \xi_{\delta_i})} \int_{\xi_{\delta_i}}^{\xi_{d_i}} Q_2^2(\xi_{d_i}, s) \tilde{r}(s) \tilde{v}(s) \mathrm{d}s$$

Then it follows by theorem 3.2 that every solution of $(E), (B_1)$ is oscillatory in *G*.

Consider the special case $H(\xi, s) = H(\xi - s)$ then

$$\frac{\partial H}{\partial \xi} = h_1(\xi - s)\sqrt{H(\xi - s)}, \quad \frac{\partial H}{\partial s} = h_2(\xi - s)\sqrt{H(\xi - s)}$$

Thus for $H = H(\xi - s) \in P$ we have $h_1(\xi - s) = h_2(\xi - s)$ and we note them by $h(\xi - s)$. The subclass containing such $H(\xi - s)$ is denoted by P_0. Applying Theorem 3.2 to P_0 we obtain the following result.

Theorem 3.4 Suppose that conditions (A_1) - (A_5) hold. If for each $T \ge t_0$ there exists $H \in P_0$ $\rho \in C^{\alpha}([t_0, \infty), (0, \infty))$ and $\xi_{c_i}, \xi_{\delta_i} \in R$ with $\xi_T \le \xi_{c_i} < \xi_{\delta_i}$ such that

$$\begin{aligned}
\int_{\xi_{c_i}}^{\xi_{\delta_i}} H(s - \xi_{c_i}) \left[\tilde{\phi}(s) + \tilde{\phi}(2\xi_{\delta_i} - s) \right] \mathrm{d}s &> \frac{1}{4M\Gamma(1-\alpha)} \int_{\xi_{c_i}}^{\xi_{\delta_i}} \left[\tilde{r}(s)\tilde{v}(s) + \tilde{r}(2\xi_{\delta_i} - s)\tilde{v}(2\xi_{\delta_i} - s) \right] h^2(s - \xi_{c_i}) \mathrm{d}s \\
&+ \frac{1}{2M\Gamma(1-\alpha)} \int_{\xi_{c_i}}^{\xi_{\delta_i}} \left[\tilde{v}(2\xi_{\delta_i} - s)\tilde{q}(2\xi_{\delta_i} - s) - \tilde{v}(s)\tilde{q}(s) \right] h(s - \xi_{c_i}) \sqrt{H(s - \xi_{c_i})} \mathrm{d}s \\
&+ \frac{1}{4M\Gamma(1-\alpha)} \int_{\xi_{c_i}}^{\xi_{\delta_i}} \left[\frac{\tilde{v}(s)\tilde{q}^2(s)}{\tilde{r}(s)} + \frac{\tilde{v}(2\xi_{\delta_i} - s)\tilde{q}^2(2\xi_{\delta_i} - s)}{\tilde{r}(2\xi_{\delta_i} - s)} \right] H(s - \xi_{c_i}) \mathrm{d}s
\end{aligned} \tag{21}$$

where $\tilde{v}(\xi)$ and $\tilde{\phi}(\xi)$ are defined as in Theorem 3.2. Then, every solution of (E) and (B_1) is oscillatory in G.

Proof. Let $\xi_{\delta_i} = \frac{\xi_{c_i} + \xi_{d_i}}{2}$, for $i = 1, 2$, that is $\xi_{d_i} = 2\xi_{\delta_i} - \xi_{c_i}$, then

$$H\left(\xi_{d_i} - \xi_{\delta_i}\right) = H\left(\xi_{\delta_i} - \xi_{c_i}\right) = H\left(\frac{\xi_{d_i} - \xi_{c_i}}{2}\right).$$

For any $w \in L'\left(\xi_{c_i}, \xi_{d_i}\right)$ we have

$$\int_{\xi_{\delta_i}}^{\xi_{d_i}} w(s)\mathrm{d}s = \int_{\xi_{c_i}}^{\xi_{\delta_i}} w\left(2\xi_{\delta_i} - s\right)\mathrm{d}s$$

$$\int_{\xi_{\delta_i}}^{\xi_{d_i}} H\left(\xi_{d_i} - s\right)\tilde{\phi}(s)\mathrm{d}s = \int_{\xi_{c_i}}^{\xi_{\delta_i}} H\left(s - \xi_{c_i}\right)\tilde{\phi}\left(2\xi_{c_i} - s\right)\mathrm{d}s$$

$$\int_{\xi_{\delta_i}}^{\xi_{d_i}} \tilde{v}(s)\tilde{q}(s)h\left(\xi_{d_i} - s\right)\sqrt{H\left(\xi_{d_i} - s\right)}\mathrm{d}s = \int_{\xi_{c_i}}^{\xi_{\delta_i}} \tilde{v}\left(2\xi_{\delta_i} - s\right)\tilde{q}\left(2\xi_{\delta_i} - s\right)h\left(s - \xi_{c_i}\right)\sqrt{H\left(s - \xi_{c_i}\right)}\mathrm{d}s$$

$$\int_{\xi_{\delta_i}}^{\xi_{d_i}} \tilde{v}(s)\tilde{r}(s)h^2\left(\xi_{d_i} - s\right)\sqrt{H\left(\xi_{d_i} - s\right)}\mathrm{d}s = \int_{\xi_{c_i}}^{\xi_{\delta_i}} \tilde{r}\left(2\xi_{\delta_i} - s\right)\tilde{v}\left(2\xi_{\delta_i} - s\right)h^2\left(s - \xi_{c_i}\right)\sqrt{H\left(s - \xi_{c_i}\right)}\mathrm{d}s$$

From (21) we have

$$\int_{\xi_{c_i}}^{\xi_{\delta_i}} H\left(s - \xi_{c_i}\right)\left[\tilde{\phi}(s) + \tilde{\phi}\left(2\xi_{\delta_i} - s\right)\right]\mathrm{d}s > \frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{c_i}}^{\xi_{\delta_i}}\left[\tilde{r}(s)\tilde{v}(s) + \tilde{r}\left(2\xi_{\delta_i} - s\right)\tilde{v}\left(2\xi_{\delta_i} - s\right)\right]h^2\left(s - \xi_{c_i}\right)\mathrm{d}s$$
$$+\frac{1}{2M\Gamma(1-\alpha)}\int_{\xi_{c_i}}^{\xi_{\delta_i}}\left[\tilde{v}\left(2\xi_{\delta_i} - s\right)\tilde{q}\left(2\xi_{\delta_i} - s\right) - \tilde{v}(s)\tilde{q}(s)\right]h\left(s - \xi_{c_i}\right)\sqrt{H\left(s - \xi_{c_i}\right)}\mathrm{d}s$$
$$+\frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{c_i}}^{\xi_{\delta_i}}\left[\frac{\tilde{v}(s)\tilde{q}^2(s)}{\tilde{r}(s)} + \frac{\tilde{v}\left(2\xi_{\delta_i} - s\right)\tilde{q}^2\left(2\xi_{\delta_i} - s\right)}{\tilde{r}\left(2\xi_{\delta_i} - s\right)}\right]H\left(s - \xi_{c_i}\right)\mathrm{d}s$$

