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Idempotent Elements of the Semigroups $B_X(D)$ Defined by Semilattices of the Class $\Sigma_3(X,8)$ When $Z_7 \neq \varnothing$

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Abstract

In the paper, complete semigroup binary relation is defined by semilattices of the class $\Sigma_3(X,8)$. We give a full description of idempotent elements of given semigroup. For the case where X is a finite set and $Z_7 \neq \varnothing$, we derive formulas by calculating the numbers of idempotent elements of the respective semigroup.

Keywords

Semilattice, Semigroup, Binary Relation, Idempotent Element

1. Introduction

Let X be an arbitrary nonempty set, D be a X-semilattice of unions, *i.e.* a nonempty set of subsets of the set X that is closed with respect to the set-theoretic operations of unification of elements from D, f be an arbitrary mapping from X into D. To each such a mapping f there corresponds a binary relation α_f on the set X that satisfies the condition $\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x))$. The set of all such α_f $(f : X \to D)$ is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by a X-semilattice of unions D (see 2.1 p. 34 of [1]).

By $\varnothing$ we denote an empty binary relation or empty subset of the set X. The condition $(x, y) \in \alpha$ will be

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written in the form $x\alpha y$. Further let $x, y \in X$, $Y \subseteq X$, $\alpha \in B_X(D)$, $T \in D$, $\varnothing \neq D' \subseteq D$ and $t \in \breve{D} = \bigcup_{Y \in D} Y$. Then by symbols we denote the following sets:

$$y\alpha = \{x \in X \mid y\alpha x\},\ Y\alpha = \bigcup_{y \in Y} y\alpha,\ V(D,\alpha) = \{Y\alpha \mid Y \in D\},$$

$$X^* = \{T \mid \varnothing \neq T \subseteq X\},\ D'_t = \{Z' \in D' \mid t \in Z'\},\ D'_T = \{Z' \in D' \mid T \subseteq Z'\}.$$

$$\ddot{D}'_T = \{Z' \in D' \mid Z' \subseteq T\},\ l(D',T) = \cup(D' \setminus D'_T),\ Y_T^\alpha = \{x \in X \mid x\alpha = T\}.$$

By symbol $\wedge(D, D_t)$ we mean an exact lower bound of the set D' in the semilattice *D*.

Definition 1.1. Let $\varepsilon \in B_X(D)$. If $\varepsilon \circ \varepsilon = \varepsilon$ or $\alpha \circ \varepsilon = \alpha$ for any $\alpha \in B_X(D)$, then ε is called an idempotent element or called right unit of the semigroup $B_X(D)$ respectively.

Definition 1.2. We say that a complete *X*-semilattice of unions *D* is an *XI*-semilattice of unions if it satisfies the following two conditions:

a) $\wedge(D, D_t) \in D$ for any $t \in \breve{D}$;

b) $Z = \bigcup_{t \in Z} \wedge(D, D_t)$ for any nonempty element *Z* of *D*. (see [2], definition 1.14.2) or see ([3], definition 1.14.2)

Definition 1.3. Let *D* be an arbitrary complete *X*-semilattice of unions, $\alpha \in B_X(D)$. If

$$V[\alpha] = \begin{cases} V(X^*,\alpha), & \text{if } \varnothing \notin D, \\ V(X^*,\alpha), & \text{if } \varnothing \in V(X^*,\alpha), \\ V(X^*,\alpha) \cup \{\varnothing\}, & \text{if } \varnothing \notin V(X^*,\alpha) \text{ and } \varnothing \in D, \end{cases}$$

then it is obvious that any binary relation α of a semigroup $B_X(D)$ can always be written in the form $\alpha = \bigcup_{T \in V[\alpha]} (Y_T^\alpha \times T)$ the sequel, such a representation of a binary relation α will be called quasinormal.

Note that for a quasinormal representation of a binary relation α, not all sets Y_T^α $(T \in V[\alpha])$ can be different from an empty set. But for this representtation the following conditions are always fulfilled:

a) $Y_T^\alpha \cap Y_{T'}^\alpha = \varnothing$, for any $T, T' \in D$ and $T \neq T'$;

b) $X = \bigcup_{T \in V[\alpha]} Y_T^\alpha$. (see [2], definition 1.11 or see [3], definition 1.11)

Definition 1.4. We say that a nonempty element *T* is a nonlimiting element of the set D' if $T \setminus l(D',T) \neq \varnothing$ and a nonempty element *T* is a limiting element of the set D' if $T \setminus l(D',T) = \varnothing$

Definition 1.5. Let us assume that by the symbol $\Sigma'_{XI}(X,D)$ denote a set of all *XI*-subsemilattices of *X*-semilatices of unions *D* every element of this set contain an empty set if $\varnothing \in D$ or denotes a set of all *XI*-subsemilatices of *D*.

Further, let $D, D' \in \Sigma'_{XI}(X,D)$ and $\vartheta_{XI} \subseteq \Sigma'_{XI}(X,D) \times \Sigma'_{XI}(X,D)$. It is assumed that $D\vartheta_{XI}D'$ iff there exist some complete isomorphism φ between the semilatices *D* and D'. One can easily verify that the binary relation ϑ_{XI} is an equivalence relation on the set $\Sigma'_{XI}(X,D)$.