$$\int_{\xi_{c_i}}^{\xi_{\delta_i}} H\left(s - \xi_{c_i}\right)\tilde{\phi}(s)\mathrm{d}s + \int_{\xi_{\delta_i}}^{\xi_{d_i}} H\left(\xi_{d_i} - s\right)\tilde{\phi}(s)\mathrm{d}s$$
$$> \frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{c_i}}^{\xi_{\delta_i}} \tilde{r}(s)\tilde{v}(s)h^2\left(s - \xi_{c_i}\right)\mathrm{d}s + \frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{\delta_i}}^{\xi_{d_i}} \tilde{r}(s)\tilde{v}(s)h^2\left(\xi_{d_i} - s\right)\mathrm{d}s$$
$$+\frac{1}{2M\Gamma(1-\alpha)}\left[\int_{\xi_{\delta_i}}^{\xi_{d_i}} \tilde{v}(s)\tilde{q}(s)h\left(\xi_{d_i} - s\right)\sqrt{H\left(\xi_{d_i} - s\right)}\mathrm{d}s - \int_{\xi_{c_i}}^{\xi_{\delta_i}} \tilde{v}(s)\tilde{q}(s)h\left(s - \xi_{c_i}\right)\sqrt{H\left(s - \xi_{c_i}\right)}\mathrm{d}s\right]$$
$$+\frac{1}{4M\Gamma(1-\alpha)}\left[\int_{\xi_{c_i}}^{\xi_{\delta_i}} \frac{\tilde{v}(s)\tilde{q}^2(s)}{\tilde{r}(s)}H\left(s - \xi_{c_i}\right)\mathrm{d}s + \int_{\xi_{\delta_i}}^{\xi_{d_i}} \frac{\tilde{v}(s)\tilde{q}^2(s)}{\tilde{r}(s)}H\left(\xi_{d_i} - s\right)\mathrm{d}s\right]$$

$$\int_{\xi_{c_i}}^{\xi_{\delta_i}} H\left(s - \xi_{c_i}\right)\tilde{\phi}(s)\mathrm{d}s + \int_{\xi_{\delta_i}}^{\xi_{d_i}} H\left(\xi_{d_i} - s\right)\tilde{\phi}(s)\mathrm{d}s$$
$$> \frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{c_i}}^{\xi_{\delta_i}}\left[\tilde{r}(s)\tilde{v}(s)h^2\left(s - \xi_{c_i}\right) - 2\tilde{v}(s)\tilde{q}(s)h\left(s - \xi_{c_i}\right)\sqrt{H\left(s - \xi_{c_i}\right)} + \tilde{v}(s)\frac{\tilde{q}^2(s)}{\tilde{r}(s)}H\left(s - \xi_{c_i}\right)\right]\mathrm{d}s$$
$$+\frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{\delta_i}}^{\xi_{d_i}}\left[\tilde{r}(s)\tilde{v}(s)h^2\left(\xi_{d_i} - s\right) + 2\tilde{v}(s)\tilde{q}(s)h\left(\xi_{d_i} - s\right)\sqrt{H\left(\xi_{d_i} - s\right)} + \tilde{v}(s)\frac{\tilde{q}^2(s)}{\tilde{r}(s)}H\left(\xi_{d_i} - s\right)\right]\mathrm{d}s$$
$$= \frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{c_i}}^{\xi_{\delta_i}} \tilde{r}(s)\tilde{v}(s)\left[h\left(s - \xi_{c_i}\right) - \frac{\tilde{q}(s)}{\tilde{r}(s)}\sqrt{H\left(s - \xi_{c_i}\right)}\right]^2$$
$$+\frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{\delta_i}}^{\xi_{d_i}} \tilde{r}(s)\tilde{v}(s)\left[h\left(\xi_{d_i} - s\right) + \frac{\tilde{q}(s)}{\tilde{r}(s)}\sqrt{H\left(\xi_{d_i} - s\right)}\right]^2\mathrm{d}s$$

since $H(\xi - s) = H(\xi, s)$ we have

$$\int_{\xi_{c_i}}^{\xi_{\delta_i}} H\left(s,\xi_{c_i}\right)\tilde{\phi}(s)\,\mathrm{d}s+\int_{\xi_{\delta_i}}^{\xi_{d_i}} H\left(\xi_{d_i},s\right)\tilde{\phi}(s)\,\mathrm{d}s$$
$$>\frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{c_i}}^{\xi_{\delta_i}}\tilde{r}(s)\tilde{v}(s)Q_1^2\left(s,\xi_{c_i}\right)\mathrm{d}s+\frac{1}{4M\Gamma(1-\alpha)}\int_{\xi_{\delta_i}}^{\xi_{d_i}}\left[\tilde{r}(s)\tilde{v}(s)Q_2^2\left(\xi_{d_i},s\right)\right]\mathrm{d}s$$

Hence every solution of $(E),(B_1)$ is oscillatory in G by Theorem 3.2.

Let $H(\xi,s)=(\xi-s)^\lambda$ $\xi\ge s\ge\xi_0$ where $\lambda>1$ is a constant. Then, the sufficient conditions (17) and (18) can be modified in the form

$$\limsup_{\xi\to\infty}\int_{\xi_i}^{\xi}(s-\xi_i)^\lambda\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\tilde{r}(s)\tilde{v}(s)\left(\frac{\lambda}{s-\xi_i}-\frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right]\mathrm{d}s>0 \tag{22}$$

$$\limsup_{\xi\to\infty}\int_{\xi_i}^{\xi}(\xi-s)^\lambda\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\tilde{r}(s)\tilde{v}(s)\left(\frac{\lambda}{\xi-s}+\frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right]\mathrm{d}s>0 \tag{23}$$

Corollary 3.1 Assume that the conditions (A_1) - (A_5) hold. Assume for each $\xi_i\ge\xi_0$ i = 1, 2 that is $\xi_2\ge\xi_1\ge\xi_0$ and for some $\lambda>1$ $\rho\in C'([\xi_0,\infty),(0,\infty))$ we have

$$\limsup_{\xi\to\infty}\frac{1}{\xi^{\lambda-1}}\int_{\xi_i}^{\xi}(s-\xi_i)^\lambda\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\tilde{r}(s)\tilde{v}(s)\left(\frac{\lambda}{s-\xi_i}-\frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right]\mathrm{d}s>0$$

and

$$\limsup_{\xi\to\infty}\frac{1}{\xi^{\lambda-1}}\int_{\xi_i}^{\xi}(\xi-s)^\lambda\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\tilde{r}(s)\tilde{v}(s)\left(\frac{\lambda}{\xi-s}+\frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right]\mathrm{d}s>0.$$

Then every solution of (E) and (B_1) is oscillatory in G.

Theorem 3.5 Suppose that the conditions (A_1) - (A_5) hold. If for each $\xi_i\ge\xi_0$ i = 1, 2 and for some $\lambda>1$ satisfies the following conditions

$$\limsup_{\xi\to\infty}\frac{1}{\xi^{\lambda-1}}\int_{\xi_i}^{\xi}(s-\xi_i)^\lambda\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\left(\tilde{q}^2(s)-\frac{2\lambda\tilde{q}(s)}{s-\xi_i}\right)\right]\mathrm{d}s>\frac{\lambda^2}{4M\Gamma(1-\alpha)(\lambda-1)}$$

and

$$\limsup_{\xi\to\infty}\frac{1}{\xi^{\lambda-1}}\int_{\xi_i}^{\xi}(\xi-s)^\lambda\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\left(\tilde{q}^2(s)+\frac{2\lambda\tilde{q}(s)}{\xi-s}\right)\right]\mathrm{d}s>\frac{\lambda^2}{4M\Gamma(1-\alpha)(\lambda-1)}$$

Then every solution of (E) and (B_1) is oscillatory in G.