Further, if *Q* ia a *XI*-subsemilattice of unions, then by the symbol $Q\vartheta_{XI}$ we denote that ϑ_{XI}-equivalence classes of the set $\Sigma'_{XI}(D)$ for each element of which there exists a complete isomorphism on the semilattice *Q*. (see [2], definition 6.3.5 or see [3], definition 6.3.5)

Theorem 1.1. A binary relation $\alpha \in B_X(D)$ is a right units of this semigroupiff α is idempotent and $D = V(D,\alpha)$ (see [2] Theorem 4.1.3 or [3] Theorem 4.1.3 or [4] Theorem 2.1).

Theorem 1.2. Let *D* be a complete *X*-semilattice of unions. The semigroup $B_X(D)$ possesses right unit iff *D* is an *XI*-semilattice of unions. (see [2] Theorem 6.1.3 or [3] Theorem 6.1.3 or [4] Theorem 2.6).

Theorem 1.3. Let *X* be a finite set and $D(\alpha)$ be the set of all those elements *T* of the semilattice $Q = V(D,\alpha) \setminus \{\varnothing\}$ which are nonlimiting elements of the set $\ddot{Q}_T$. A binary relation α having a quasinormal representation $\alpha = \bigcup_{T \in V(D,\alpha)} (Y_T^\alpha \times T)$ is an idempotent element of this semigroupiff

a) $V(D,\alpha)$ is complete *XI*-semilattise of unions;

b) $\bigcup_{T' \in \ddot{D}(\alpha)_T} Y_{T'}^\alpha \supseteq T$ for any $T \in D(\alpha)$;

c) $Y_T^{\alpha} \cap T \neq \varnothing$ for any nonlimiting element of the set $\ddot{D}(\alpha)_T$ (see [2] Theorem 6.3.9 or [3] Theorem 6.3.9).

Theorem 1.4. Let *D*, $\Sigma(D)$, $E_X^{(r)}(D')$ and *I* denote respectively the complete *X*-semilattice of unions, the set of all *XI*-subsemilatices of the semilattice *D*, the set of all right units of the semigroup $B_X(D')$ and the set of allidempotents of the semigroup $B_X(D)$. Then for the sets $E_X^{(r)}(D')$ and *I* the following statements are true:

a) if $\varnothing \in D$ and $\Sigma_{\varnothing}(D) = \{D' \in \Sigma(D) / \varnothing \in D'\}$ then

1) $E_X^{(r)}(D') \cap E_X^{(r)}(D'') = \varnothing$ for any elements D' and D'' of the set $\Sigma_{\varnothing}(D)$ that satisfy the condition $D' \neq D''$;

2) $I = \bigcup_{D' \in \Sigma(D)} E_X^{(r)}(D')$;

3) The equality $|I| = \sum_{D' \in \Sigma(D)} \left|E_X^{(r)}(D')\right|$ is fulfilled for the finite set *X*.

b) if $\varnothing \notin D$, then

1) $E_X^{(r)}(D') \cap E_X^{(r)}(D'') = \varnothing$ for any elements D' and D'' of the set $\Sigma(D)$ that satisfy the condition $D' \neq D''$;

2) $I = \bigcup_{D' \in \Sigma(D)} E_X^{(r)}(D')$;

3) The equality $|I| = \sum_{D' \in \Sigma(D)} \left|E_X^{(r)}(D')\right|$ is fulfilled for the finite set *X*. (see [2] Theorem 6.2.3 or [3] Theorem 6.2.3 or [4] Theorem 6).

Lemma 1.1. Let $Y = \{y_1, y_2, \cdots, y_k\}$ and $D_j = \{T_1, \cdots, T_j\}$ be some sets, where $k \geq 1$ and $j \geq 1$. Then the number $s(k, j)$ of all possible mappings of the set *Y* on any such subset of the set D_j' that $T_j \in D_j'$ can be calculated by the formula $s(k, j) = j^k - (j-1)^k$ (see [2] Corollary 1.18.1 or [3], Corollary 1.18.1 or [4] equality 6.9).

Lemma 1.2. Let $D_j = \{T_1, T_2, \cdots, T_j\}$, *X*, *Y* are tree nonempty set and $Y \subseteq X$. *f* be a mapping of the set *X* in the set D_j which satisfies the conditions $f(y) = T_j$ for some $y \in Y$, Then number such mappings of the set *X* in the set D_j is equal to $s = j^{|X \setminus Y|} \cdot \left(j^{|Y|} - (j-1)^{|y|}\right)$ (see [2] Theorem 1.18.2 or [3] Theorem 1.18.2).

Lemma 1.3. Let *D* by a complete *X*-semilattice of unions. If a binary relation ε of the form $\varepsilon = \bigcup_{t \in \breve{D}} \left(\{t\} \times \wedge(D, D_t)\right) \cup \left((X \setminus \breve{D}) \times \breve{D}\right)$ is right unit of the semigroup $B_X(D)$, then ε is the greatest right unit of that semigroup (see [2], Lemma 12.1.2 or [3], lemma 1.1.2).