Proof. Clearly $h_1(s,\xi_i)=\lambda(s-\xi_i)^{\lambda/2-1}$, $h_2(\xi,s)=\lambda(\xi-s)^{\lambda/2-1}$.

Note that

$$\limsup_{\xi\to\infty}\frac{1}{4M\Gamma(1-\alpha)\xi^{\lambda-1}}\int_{\xi_i}^{\xi}h_1^2(s-\xi_i)\,\mathrm{d}s=\limsup_{\xi\to\infty}\frac{1}{4M\Gamma(1-\alpha)\xi^{\lambda-1}}\int_{\xi_i}^{\xi}\lambda^2(s-\xi_i)^{\lambda-2}\,\mathrm{d}s$$
$$=\frac{\lambda^2}{4M\Gamma(1-\alpha)(\lambda-1)}$$

and

$$\limsup_{\xi\to\infty}\frac{1}{4M\Gamma(1-\alpha)\xi^{\lambda-1}}\int_{\xi_i}^{\xi}h_2^2(\xi-s)\,\mathrm{d}s=\limsup_{\xi\to\infty}\frac{1}{4M\Gamma(1-\alpha)\xi^{\lambda-1}}\int_{\xi_i}^{\xi}\lambda^2(\xi-s)^{\lambda-2}\,\mathrm{d}s$$
$$=\frac{\lambda^2}{4M\Gamma(1-\alpha)(\lambda-1)}$$

Consider

$$\limsup_{\xi\to\infty}\frac{1}{\xi^{\lambda-1}}\int_{\xi_i}^{\xi}\left[H(s-\xi_i)\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}Q_1^2(s-\xi_i)\right]ds>0$$

$$\limsup_{\xi\to\infty}\frac{1}{\xi^{\lambda-1}}\int_{\xi_i}^{\xi}H(s-\xi_i)\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\left[\frac{h_1(s-\xi_i)}{\sqrt{H(s-\xi_i)}}-\tilde{q}(s)\right]^2\right]ds>0$$

$$\limsup_{\xi\to\infty}\frac{1}{\xi^{\lambda-1}}\int_{\xi_i}^{\xi}(s-\xi_i)^{\lambda}\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\left(\tilde{q}^2(s)-2\frac{\lambda\tilde{q}(s)}{(s-\xi_i)}\right)\right]ds$$

$$>\limsup_{\xi\to\infty}\frac{1}{4M\Gamma(1-\alpha)\xi^{\lambda-1}}\int_{\xi_i}^{\xi}h_1^2(s-\xi_i)ds>\frac{\lambda^2}{4M\Gamma(1-\alpha)(\lambda-1)}$$

$$\limsup_{\xi\to\infty}\frac{1}{\xi^{\lambda-1}}\int_{\xi_i}^{\xi}(s-\xi_i)^{\lambda}\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\left(\tilde{q}^2(s)-2\frac{\lambda\tilde{q}(s)}{(s-\xi_i)}\right)\right]ds>\frac{\lambda^2}{4M\Gamma(1-\alpha)(\lambda-1)}$$

Similarly we can prove other inequality

Next we consider $H(\xi,s)=\left[R(\xi)-R(s)\right]^{\lambda}$, where λ is a constant and $R(\xi)=\int_{\xi_i}^{\xi}\frac{1}{\tilde{r}(s)}ds$ and $\lim_{\xi\to\infty}R(\xi)=\infty$.

Theorem 3.6 Assume that the conditions (A_1) - (A_5) hold. If for each $\xi_i\ge\xi_0$ i = 1, 2 and for some $\lambda>1$ $\rho\in C'([\xi_0,\infty),(0,\infty))$ such that

$$\limsup_{\xi\to\infty}\frac{1}{R^{\lambda-1}(\xi)}\int_{\xi_i}^{\xi}(R(s)-R(\xi_i))^{\lambda}\left[\tilde{\phi}(s)-\frac{\tilde{v}(s)}{4M\Gamma(1-\alpha)\tilde{r}(s)}\left(\frac{\lambda}{R(s)-R(\xi_i)}-\tilde{q}(s)\right)^2\right]ds>0$$

and

$$\limsup_{\xi\to\infty}\frac{1}{R^{\lambda-1}(\xi)}\int_{\xi_i}^{\xi}(R(\xi)-R(s))^{\lambda}\left[\tilde{\phi}(s)-\frac{\tilde{v}(s)}{4M\Gamma(1-\alpha)\tilde{r}(s)}\left(\frac{\lambda}{R(\xi)-R(s)}+\tilde{q}(s)\right)^2\right]ds>0$$

Then every solution of (E) and (B_1) is oscillatory in G.

Proof. From (17)

$$\limsup_{\xi\to\infty}\int_{\xi_i}^{\xi}\left[H(s,\xi_i)\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\tilde{r}(s)\tilde{v}(s)Q_1^2(s,\xi_i)\right]ds>0$$

$$\limsup_{\xi\to\infty}\int_{\xi_i}^{\xi}H(s,\xi_i)\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\frac{\tilde{r}(s)\tilde{v}(s)}{\tilde{r}^2(s)}\left[\frac{h_1(s,\xi_i)\tilde{r}(s)}{\sqrt{H(s,\xi_i)}}-\tilde{q}(s)\right]^2\right]ds>0$$

$$\limsup_{\xi\to\infty}\int_{\xi_i}^{\xi}(R(s)-R(\xi_i))^{\lambda}\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\frac{\tilde{v}(s)}{\tilde{r}(s)}\left(\frac{\lambda\left[R(s)-R(\xi_i)\right]^{\lambda/2-1}\tilde{r}(s)}{\tilde{r}(s)(R(s)-R(\xi_i))^{\lambda/2}}-\tilde{q}(s)\right)^2\right]ds>0$$

$$\limsup_{\xi\to\infty}\int_{\xi_i}^{\xi}(R(s)-R(\xi_i))^{\lambda}\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\frac{\tilde{v}(s)}{\tilde{r}(s)}\left(\frac{\lambda}{R(s)-R(\xi_i)}-\tilde{q}(s)\right)^2\right]ds>0$$

$$\limsup_{\xi\to\infty}\frac{1}{R^{\lambda-1}(\xi)}\int_{\xi_i}^{\xi}(R(s)-R(\xi_i))^{\lambda}\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\frac{\tilde{v}(s)}{\tilde{r}(s)}\left(\frac{\lambda}{R(s)-R(\xi_i)}-\tilde{q}(s)\right)^2\right]ds>0$$

Similarly we can prove that

$$\limsup_{\xi\to\infty}\frac{1}{R^{\lambda-1}(\xi)}\int_{\xi_i}^{\xi}\left(R(\xi)-R(s)\right)^{\lambda}\left[\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\frac{\tilde{v}(s)}{\tilde{r}(s)}\left(\frac{\lambda}{R(\xi)-R(s)}+\tilde{q}(s)\right)^2\right]\mathrm{d}s>0$$

If we choose $H(\xi,s)=\left(\log\left(\frac{\xi}{s}\right)\right)^n, \xi>s>\xi_0$ and $H(\xi,s)=\left(\int_s^{\xi}\frac{\mathrm{d}u}{\theta(u)}\right)^n$ we have the following corollaries.