Theorem 1.5. Let $D = \{\breve{D}, Z_1, Z_2, \cdots, Z_{n-1}\}$ be some finite *X*-semilattice of unions and $C(D) = \{P_0, P_1, P_2, \cdots, P_{n-1}\}$ be the family of sets of pairwise nonintersecting subsets of the set *X*. If φ is a mapping of the semilattice *D* on the family of sets $C(D)$ which satisfies the condition $\varphi(\breve{D}) = P_0$ and $\varphi(Z_i) = P_i$ for any $i = 1, 2, \cdots, n-1$ and $\hat{D}_Z = D \setminus \{T \in D \mid Z \subseteq T\}$, then the following equalities are valid:

$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{n-1}, \; Z_i = P_0 \cup \bigcup_{T \in \hat{D}_{Z_i}} \varphi(T). \tag{1.1}$$

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice *D* are represented in the form (1.1), then among the parameters P_i $(i = 0, 1, 2, \cdots, n-1)$ there exist such parameters that cannot be empty sets. Such sets P_i $(0 < i \leq n-1)$ are called basis sources, whereas sets P_j $(0 \leq j \leq n-1)$ which can be empty sets too are called completeness sources.

The number the basis sources we denote by symbol δ.

It is proved that under the mapping φ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see [2], 11.4 or [3], 11.4 or [5]).

Theorem 1.6. Let *X* be a finite set; δ and *q* are respectively the number of basic sources and the number of all automorphisms of the semilattice *D*. If $|X| = n \geq \delta$ and $|\Sigma_n(X, m)| = s$, then

$$s=\frac{1}{q}\cdot\sum_{p=\delta}^{m}\left(\sum_{i=1}^{p+1}\left(\frac{(-1)^{p+i+1}\cdot C_{m-\delta}^{p-\delta}\cdot C_{p}^{\delta}\cdot(\delta!)\cdot((p-\delta)!)\cdot i^{n}}{(i-1)!\cdot(p-i+1)!}\right)\right),$$

where $C_j^k=\frac{j!}{k!\cdot(j-k)!}$ (see [2] Theorem 11.5.1 or [3] Theorem 11.5.4).

we give complete classification all XI-subsemilattices of the semilatticeopf the class $\Sigma_3(X,8)$ we derive formulas by calculation the numbers of the semilattices of the given class.

2. Results

In this subsection it is assumed that $Z_7\neq\varnothing$ and we characterize the idemtpotent elements of the complete semigroup of binary relations which are defined by semilattices of the class $\Sigma_3(X,8)$.

By the symbol $\Sigma_3(X,8)$ we denote the class of all *X*-semilattices of unions whose every element is isomorphic to *X*-semilattice of the form $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$, where

$$\begin{aligned}
&Z_7\subset Z_6\subset Z_4\subset Z_2\subset\breve{D},\ \ Z_7\subset Z_6\subset Z_4\subset Z_1\subset\breve{D},\\
&Z_7\subset Z_5\subset Z_4\subset Z_2\subset\breve{D},\ \ Z_7\subset Z_5\subset Z_4\subset Z_1\subset\breve{D},\\
&Z_7\subset Z_5\subset Z_3\subset Z_1\subset\breve{D};\\
&Z_1\setminus Z_2\neq\varnothing,\ \ Z_2\setminus Z_1\neq\varnothing,\ \ Z_3\setminus Z_2\neq\varnothing,\ \ Z_2\setminus Z_3\neq\varnothing,\\
&Z_3\setminus Z_4\neq\varnothing,\ \ Z_4\setminus Z_3\neq\varnothing,\ \ Z_3\setminus Z_6\neq\varnothing,\ \ Z_6\setminus Z_3\neq\varnothing,\\
&Z_5\setminus Z_6\neq\varnothing,\ \ Z_6\setminus Z_5\neq\varnothing;
\end{aligned}\tag{2.1}$$

The semilattice satisfying the conditions (2.1) is shown in **Figure 1**.

It is further assumed that $C(D)=\{P_0,P_1,P_2,P_3,P_4,P_5,P_6,P_7\}$ is some set of pairwise nonitersecting subsets of the set X, then formal equalities for the element of the considered semilattice have the form

$$\begin{aligned}
\breve{D}&=P_0\cup P_1\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\\
Z_1&=P_0\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\\
Z_2&=P_0\cup P_1\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\\
Z_3&=P_0\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7\\
Z_4&=P_0\cup P_3\cup P_5\cup P_6\cup P_7\\
Z_5&=P_0\cup P_6\cup P_7\\
Z_6&=P_0\cup P_3\cup P_5\cup P_7\\
Z_7&=P_0
\end{aligned}\tag{2.2}$$

where $|P_0|\geq 0,|P_4|\geq 0,|P_5|\geq 0,|P_7|\geq 0,|P_1|\geq 1,|P_2|\geq 1,|P_3|\geq 1,|P_6|\geq 1,$ thus the elements P_0,P_4,P_5 and P_7 are the sources of completeness, while the elements P_1,P_2,P_3,P_6 are the basis sources of the *X*-semilattice of unions *D*

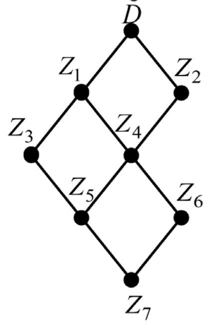

Figure 1. Diagram of *D*.

Lemma 2.1. Let $D \in \Sigma_3(X,8)$, $|\Sigma_3(X,8)| = s$ and $|X| \ge \delta \ge 4$. If X be a finite set, then

$$s = \frac{1}{2} \cdot \left(5^n - 4 \cdot 6^n + 6 \cdot 7^n - 4 \cdot 8^n + 9^n\right).$$

Proof. In this case we have: $m = 8$, $\delta = 4$ and $q = 2$, while the given semilattice D has only one identity automorphism. Of this and by Theorem 1.6 follows

$$s = \frac{1}{2} \cdot \sum_{p=4}^{8} \left(\sum_{i=1}^{p+1} \left(\frac{(-1)^{p+i+1} \cdot C_4^{p-4} \cdot C_p^4 \cdot (4!) \cdot ((p-4)!) \cdot i^n}{(i-1)! \cdot (p-i+1)!} \right) \right),$$

where $C_j^k = \dfrac{j!}{k! \cdot (j-k)!}$. Therefore the equality $s = \frac{1}{2} \cdot \left(5^n - 4 \cdot 6^n + 6 \cdot 7^n - 4 \cdot 8^n + 9^n\right)$ is true.

The Lemma is proved.

Example 2.1. Let $n = 4,5,6,7,8,9,10,$ then: $s = 24, 840, 17760, 147000, 2099412, 27156780, 327284760$ and $|B_X(D)| = 4096, 32768, 262144, 2097152, 16777216, 134217728, 1073741824$.

The number obtained show that if, for instance $|X| = 10$, than the number of elements in the class of semigroups, where each element is defined by some semilattice of the class is equal to $\Sigma_3(X,8)$ 327284760, while the number of elements in each semigroup belonging to this class is equal to 1072741824.

Let us define all subsemilattice of the semilattice *D*.

Lemma 2.2. Let $D \in \Sigma_3(X,8)$. Then the following sets exhaust all subsemilattices of the semilattice $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$

1) $\{Z_7\}, \{Z_6\}, \{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\breve{D}\}$.

(see diagram 1 of the **Figure 2**);

2) $\{Z_7, Z_1\}, \{Z_7, Z_2\}, \{Z_7, Z_3\}, \{Z_7, Z_4\}, \{Z_7, Z_5\}, \{Z_7, Z_6\}, \{Z_7, \breve{D}\}, \{Z_6, Z_4\}, \{Z_6, Z_2\},$
$\{Z_6, Z_1\}, \{Z_6, \breve{D}\}, \{Z_5, Z_4\}, \{Z_5, Z_3\}, \{Z_5, Z_2\}, \{Z_5, Z_1\}, \{Z_5, \breve{D}\}, \{Z_4, Z_2\}, \{Z_4, Z_1\},$
$\{Z_4, \breve{D}\}, \{Z_3, Z_1\}, \{Z_3, \breve{D}\}, \{Z_2, \breve{D}\}, \{Z_1, \breve{D}\}$.

(see diagram 2 of the **Figure 2**);

3) $\{Z_7, Z_6, Z_4\}, \{Z_7, Z_5, Z_2\}, \{Z_7, Z_6, Z_1\}, \{Z_7, Z_6, \breve{D}\}, \{Z_7, Z_5, Z_4\}, \{Z_7, Z_5, Z_3\}, \{Z_7, Z_5, Z_2\},$
$\{Z_7, Z_5, Z_1\}, \{Z_7, Z_5, \breve{D}\}, \{Z_7, Z_4, Z_2\}$ $\{Z_7, Z_4, Z_1\}, \{Z_7, Z_4, \breve{D}\}, \{Z_7, Z_3, Z_1\}, \{Z_7, Z_3, \breve{D}\},$
$\{Z_7, Z_2, \breve{D}\}, \{Z_7, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_2\}, \{Z_6, Z_4, Z_1\}, \{Z_6, Z_4, \breve{D}\}, \{Z_6, Z_2, \breve{D}\}, \{Z_6, Z_1, \breve{D}\},$
$\{Z_5, Z_4, Z_2\}, \{Z_5, Z_4, Z_1\}, \{Z_5, Z_4, \breve{D}\}, \{Z_5, Z_3, Z_1\}, \{Z_5, Z_3, \breve{D}\}, \{Z_5, Z_2, \breve{D}\}, \{Z_5, Z_1, \breve{D}\},$
$\{Z_4, Z_2, \breve{D}\}, \{Z_4, Z_1, \breve{D}\}, \{Z_3, Z_1, \breve{D}\}$.

(see diagram 3 of the **Figure 2**);

4) $\{Z_7, Z_6, Z_4, Z_2\}, \{Z_7, Z_6, Z_4, Z_1\}, \{Z_7, Z_6, Z_4, \breve{D}\}, \{Z_7, Z_6, Z_2, \breve{D}\}, \{Z_7, Z_6, Z_1, \breve{D}\},$
$\{Z_7, Z_5, Z_4, Z_2\}, \{Z_7, Z_5, Z_4, Z_1\}, \{Z_7, Z_5, Z_4, \breve{D}\}, \{Z_7, Z_5, Z_3, Z_1\}, \{Z_7, Z_5, Z_3, \breve{D}\},$
$\{Z_7, Z_5, Z_2, \breve{D}\}, \{Z_7, Z_5, Z_1, \breve{D}\}, \{Z_7, Z_4, Z_2, \breve{D}\}, \{Z_7, Z_4, Z_1, \breve{D}\}, \{Z_7, Z_3, Z_1, \breve{D}\},$
$\{Z_5, Z_4, Z_2, \breve{D}\}, \{Z_5, Z_3, Z_1, \breve{D}\}, \{Z_5, Z_4, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_2, \breve{D}\}, \{Z_6, Z_4, Z_1, \breve{D}\}$.