Corollary 3.2 Suppose that the conditions (A_1) - (A_5) hold. Assume for each $\xi_i\ge\xi_0$ i = 1, 2 that is $\xi_2\ge\xi_1\ge\xi_0$ and for some $n>1$ $\rho\in C'([\xi_0,\infty),(0,\infty))$ we have

$$\limsup_{\xi\to\infty}\int_{\xi_i}^{\xi}\left(\log\left(\frac{s}{\xi_i}\right)\right)^n\left\{\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\left(\frac{n}{s\log\left(\frac{s}{\xi_i}\right)}-\frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right\}\mathrm{d}s>0$$

and

$$\limsup_{\xi\to\infty}\int_{\xi_i}^{\xi}\left(\log\left(\frac{\xi}{s}\right)\right)^n\left\{\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\left(\frac{n}{s\log\left(\frac{\xi}{s}\right)}+\frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right\}\mathrm{d}s>0$$

Then every solution of (E) and (B_1) is oscillatory in G.

Corollary 3.3 Suppose that the conditions (A_1) - (A_5) hold. Assume for each $\xi_i\ge\xi_0$ $i=1,2$ that $\xi_2\ge\xi_1\ge\xi_0$ and for some $n>1$ $\rho\in C'([\xi_0,\infty),(0,\infty))$ we have

$$\limsup_{\xi\to\infty}\int_{\xi_i}^{\xi}\left(\int_{\xi_i}^{s}\frac{\mathrm{d}u}{\theta(u)}\right)^n\left\{\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\left(\frac{n}{\theta(s)\int_{\xi_i}^{s}\frac{\mathrm{d}u}{\theta(u)}}-\frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right\}\mathrm{d}s>0$$

and

$$\limsup_{\xi\to\infty}\int_{\xi_i}^{\xi}\left(\int_{\xi}^{s}\frac{\mathrm{d}u}{\theta(u)}\right)^n\left\{\tilde{\phi}(s)-\frac{1}{4M\Gamma(1-\alpha)}\left(\frac{n}{\theta(s)\int_{\xi}^{s}\frac{\mathrm{d}u}{\theta(u)}}+\frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right\}\mathrm{d}s>0$$

Then every solution of (E) and (B_1) is oscillatory in G.

4. Oscillation without Monotonicity of $f(x)$ of (E) and (B_1)

We now consider non monotonous situation

$$(A_6)\quad \frac{f(u)}{u}\ge M_1>0 \quad \text{where } M_1 \text{ is a constant.}$$

Theorem 4.1 Suppose that the conditions (A_1) - (A_4) and (A_6) hold. Assume that for any $T_0\ge t_0$ there exist c_i, δ_i, d_i for $i=1,2$ such that $T_0\le c_1<\delta_1<d_1\le c_2<\delta_2<d_2$, $t\in[c_1,d_1]\cup[c_2,d_2]$ satisfying

$$F(t)=\begin{cases}\le 0, & t\in[c_1,d_1]\\ \ge 0, & t\in[c_2,d_2].\end{cases} \tag{24}$$

If there exist $\xi_{\delta_i} \in \left(\xi_{c_i}, \xi_{d_i}\right)$ $H \in P$ and $\rho \in C^{\alpha}\left(\left[t_0, \infty\right), R_+\right)$ such that

$$\begin{aligned}
&\frac{1}{H\left(\xi_{\delta_i},\xi_{c_i}\right)}\int_{\xi_{c_i}}^{\xi_{\delta_i}} H\left(s,\xi_{c_i}\right)\tilde{\phi}(s)\mathrm{d}s+\frac{1}{H\left(\xi_{d_i},\xi_{\delta_i}\right)}\int_{\xi_{\delta_i}}^{\xi_{d_i}} H\left(\xi_{d_i},s\right)\tilde{\phi}(s)\mathrm{d}s \\
&>\frac{1}{4\Gamma(1-\alpha)H\left(\xi_{\delta_i},\xi_{c_i}\right)}\int_{\xi_{c_i}}^{\xi_{\delta_i}}\tilde{r}(s)\tilde{v}(s)Q_1^2\left(s,\xi_{c_i}\right)\mathrm{d}s \\
&+\frac{1}{4\Gamma(1-\alpha)H\left(\xi_{d_i},\xi_{\delta_i}\right)}\int_{\xi_{\delta_i}}^{\xi_{d_i}}\tilde{r}(s)\tilde{v}(s)Q_2^2\left(\xi_{d_i},s\right)\mathrm{d}s, \quad \text{for } i=1,2,
\end{aligned} \tag{25}$$

where $\tilde{v}$ and $\tilde{\phi}$ are defined as

$$\tilde{v}(\xi)=\exp\left(-2\int_{\xi_0}^{\xi}\tilde{\rho}(s)\mathrm{d}s\right)$$

$$\tilde{\phi}(\xi)=\tilde{v}(\xi)\left\{LM_1\tilde{p}(\xi)-\tilde{q}(\xi)\tilde{\rho}(\xi)+\Gamma(1-\alpha)\tilde{r}(\xi)\tilde{\rho}^2(\xi)-\left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right\}.$$

Then every solution of (E), (B_1) is oscillatory in G.

Proof. Suppose to the contrary that $u(x,t)$ be a non oscillatory solution of the problem (E), (B_1) say $u(x,t)\neq 0$ in $\Omega\times\left[T_0,\infty\right)$ for some $T_0\geq t_0$. Define the Riccati transformation function

$$w(t)=v(t)r(t)\left[\frac{D_+^{\alpha}U(t)}{K(t)}+\rho(t)\right], \quad t\geq T_0.$$

Then for $t\geq T_0$

$$D_+^{\alpha}w(t)=D_+^{\alpha}v(t)\frac{w(t)}{v(t)}+v(t)\left\{\frac{D_+^{\alpha}\left(r(t)D_+^{\alpha}U(t)\right)}{K(t)}-r(t)\frac{D_+^{\alpha}U(t)}{K^2(t)}D_+^{\alpha}K(t)+D_+^{\alpha}\left(\rho(t)r(t)\right)\right\}.$$

By using $\dfrac{f\left(K(t)\right)}{K(t)}\geq M_1$ and inequality (4) we get

$$\begin{aligned}
&D_+^{\alpha}w(t)\leq D_+^{\alpha}v(t)\frac{w(t)}{v(t)} \\
&+v(t)\left[-\frac{q(t)D_+^{\alpha}U(t)}{K(t)}-M_1Lp(t)-r(t)\frac{D_+^{\alpha}U(t))}{K^2(t)}D_+^{\alpha}\left(K(t)\right)+D_+^{\alpha}\left(\rho(t)r(t)\right)\right]+\frac{F(t)v(t)}{K(t)}.
\end{aligned} \tag{26}$$

By assumption, if $u(x,t)>0$ then we can choose $c_1,d_1\geq T_0$ with $c_1<d_1$ such that $F(t)\leq 0$ on the interval $\left[c_1,d_1\right]$. If $u(x,t)<0$ then we can choose $c_2,d_2\geq T_0$ with $c_2<d_2$ such that $F(t)\geq 0$ On the interval $\left[c_2,d_2\right]$ So

$$\frac{F(t)v(t)}{K(t)}\leq 0 \quad t\in\left[c_i,d_i\right], \quad i=1,2.$$

Therefore inequality (26) becomes

$$\begin{aligned}
&D_+^{\alpha}w(t)\leq D_+^{\alpha}v(t)\frac{w(t)}{v(t)}+v(t)\left[-\frac{q(t)D_+^{\alpha}U(t)}{K(t)}-M_1Lp(t)-r(t)\frac{D_+^{\alpha}U(t)}{K^2(t)}D_+^{\alpha}K(t)+D_+^{\alpha}\left(\rho(t)r(t)\right)\right] \\
&t\in\left[c_i,d_i\right], \quad i=1,2.
\end{aligned} \tag{27}$$