(see diagram 4 of the **Figure 2**);

5) $\{Z_7, Z_6, Z_4, Z_2, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_1, \breve{D}\}, \{Z_7, Z_5, Z_4, Z_1, \breve{D}\}, \{Z_7, Z_5, Z_4, Z_2, \breve{D}\},$
$\{Z_7, Z_5, Z_3, Z_1, \breve{D}\}$.

(see diagram 5 of the **Figure 2**);

6) $\{Z_7, Z_4, Z_3, Z_1\}, \{Z_7, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_5, Z_4\}, \{Z_4, Z_2, Z_1, \breve{D}\},$
$\{Z_6, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_3, Z_1\}, \{Z_7, Z_3, Z_2, \breve{D}\}, \{Z_5, Z_3, Z_2, \breve{D}\}.$

(see diagram 6 of the **Figure 2**);

7) $\{Z_7, Z_6, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_5, Z_4, Z_3, Z_1\}, \{Z_7, Z_5, Z_3, Z_2, \breve{D}\}, \{Z_7, Z_5, Z_2, Z_1, \breve{D}\},$
$\{Z_7, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_5, Z_4, Z_2, Z_1, \breve{D}\}.$

(see diagram 7 of the **Figure 2**);

8) $\{Z_7, Z_6, Z_4, Z_2, Z_1, \breve{D}\}; \{Z_7, Z_5, Z_4, Z_2, Z_1, \breve{D}\}.$

(see diagram 8 of the **Figure 2**);

9) $\{Z_7, Z_5, Z_4, Z_3, Z_1, \breve{D}\}.$

(see diagram 9 of the **Figure 2**);

10) $\{Z_7, Z_6, Z_5, Z_4, Z_2\}, \{Z_7, Z_6, Z_5, Z_4, Z_1\}, \{Z_7, Z_6, Z_5, Z_4, \breve{D}\},$
$\{Z_7, Z_6, Z_3, Z_1, \breve{D}\}, \{Z_5, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_7, Z_4, Z_3, Z_1, \breve{D}\},$

(see diagram 10 of the **Figure 2**);

11) $\{Z_7, Z_6, Z_5, Z_4, Z_2, \breve{D}\}, \{Z_7, Z_6, Z_5, Z_4, Z_1, \breve{D}\}.$

(see diagram 11 of the **Figure 2**);

12) $\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_1\}, \{Z_7, Z_6, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_4, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}.$

(see diagram 12 of the **Figure 2**);

13) $\{Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}.$

(see diagram 13 of the **Figure 2**);

14) $\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\},$

(see diagram 14 of the **Figure 2**);

15) $\{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\},$

(see diagram 15 of the **Figure 2**);

16) $\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}.$

(see diagram 16 of the **Figure 2**);

17) $\{Z_3, Z_2, \breve{D}\}, \{Z_2, Z_1, \breve{D}\}. \{Z_6, Z_5, Z_4\}, \{Z_6, Z_3, Z_1\}, \{Z_4, Z_3, Z_1\}.$

(see diagram 17 of the **Figure 2**);

18) $\{Z_6, Z_5, Z_4, \breve{D}\}, \{Z_6, Z_5, Z_4, Z_2\}, \{Z_6, Z_5, Z_4, Z_1\}, \{Z_4, Z_3, Z_1, \breve{D}\}, \{Z_6, Z_3, Z_1, \breve{D}\}.$

(see diagram 18 of the **Figure 2**);

19) $\{Z_3, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_3, Z_1\}.$

(see diagram 19 of the **Figure 2**);

20) $\{Z_6, Z_5, Z_4, Z_2, \breve{D}\}, \{Z_6, Z_5, Z_4, Z_1, \breve{D}\},$

(see diagram 20 of the **Figure 2**);

21) $\{Z_6, Z_4, Z_3, Z_1, \breve{D}\},$

(see diagram 21 of the **Figure 2**);

22) $\{Z_7, Z_6, Z_4, Z_3, Z_1\}, \{Z_5, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_3, Z_2, Z_1, \breve{D}\},$

(see diagram 22 of the **Figure 2**);

23) $\{Z_6, Z_5, Z_4, Z_3, Z_1\}, \{Z_6, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_4, Z_3, Z_2, Z_1, \breve{D}\},$

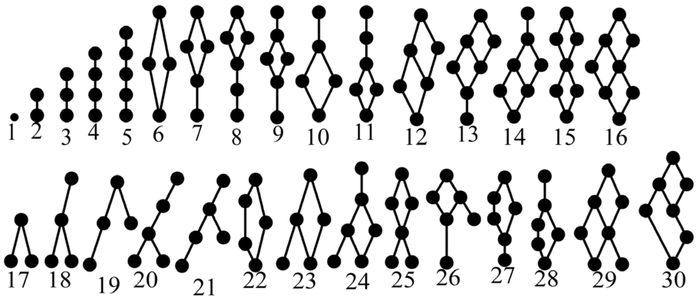

Figure 2. All diagrams of subsemilattices of the semilattice *D*.