Let $w(t)=\tilde{w}(\xi)$, $v(t)=\tilde{v}(\xi)$, $q(t)=\tilde{q}(\xi)$, $U(t)=\tilde{U}(\xi)$, $p(t)=\tilde{p}(\xi)$, $K(t)=\tilde{K}(\xi)$, $r(t)=\tilde{r}(\xi)$. Then $D_+^{\alpha}w(t)=\tilde{w}'(\xi)$, $D_+^{\alpha}U(t)=\tilde{U}'(\xi)$, $D_+^{\alpha}K(t)=\tilde{K}'(\xi)$, $D_+^{\alpha}\left(r(t)\rho(t)\right)=\left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'$, so (27) is transformed into

$$\tilde{w}'(\xi) \le \tilde{v}'(\xi)\frac{\tilde{w}(\xi)}{\tilde{v}(\xi)} + \tilde{v}(\xi)\left[\frac{-\tilde{q}(\xi)\tilde{U}'(\xi)}{\tilde{K}(\xi)} - M_1 L\tilde{p}(\xi) - \tilde{r}(\xi)\frac{\tilde{U}'(\xi)}{\tilde{K}^2(\xi)}\tilde{K}'(\xi) + \left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right]$$

$$\le -2\tilde{\rho}(\xi)\tilde{v}(\xi)\frac{\tilde{w}(\xi)}{\tilde{v}(\xi)}$$

$$+\tilde{v}(\xi)\left[-\tilde{q}(\xi)\left(\frac{\tilde{w}(\xi)}{\tilde{v}(\xi)\tilde{r}(\xi)} - \tilde{\rho}(\xi)\right) - M_1 L\tilde{p}(\xi) - \tilde{r}(\xi)\left(\frac{\tilde{w}(\xi)}{\tilde{r}(\xi)\tilde{v}(\xi)} - \tilde{\rho}(\xi)\right)\Gamma(1-\alpha)\frac{\tilde{U}'(\xi)}{\tilde{K}(\xi)} + \left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right]$$

$$\le -2\tilde{\rho}(\xi)\tilde{w}(\xi) - \frac{\tilde{q}(\xi)\tilde{w}(\xi)}{\tilde{v}(\xi)\tilde{r}(\xi)}$$

$$+\tilde{v}(\xi)\left[\tilde{q}(\xi)\tilde{\rho}(\xi) - M_1 L\tilde{p}(\xi) - \tilde{r}(\xi)\Gamma(1-\alpha)\left(\frac{\tilde{w}(\xi)}{\tilde{r}(\xi)\tilde{v}(\xi)} - \tilde{\rho}(\xi)\right)^2 + \left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right]$$

$$\le -2\tilde{\rho}(\xi)\tilde{w}(\xi)\left[1-\Gamma(1-\alpha)\right] - \frac{\tilde{q}(\xi)\tilde{w}(\xi)}{\tilde{r}(\xi)}$$

$$-\tilde{v}(\xi)\left[-\tilde{q}(\xi)\tilde{\rho}(\xi) + LM_1\tilde{p}(\xi) + \tilde{r}(\xi)\Gamma(1-\alpha)\tilde{\rho}^2(\xi) - \left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right] - \Gamma(1-\alpha)\frac{\tilde{w}^2(\xi)}{\tilde{r}(\xi)\tilde{v}(\xi)}$$

$$\tilde{w}'(\xi) \le -\tilde{q}(\xi)\frac{\tilde{w}(\xi)}{\tilde{r}(\xi)} - \tilde{\phi}(\xi) - \Gamma(1-\alpha)\frac{\tilde{w}^2(\xi)}{\tilde{r}(\xi)\tilde{v}(\xi)},$$

where

$$\tilde{\phi}(\xi) = \tilde{v}(\xi)\left[LM_1\tilde{p}(\xi) - \tilde{q}(\xi)\tilde{\rho}(\xi) + \tilde{r}(\xi)\Gamma(1-\alpha)\tilde{\rho}^2(\xi) - \left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right]$$

that is

$$\tilde{\phi}(\xi) \le -\tilde{w}'(\xi) - \tilde{q}(\xi)\frac{\tilde{w}(\xi)}{\tilde{r}(\xi)} - \Gamma(1-\alpha)\frac{\tilde{w}^2(\xi)}{\tilde{r}(\xi)\tilde{v}(\xi)}, \quad \xi \in \left[\xi_{c_i}, \xi_{d_i}\right], \ i = 1,2.$$

The remaining part of the proof is the same as that of theorem 3.2 in section 3, and hence omitted.

Corollary 4.1 Suppose that the conditions (A_1) - (A_4) and (A_6) hold. Assume for each $\xi_i \ge \xi_0$ $i = 1,2$ that is $\xi_2 \ge \xi_1 \ge \xi_0$ and for some $\lambda > 1$ $\rho \in C'\left(\left[\xi_0,\infty\right),(0,\infty)\right)$ we have

$$\limsup_{\xi\to\infty}\frac{1}{\xi^{\lambda-1}}\int_{\xi_i}^{\xi}(s-\xi_i)^{\lambda}\left[\tilde{\phi}(s) - \frac{1}{4\Gamma(1-\alpha)}\tilde{r}(s)\tilde{v}(s)\left(\frac{\lambda}{s-\xi_i} - \frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right]\mathrm{d}s > 0$$

and

$$\limsup_{\xi\to\infty}\frac{1}{\xi^{\lambda-1}}\int_{\xi_i}^{\xi}(\xi-s)^{\lambda}\left[\tilde{\phi}(s) - \frac{1}{4\Gamma(1-\alpha)}\tilde{r}(s)\tilde{v}(s)\left(\frac{\lambda}{\xi-s} + \frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right]\mathrm{d}s > 0.$$

Then every solution of (E) and (B_1) is oscillatory in G.

5. Oscillation with and without Monotonicity of $f(x)$ of (E) and (B_2)

In this section, we establish sufficient conditions for the oscillation of all solutions of (E), (B_2). For this, we need the following:

The smallest eigen value β_0 of the Dirichlet problem

$$\Delta\omega(x) + \beta\omega(x) = 0 \text{ in } \Omega$$

$$\omega(x) = 0 \text{ on } \partial\Omega,$$

is positive and the corresponding eigen function $\phi(x)$ is positive in $\Omega.$.

Theorem 5.1 Let all the conditions of Theorem 3.2 be hold. Then every solution of (E) and (B_2) is oscillatory in G.