(see diagram 23 of the **Figure 2**);

24) $\{Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$,

(see diagram 24 of the **Figure 2**);

25) $\{Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$,

(see diagram 25 of the **Figure 2**);

26) $\{Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$,

(see diagram 26 of the **Figure 2**);

27) $\{Z_7, Z_5, Z_3, Z_2, Z_1, \breve{D}\}$;

(see diagram 27 of the **Figure 2**);

28) $\{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\}$,

(see diagram 28 of the **Figure 2**);

29) $\{Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$,

(see diagram 29 of the **Figure 2**);

30) $\{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$,

(see diagram 30 of the **Figure 2**);

Proof. It is easy to see that, the sets $\{Z_7\}, \{Z_6\}, \{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\breve{D}\}$ are subsemilattices of the semilattice *D*.

The number subsets of the semilattise *D*, which contain two element is equal to $C_8^2 = 28$. They are:

$$\{Z_7, Z_1\}, \{Z_7, Z_2\}, \{Z_7, Z_3\}, \{Z_7, Z_4\}, \{Z_7, Z_5\}, \{Z_7, Z_6\}, \{Z_7, \breve{D}\}, \{Z_5, \breve{D}\},$$

$$\{Z_6, Z_4\}, \{Z_6, Z_2\}, \{Z_6, Z_1\}, \{Z_6, \breve{D}\}, \{Z_3, \breve{D}\}, \{Z_5, Z_4\}, \{Z_5, Z_3\}, \{Z_5, Z_2\},$$

$$\{Z_5, Z_1\}, \{Z_4, Z_2\}, \{Z_4, Z_1\}, \{Z_4, \breve{D}\}, \{Z_3, Z_1\}, \{Z_2, \breve{D}\}, \{Z_1, \breve{D}\},$$

$$\{Z_2, Z_1\}, \{Z_6, Z_5\}, \{Z_6, Z_3\}, \{Z_4, Z_3\}, \{Z_3, Z_2\}.$$

It is easy to see that, last five sats are not subsemilattices of the semilattice *D*.

The number subsets of the semilattise *D*, which contain tree element is equal to $C_8^3 = 56$. They are:

$$\{Z_7, Z_6, Z_4\}, \{Z_7, Z_5, Z_2\}, \{Z_7, Z_6, Z_1\}, \{Z_7, Z_6, \breve{D}\}, \{Z_7, Z_5, Z_4\}, \{Z_7, Z_5, Z_3\},$$

$$\{Z_7, Z_5, Z_2\}, \{Z_7, Z_5, Z_1\}, \{Z_7, Z_5, \breve{D}\}, \{Z_7, Z_4, Z_2\}, \{Z_7, Z_4, Z_1\}, \{Z_7, Z_4, \breve{D}\},$$

$$\{Z_7, Z_3, Z_1\}, \{Z_7, Z_3, \breve{D}\}, \{Z_7, Z_2, \breve{D}\}, \{Z_7, Z_1, \breve{D}\}, \{Z_6, Z_5, Z_4\}, \{Z_6, Z_4, Z_2\},$$

$$\{Z_6, Z_4, Z_1\}, \{Z_6, Z_4, \breve{D}\}, \{Z_6, Z_3, Z_1\}, \{Z_6, Z_2, \breve{D}\}, \{Z_6, Z_1, \breve{D}\}, \{Z_5, Z_4, Z_2\},$$

$$\{Z_5, Z_4, Z_1\}, \{Z_5, Z_4, \breve{D}\}, \{Z_5, Z_3, Z_1\}, \{Z_5, Z_3, \breve{D}\}, \{Z_5, Z_2, \breve{D}\}, \{Z_5, Z_1, \breve{D}\},$$

$$\{Z_4,Z_3,Z_1\},\{Z_4,Z_2,\breve{D}\},\{Z_4,Z_1,\breve{D}\},\{Z_3,Z_2,\breve{D}\},\{Z_3,Z_1,\breve{D}\},\{Z_2,Z_1,\breve{D}\},$$
$$\{Z_4,Z_2,Z_1\},\{Z_3,Z_2,Z_1\},\{Z_7,Z_6,Z_5\},\{Z_7,Z_6,Z_3\},\{Z_7,Z_4,Z_3\},\{Z_7,Z_3,Z_2\},$$
$$\{Z_7,Z_2,Z_1\},\{Z_6,Z_5,Z_3\},\{Z_6,Z_5,Z_2\},\{Z_6,Z_5,Z_1\},\{Z_6,Z_5,\breve{D}\},\{Z_6,Z_4,Z_3\},$$
$$\{Z_5,Z_3,Z_2\},\{Z_6,Z_3,Z_2\},\{Z_6,Z_3,\breve{D}\},\{Z_6,Z_2,Z_1\},\{Z_5,Z_4,Z_3\},\{Z_5,Z_2,Z_1\},$$
$$\{Z_4,Z_3,Z_2\},\{Z_4,Z_3,\breve{D}\},$$

It is easy to see that, last twenty sats are not subsemilattices of the semilattice *D*.