Proof. Suppose to the contrary that there is a non oscillatory solution $u(x,t)$ of the problem (E) and (B_2) which has no zero in $\Omega\times[t_0,\infty)$ for some $t_0>0$. Without loss of generality, we may assume that $u(x,t)>0$ in $\Omega\times[t_1,\infty)$, $t_1\ge t_0$. Multiplying both sides of the Equation (E) by $\phi(x)>0$ and then integrating with respect to x over $\Omega.$, we obtain for $t\ge t_1$,

$$\begin{aligned}&\int_\Omega D^\alpha_{+,t}\left[r(t)D^\alpha_{+,t}u(x,t)\right]\phi(x)\mathrm{d}x+\int_\Omega q(x,t)D^\alpha_{+,t}u(x,t)\phi(x)\mathrm{d}x\\&+\int_\Omega p(x,t)\,f\left(\int_0^t(t-s)^{-\alpha}u(x,s)\mathrm{d}s\right)g\left(D^\alpha_{+,t}u(x,t)\right)\phi(x)\mathrm{d}x\\&=a(t)\int_\Omega\Delta u(x,t)\phi(x)\mathrm{d}x+\int_\Omega F(x,t)\phi(x)\mathrm{d}x.\end{aligned}\tag{28}$$

Using Green's formula and boundary condition (B_2), it follows that

$$\int_\Omega\Delta u(x,t)\phi(x)\mathrm{d}x=\int_{\partial\Omega}u(x,t)\Delta\phi(x)\mathrm{d}x=-\beta_0\int_{\partial\Omega}u(x,t)\phi(x)\mathrm{d}x\le0,\quad t\ge t_1,\tag{29}$$

$$\int_\Omega q(x,t)D^\alpha_{+,t}u(x,t)\phi(x)\mathrm{d}x\ge q(t)D^\alpha_+\left(\int_\Omega u(x,t)\phi(x)\mathrm{d}x\right),\quad t\ge t_1.\tag{30}$$

By using Jensen's inequality and (A_2) we get

$$\begin{aligned}&\int_\Omega p(x,t)\,f\left(\int_0^t(t-s)^{-\alpha}u(x,s)\mathrm{d}s\right)g\left(D^\alpha_{+,t}u(x,t)\right)\phi(x)\mathrm{d}x\\&\ge p(t)\int_\Omega f\left(\int_0^t(t-s)^{-\alpha}u(x,s)\phi(x)\mathrm{d}s\right)g\left(D^\alpha_{+,t}u(x,t)\right)\mathrm{d}x\\&\ge p(t)\int_\Omega\phi(x)\mathrm{d}xf\left(\int_0^t(t-s)^{-\alpha}\left(\int_\Omega u(x,s)\phi(x)\mathrm{d}x\right)\left(\int_\Omega\phi(x)\mathrm{d}x\right)^{-1}\mathrm{d}s\right)g\left(D^\alpha_{+,t}u(x,t)\right).\end{aligned}$$

Set

$$U(t)=\int_\Omega u(x,t)\phi(x)\mathrm{d}x\left(\int_\Omega\phi(x)\mathrm{d}x\right)^{-1}.\tag{31}$$

Therefore,

$$\int_\Omega p(x,t)\,f\left(\int_0^t(t-s)^{-\alpha}u(x,s)\mathrm{d}s\right)g\left(D^\alpha_{+,t}u(x,t)\right)\phi(x)\mathrm{d}x\ge p(t)\int_\Omega\phi(x)\mathrm{d}xf\left(\int_0^t(t-s)^{-\alpha}U(s)\mathrm{d}s\right)g\left(D^\alpha_{+,t}u(x,t)\right).$$

By using $g\left(D^\alpha_{+,t}u(x,t)\right)\ge L>0$ we have

$$\int_\Omega p(x,t)\,f\left(\int_0^t(t-s)^{-\alpha}u(x,s)\mathrm{d}s\right)g\left(D^\alpha_{+,t}u(x,t)\right)\mathrm{d}x\ge p(t)\,f\left(K(t)\right)L\int_\Omega\phi(x)\mathrm{d}x,\quad t\ge t_1.\tag{32}$$

In view of (31), (29)-(30), (32), (28) yield

$$D^\alpha_+\left[r(t)D^\alpha_+U(t)\right]+q(t)D^\alpha_+U(t)+Lp(t)\,f\left(K(t)\right)\le\frac{1}{\int_\Omega\phi(x)\mathrm{d}x}\int_\Omega F(x,t)\phi(x)\mathrm{d}x$$

Take $F(t)=\frac{1}{\int_\Omega\phi(x)\mathrm{d}x}\int_\Omega F(x,t)\phi(x)\mathrm{d}x$ therefore

$$D^\alpha_+\left[r(t)D^\alpha_+U(t)\right]+q(t)D^\alpha_+U(t)+Lp(t)\,f\left(K(t)\right)-F(t)\le0,\quad t\ge t_1.$$

Rest of the proof is similar to that of Theorem 3.2 and hence the details are omitted.

Remark 5.1 If the differential inequality

$$\left(\tilde{r}(\xi)\tilde{U}'(\xi)\right)+\tilde{q}(\xi)\tilde{U}'(\xi)+\tilde{p}(\xi)\,f\left(\tilde{K}(\xi)\right)L-\tilde{F}(\xi)\le0$$

has no eventually positive solution then every solution of (E) and (B_2) is oscillatory in $G_{\xi_1}=\Omega\times[\xi_1,\infty)$

where $\xi_1 \ge 0$.

Theorem 5.2 Let the conditions of Theorem 3.3 hold. Then every solution of (E) and (B_2) is oscillatory in G.

Theorem 5.3 Let the conditions of Theorem 3.4 hold. Then every solution of (E) and (B_2) is oscillatory in G.

Corollary 5.1 Let the conditions of Corollary 3.1 hold. Then every solution of (E) and (B_2) is oscillatory in G.

Theorem 5.4 Let the conditions of Theorem 3.5 hold. Then every solution of (E) and (B_2) is oscillatory in G.

Theorem 5.5 Let the conditions of Theorem 3.6 hold. Then every solution of (E) and (B_2) is oscillatory in G.

Corollary 5.2 Let the conditions of Corollary 3.2 hold. Then every solution of (E) and (B_2) is oscillatory in G.

Corollary 5.3 Let the conditions of Corollary 3.3 hold. Then every solution of (E) and (B_2) is oscillatory in G.

Theorem 5.6 Let all the conditions of Theorem 4.1 be hold. Then every solution of (E), (B_2) is oscillatory in G.

Corollary 5.4 Let the conditions of Corollary 4.1 hold. Then every solution of (E) and (B_2) is oscillatory in G.

6. Examples

In this section, we give some examples to illustrate our results established in Sections 3 and 4.

Example 6.1 Consider the fractional partial differential equation

$$
\begin{aligned}
&D_{+,t}^{\frac{1}{3}}\left(t^2 D_{+,t}^{\frac{1}{3}}u(x,t)\right)-\frac{\sqrt{3}}{2}t^2 D_{+,t}^{\frac{1}{3}}u(x,t)\\
&+\frac{1}{\sqrt{2\pi}\left(\cos t C(x)+\sin t S(x)\right)\left(1+\sin^2 x\left(\frac{\sqrt{3}\cos t-\sin t}{2}\right)^2\right)}\int_0^t (t-s)^{\frac{-1}{3}}u(x,s)\,\mathrm{d}s\left(1+\left[D_{+,t}^{\frac{1}{3}}u(x,t)\right]^2\right) \qquad (E_1)\\
&=\frac{t^2}{4}\Delta u(x,t)+\frac{27}{20\pi}\Gamma\left(\frac{1}{3}\right)t^{\frac{5}{3}}\sin x\cos t-\left(\frac{9\sqrt{3}}{20\pi}\Gamma\left(\frac{1}{3}\right)t^{\frac{5}{3}}+\frac{\sqrt{3}}{4}t^2\right)\sin x\sin t+1
\end{aligned}
$$

for $(x,t)\in(0,\pi)\times[0,\infty)$, with the boundary condition

$$u(0,t)=u(\pi,t)=0,\quad t\ge 0. \tag{33}$$

Here

$$\alpha=\frac{1}{3},\ N=1,\ r(t)=t^2,\ q(x,t)=-\frac{\sqrt{3}}{2}t^2,\ p(x,t)=\frac{1}{\sqrt{2\pi}\left(\cos t C(x)+\sin t S(x)\right)\left(1+\sin^2 x\left(\frac{\sqrt{3}\cos t-\sin t}{2}\right)^2\right)},$$

where $C(x)$ and $S(x)$ are the Fresnel integrals namely

$$C(x)=\int_0^x \cos\left(\frac{1}{2}\pi t^2\right)\mathrm{d}t,\quad S(x)=\int_0^x \sin\left(\frac{1}{2}\pi t^2\right)\mathrm{d}t$$

$$f(K(t))=K(t),\ K(t)=\int_0^t (t-s)^{\frac{-1}{3}}u(x,s)\,\mathrm{d}s,\ g\left(D_{+,t}^{\frac{1}{3}}u(x,t)\right)=1+\left[D_{+,t}^{\frac{1}{3}}u(x,t)\right]^2,\ a(t)=\frac{t^2}{4}$$

and

$$F(x,t)=\frac{27}{20\pi}\Gamma\left(\frac{1}{3}\right)t^{\frac{5}{3}}\sin x\cos t-\left(\frac{9\sqrt{3}}{20\pi}\Gamma\left(\frac{1}{3}\right)t^{\frac{5}{3}}+\frac{\sqrt{3}}{4}t^2\right)\sin x\sin t+1.$$

It is easy to see that $q(t)=\min_{x\in\Omega} q(x,t)=-\frac{\sqrt{3}}{2}t^2$. But $|C(x)|\le\pi$ and $|S(x)|\le\pi$. Therefore

$$p(t)=\min_{x\in\Omega} p(x,t)=\frac{1}{\pi\sqrt{2\pi}\left(\cos t+\sin t\right)}$$

we take $\lambda = 2$ and $\tilde{v}(\xi) = \frac{1}{\xi^2}$ so that $\tilde{\rho}(\xi) = \frac{1}{\xi}$. It is clear that the conditions (A_1) - (A_5) hold. We may observe that

$$f'(K(t)) = K'(t) = \Gamma\left(1-\frac{1}{3}\right) D_{+,t}^{\frac{1}{3}} u(x,t) = \frac{\pi}{\Gamma\left(\frac{1}{3}\right)\sqrt{3}} \sin x\left(\sqrt{3}\cos t - \sin t\right)$$

Using the property, $|a-b| \geq \big||a|-|b|\big|$ we get

$$f'(K(t)) > \frac{\pi}{\Gamma\left(\frac{1}{3}\right)\sqrt{3}}\left(\sqrt{3}-1\right) > 1 = M,\ g\left(D_{+,t}^{\frac{1}{3}} u(x,t)\right) = 1 + \left[D_{+,t}^{\frac{1}{3}} u(x,t)\right]^2 > 1 = L.$$

$$\tilde{\phi}(\xi) = \tilde{v}(\xi)\left\{L\tilde{p}(\xi) - \tilde{q}(\xi)\tilde{\rho}(\xi) + M\Gamma(1-\alpha)\tilde{r}(\xi)\tilde{\rho}^2(\xi) - \left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right\}$$

$$= \frac{1}{\xi^2\left(\pi\sqrt{2\pi}\right)(\cos\xi + \sin\xi)} + \frac{\sqrt{3}}{2\xi} + \left(\frac{2\pi}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)} - 1\right)\frac{1}{\xi^2}.$$

Consider

$$\limsup_{\xi\to\infty} \frac{1}{\xi^{\lambda-1}} \int_{\xi_i}^{\xi} (s-\xi_i)^{\lambda} \left[\tilde{\phi}(s) - \frac{1}{4M\Gamma(1-\alpha)}\tilde{r}(s)\tilde{v}(s)\left(\frac{\lambda}{s-\xi_i} - \frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right] \mathrm{d}s$$

$$= \limsup_{\xi\to\infty} \frac{1}{\xi} \int_{\xi_i}^{\xi} (s-\xi_i)^2 \left[\frac{1}{s^2\left(\pi\sqrt{2\pi}\right)(\cos s + \sin s)} + \frac{\sqrt{3}}{2s} + \left(\frac{2\pi}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)} - 1\right)\frac{1}{s^2} - \frac{1}{4\Gamma\left(1-\frac{1}{3}\right)}\left(\frac{2}{s-\xi_i} + \frac{\sqrt{3}}{2}\right)^2\right] \mathrm{d}s$$

$$> \limsup_{\xi\to\infty} \frac{1}{\xi\Gamma\left(\frac{2}{3}\right)} \int_{\xi_i}^{\xi} (s-\xi_i)^2 \left(\frac{1}{s-\xi_i} + \frac{\sqrt{3}}{4}\right)^2 \mathrm{d}s$$

$$> \limsup_{\xi\to\infty} \frac{1}{\xi} \frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)}{32\pi} \int_{\xi_i}^{\xi} \left(4 + \sqrt{3}s - \sqrt{3}\xi_i\right)^2 \mathrm{d}s$$

$$> \limsup_{\xi\to\infty} \frac{1}{\xi} \frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)}{32\pi} \int_{\xi_i}^{\xi} s^2 \mathrm{d}s = \infty$$

and

$$\limsup_{\xi\to\infty} \frac{1}{\xi^{\lambda-1}} \int_{\xi_i}^{\xi} (\xi-s)^{\lambda} \left[\tilde{\phi}(s) - \frac{1}{4M\Gamma(1-\alpha)}\tilde{r}(s)\tilde{v}(s)\left(\frac{\lambda}{\xi-s} + \frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right] \mathrm{d}s$$

$$= \limsup_{\xi\to\infty} \frac{1}{\xi} \int_{\xi_i}^{\xi} (\xi-s)^2 \left[\frac{1}{s^2\left(\pi\sqrt{2\pi}\right)(\cos s + \sin s)} + \frac{\sqrt{3}}{2s} + \left(\frac{2\pi}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)} - 1\right)\frac{1}{s^2} - \frac{1}{4\Gamma\left(1-\frac{1}{3}\right)}\left(\frac{2}{\xi-s} - \frac{\sqrt{3}}{2}\right)^2\right] \mathrm{d}s = \infty.$$

Thus all conditions of Corollary 3.1 are satisfied. Hence every solution of (E_1), (33) oscillates in $(0,\pi)\times[0,\infty)$.

In fact $u(x,t)=\sin x\cos t$ is such a solution of the problem (E_1) and (33).