The number subsets of the semilattise *D*, which contain four element is equal to $C_8^4=70$. They are:

$$\{Z_7,Z_6,Z_5,Z_4\},\{Z_7,Z_6,Z_4,Z_2\},\{Z_7,Z_6,Z_4,Z_1\},\{Z_7,Z_5,Z_4,Z_2\},\{Z_7,Z_6,Z_4,\breve{D}\},$$
$$\{Z_7,Z_6,Z_3,\breve{D}\},\{Z_7,Z_6,Z_3,Z_1\},\{Z_7,Z_5,Z_3,Z_1\},\{Z_7,Z_6,Z_2,\breve{D}\},\{Z_7,Z_6,Z_1,\breve{D}\},$$
$$\{Z_7,Z_5,Z_4,Z_1\},\{Z_7,Z_5,Z_4,\breve{D}\},\{Z_7,Z_5,Z_3,\breve{D}\},\{Z_7,Z_5,Z_2,\breve{D}\},\{Z_7,Z_5,Z_1,\breve{D}\},$$
$$\{Z_7,Z_4,Z_3,Z_1\},\{Z_7,Z_4,Z_2,\breve{D}\},\{Z_7,Z_4,Z_1,\breve{D}\},\{Z_7,Z_3,Z_2,\breve{D}\},\{Z_7,Z_3,Z_1,\breve{D}\},$$
$$\{Z_7,Z_2,Z_1,\breve{D}\},\{Z_6,Z_2,Z_1,\breve{D}\},\{Z_5,Z_4,Z_3,Z_1\},\{Z_5,Z_4,Z_2,\breve{D}\},\{Z_6,Z_5,Z_4,Z_2\},$$
$$\{Z_6,Z_5,Z_4,\breve{D}\},\{Z_6,Z_5,Z_4,Z_1\},\{Z_6,Z_4,Z_3,Z_1\},\{Z_6,Z_5,Z_3,Z_1\},\{Z_6,Z_4,Z_2,\breve{D}\},$$
$$\{Z_6,Z_4,Z_1,\breve{D}\},\{Z_6,Z_3,Z_1,\breve{D}\},\{Z_4,Z_3,Z_1,\breve{D}\},\{Z_4,Z_2,Z_1,\breve{D}\},\{Z_5,Z_3,Z_2,\breve{D}\},$$
$$\{Z_5,Z_4,Z_1,\breve{D}\},\{Z_5,Z_3,Z_1,\breve{D}\},$$
$$\{Z_7,Z_6,Z_2,Z_1\},\{Z_7,Z_5,Z_4,Z_3\},\{Z_7,Z_5,Z_3,Z_2\},\{Z_7,Z_6,Z_3,Z_2\}\{Z_7,Z_6,Z_5,Z_3\},$$
$$\{Z_7,Z_5,Z_2,Z_1\},\{Z_7,Z_4,Z_3,Z_2\},\{Z_7,Z_4,Z_3,\breve{D}\},\{Z_7,Z_4,Z_2,Z_1\},\{Z_7,Z_6,Z_4,Z_3\},$$
$$\{Z_7,Z_3,Z_2,Z_1\},\{Z_7,Z_6,Z_5,Z_2\},\{Z_7,Z_6,Z_5,Z_1\},\{Z_7,Z_6,Z_5,\breve{D}\},\{Z_6,Z_5,Z_4,Z_3\},$$
$$\{Z_6,Z_5,Z_3,Z_2\},\{Z_6,Z_5,Z_2,Z_1\},\{Z_6,Z_5,Z_2,\breve{D}\},\{Z_6,Z_5,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_2\},$$
$$\{Z_6,Z_4,Z_3,\breve{D}\},\{Z_6,Z_3,Z_2,Z_1\},\{Z_6,Z_3,Z_2,\breve{D}\},\{Z_5,Z_4,Z_3,Z_2\},\{Z_5,Z_4,Z_3,\breve{D}\},$$
$$\{Z_5,Z_4,Z_2,Z_1\},\{Z_5,Z_3,Z_2,Z_1\},\{Z_5,Z_2,Z_1,\breve{D}\},\{Z_4,Z_3,Z_2,Z_1\},\{Z_4,Z_3,Z_2,\breve{D}\},$$
$$\{Z_3,Z_2,Z_1,\breve{D}\},\{Z_6,Z_5,Z_3,\breve{D}\},\{Z_6,Z_4,Z_2,Z_1\},$$

is easy to see that, last 33 sats are not subsemilattices of the semilattice *D*.