Example 6.2 Consider the fractional partial differential equation

$$\begin{aligned}&D_{+,t}^{\frac{1}{3}}\left(tD_{+,t}^{\frac{1}{3}}u(x,t)\right)-\frac{\sqrt{3}}{2}tD_{+,t}^{\frac{1}{3}}u(x,t)\\&+\frac{3\left(1+\cos^2 x\sin^2 t\right)}{\sqrt{2\pi}\left(\cos tC(x)+\sin tS(x)\right)\left(4+\cos^2 x\sin^2 t\right)\left(1+\cos^2 x\left(\frac{\sqrt{3}\sin t+\cos t}{2}\right)^2\right)}\int_0^t (t-s)^{\frac{-1}{3}}u(x,s)\mathrm{d}s\\&\left(\frac{1}{3}+\frac{1}{1+\cos^2 x\sin^2 t}\right)\left(1+\left(D_{+,t}^{\frac{1}{3}}u(x,t)\right)^2\right)\\&=\frac{t}{4}\Delta u(x,t)+\frac{9}{8\pi}\Gamma\left(\frac{1}{3}\right)t^{\frac{2}{3}}\cos x\sin t+\left(\frac{3\sqrt{3}}{8\pi}\Gamma\left(\frac{1}{3}\right)t^{\frac{2}{3}}+\frac{\sqrt{3}}{4}t\right)\cos x\cos t+1\end{aligned} \quad (E_2)$$

for $(x,t)\in(0,\pi)\times[0,\infty)$, with the boundary condition

$$u_x(0,t)=u_x(\pi,t)=0,\quad t\geq 0 \quad (34)$$

Here

$$\alpha=\frac{1}{3}, N=1, r(t)=t, q(x,t)=-\frac{\sqrt{3}}{2}t,$$

$$p(x,t)=\frac{3\left(1+\cos^2 x\sin^2 t\right)}{\sqrt{2\pi}\left(\cos tC(x)+\sin tS(x)\right)\left(4+\cos^2 x\sin^2 t\right)\left(1+\cos^2 x\left(\frac{\sqrt{3}\sin t+\cos t}{2}\right)^2\right)},$$

where $C(x)$ and $S(x)$ are as in Example 1.

$$f(K(t))=K(t)\left(\frac{1}{3}+\frac{1}{1+\cos^2 x\sin^2 t}\right),$$

$$K(t)=\int_0^t (t-s)^{\frac{-1}{3}}u(x,s)\mathrm{d}s,$$

$$g\left(D_{+,t}^{\frac{1}{3}}u(x,t)\right)=1+\left(D_{+,t}^{\frac{1}{3}}u(x,t)\right)^2,$$

$$a(t)=\frac{t}{4}$$

and

$$F(x,t)=\frac{9}{8\pi}\Gamma\left(\frac{1}{3}\right)t^{\frac{2}{3}}\cos x\sin t+\left(\frac{3\sqrt{3}}{8\pi}\Gamma\left(\frac{1}{3}\right)t^{\frac{2}{3}}+\frac{\sqrt{3}}{4}t\right)\cos x\cos t+1.$$

It is easy to see that $q(t)=-\frac{\sqrt{3}}{2}t$, $p(t)=\frac{3}{(2\pi)^{\frac{3}{2}}[\cos t+\sin t]}$ we take $\lambda=2$ and $\tilde{v}(\xi)=\frac{1}{\xi^2}$ so that $\tilde{\rho}(\xi)=\frac{1}{\xi}$. It is clear that the conditions (A_1) - (A_4) and (A_6) hold. We may observe that

$$\frac{f(K(t))}{K(t)}=\left(\frac{1}{3}+\frac{1}{1+\cos^2 x\sin^2 t}\right)>\frac{1}{3}=M_1$$

$$g\left(D_{+,t}^{\frac{1}{3}}u(x,t)\right)=1+\left(D_{+,t}^{\frac{1}{3}}u(x,t)\right)^2>1=L.$$

$$\tilde{\phi}(\xi)=\tilde{v}(\xi)\left\{LM_1\tilde{p}(\xi)-\tilde{q}(\xi)\tilde{\rho}(\xi)+\Gamma(1-\alpha)\tilde{r}(\xi)\tilde{\rho}^2(\xi)-\left(\tilde{r}(\xi)\tilde{\rho}(\xi)\right)'\right\}$$
$$=\frac{1}{\xi^2(2\pi)^{\frac{3}{2}}(\cos\xi+\sin\xi)}+\frac{\sqrt{3}}{2\xi^2}+\frac{2\pi}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)\xi^3}.$$

Consider

$$\limsup_{\xi\to\infty}\frac{1}{\xi^{\lambda-1}}\int_{\xi_i}^{\xi}(s-\xi_i)^{\lambda}\left[\tilde{\phi}(s)-\frac{1}{4\Gamma(1-\alpha)}\tilde{r}(s)\tilde{v}(s)\left(\frac{\lambda}{s-\xi_i}-\frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right]ds$$
$$=\limsup_{\xi\to\infty}\frac{1}{\xi}\int_{\xi_i}^{\xi}(s-\xi_i)^2\left[\frac{1}{s^2\left(2\pi^{\frac{3}{2}}\right)(\cos s+\sin s)}+\frac{\sqrt{3}}{2s^2}+\frac{2\pi}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)s^3}-\frac{1}{4\Gamma\left(1-\frac{1}{3}\right)}\frac{1}{s}\left(\frac{2}{s-\xi_i}+\frac{\sqrt{3}}{2}\right)^2\right]ds$$
$$>\limsup_{\xi\to\infty}\frac{1}{\xi\Gamma\left(\frac{2}{3}\right)}\int_{\xi_i}^{\xi}(s-\xi_i)^2\frac{1}{s}\left(\frac{1}{s-\xi_i}+\frac{\sqrt{3}}{4}\right)^2ds$$
$$>\limsup_{\xi\to\infty}\frac{1}{\xi}\frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)}{32\pi}\int_{\xi_i}^{\xi}\frac{1}{s}\left(4+\sqrt{3}s-\sqrt{3}\xi_i\right)^2ds$$
$$>\limsup_{\xi\to\infty}\frac{1}{\xi}\frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)}{32\pi}\int_{\xi_i}^{\xi}\frac{1}{s}s^2ds=\infty$$

and

$$\limsup_{\xi\to\infty}\frac{1}{\xi^{\lambda-1}}\int_{\xi_i}^{\xi}(\xi-s)^{\lambda}\left[\tilde{\phi}(s)-\frac{1}{4\Gamma(1-\alpha)}\tilde{r}(s)\tilde{v}(s)\left(\frac{\lambda}{\xi-s}+\frac{\tilde{q}(s)}{\tilde{r}(s)}\right)^2\right]ds$$
$$=\limsup_{\xi\to\infty}\frac{1}{\xi}\int_{\xi_i}^{\xi}(\xi-s)^2\left[\frac{1}{s^2(2\pi)^{\frac{3}{2}}(\cos s+\sin s)}+\frac{\sqrt{3}}{2s^2}+\frac{2\pi}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)s^3}-\frac{1}{4\Gamma\left(1-\frac{1}{3}\right)}\frac{1}{s}\left(\frac{2}{\xi-s}-\frac{\sqrt{3}}{2}\right)^2\right]ds=\infty.$$

Thus, all the conditions of Corollary 4.1 are satisfied. Therefore, every solution of (E_2), (34) oscillates in $(0,\pi)\times[0,\infty)$. In fact, $u(x,t)=\cos x\sin t$ is such a solution of the problem (E_2) and (34).

Acknowledgements

The authors thank "Prof. E. Thandapani" for his support to complete the paper. Also the authors express their sincere thanks to the referee for valuable suggestions.

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