The number subsets of the semilattise *D*, which contain five element is equal to $C_8^5=56$. They are:

$$\{Z_7,Z_6,Z_5,Z_4,Z_2\},\{Z_7,Z_6,Z_5,Z_4,Z_1\},\{Z_7,Z_6,Z_5,Z_4,\breve{D}\},\{Z_7,Z_6,Z_4,Z_1,\breve{D}\},$$
$$\{Z_7,Z_6,Z_4,Z_2,\breve{D}\},\{Z_7,Z_6,Z_3,Z_1,\breve{D}\},\{Z_7,Z_6,Z_2,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_3,Z_1\},$$
$$\{Z_7,Z_5,Z_4,Z_2,\breve{D}\},\{Z_7,Z_5,Z_4,Z_1,\breve{D}\},\{Z_7,Z_5,Z_3,Z_2,\breve{D}\},\{Z_7,Z_5,Z_3,Z_1,\breve{D}\},$$
$$\{Z_7,Z_5,Z_2,Z_1,\breve{D}\},\{Z_7,Z_4,Z_3,Z_1,\breve{D}\},\{Z_7,Z_4,Z_2,Z_1,\breve{D}\},\{Z_7,Z_3,Z_2,Z_1,\breve{D}\},$$
$$\{Z_6,Z_5,Z_4,Z_3,Z_1\},\{Z_6,Z_5,Z_4,Z_2,\breve{D}\},\{Z_6,Z_5,Z_4,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_1,\breve{D}\},$$
$$\{Z_6,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_3,Z_2,Z_1,\breve{D}\},\{Z_5,Z_4,Z_3,Z_1,\breve{D}\},\{Z_5,Z_4,Z_2,Z_1,\breve{D}\},$$
$$\{Z_5,Z_3,Z_2,Z_1,\breve{D}\},\{Z_4,Z_3,Z_2,Z_1,\breve{D}\},\{Z_7,Z_6,Z_4,Z_3,Z_1\},$$

$$\{Z_7, Z_6, Z_5, Z_3, Z_2\}, \{Z_7, Z_6, Z_5, Z_3, Z_1\}, \{Z_7, Z_6, Z_5, Z_3, \breve{D}\}, \{Z_7, Z_6, Z_5, Z_2, Z_1\},$$
$$\{Z_7, Z_6, Z_5, Z_2, \breve{D}\}, \{Z_7, Z_6, Z_5, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_3, Z_2\}, \{Z_7, Z_6, Z_4, Z_3, \breve{D}\},$$
$$\{Z_7, Z_6, Z_4, Z_2, Z_1\}, \{Z_7, Z_6, Z_3, Z_2, Z_1\}, \{Z_7, Z_6, Z_3, Z_2, \breve{D}\}, \{Z_7, Z_6, Z_5, Z_4, Z_3\},$$
$$\{Z_7, Z_5, Z_4, Z_3, Z_2\}, \{Z_7, Z_5, Z_4, Z_3, \breve{D}\}, \{Z_7, Z_5, Z_4, Z_2, Z_1\}, \{Z_7, Z_5, Z_3, Z_2, Z_1\},$$
$$\{Z_7, Z_4, Z_3, Z_2, Z_1\}, \{Z_7, Z_4, Z_3, Z_2, \breve{D}\}, \{Z_6, Z_5, Z_4, Z_3, Z_2\}, \{Z_6, Z_5, Z_4, Z_3, \breve{D}\},$$
$$\{Z_6, Z_5, Z_4, Z_2, Z_1\}, \{Z_6, Z_5, Z_3, Z_2, Z_1\}, \{Z_6, Z_5, Z_3, Z_2, \breve{D}\}, \{Z_6, Z_5, Z_3, Z_1, \breve{D}\},$$
$$\{Z_6, Z_5, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_3, Z_2, Z_1\}, \{Z_6, Z_4, Z_3, Z_2, \breve{D}\}, \{Z_5, Z_4, Z_3, Z_2, Z_1\},$$
$$\{Z_5, Z_4, Z_3, Z_2, \breve{D}\}$$

is easy to see that, last 29 sats are not subsemilattices of the semilattice *D*.

The number subsets of the semilattise *D*, which contain six element is equal to $C_8^6 = 28$. They are:

$$\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_1\}, \{Z_7, Z_6, Z_5, Z_4, Z_2, \breve{D}\}, \{Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$$
$$\{Z_7, Z_6, Z_5, Z_4, Z_1, \breve{D}\}, \{Z_7, Z_5, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_7, Z_5, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_7, Z_5, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_5, Z_4, Z_3, \breve{D}\},$$
$$\{Z_7, Z_6, Z_5, Z_3, Z_2, Z_1\}, \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1\}, \{Z_7, Z_6, Z_5, Z_3, Z_2, \breve{D}\},$$
$$\{Z_7, Z_6, Z_5, Z_3, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_5, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1\},$$
$$\{Z_7, Z_6, Z_4, Z_3, Z_2, \breve{D}\}, \{Z_7, Z_5, Z_4, Z_3, Z_2, Z_1\} \{Z_7, Z_5, Z_4, Z_3, Z_2, \breve{D}\},$$
$$\{Z_6, Z_5, Z_4, Z_3, Z_2, Z_1\}, \{Z_6, Z_5, Z_4, Z_3, Z_2, \breve{D}\}, \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2\},$$
$$\{Z_6, Z_5, Z_3, Z_2, Z_1, \breve{D}\}.$$

is easy to see that, last 13 sats are not subsemilattices of the semilattice *D*.

The number subsets of the semilattise *D*, which contain seven element is equal to $C_8^7 = 8$. They are:

$$\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\},$$
$$\{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1\}, \{Z_7, Z_6, Z_5, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, \breve{D}\}$$

is easy to see that, last 3 sats are not subsemilattices of the semilattice *D*.

From the proven lemma it follows that diagrams shown in **Figure 2**, exhaust all diagrams of subsemilattices of the semilattice *D*.

Lemma 2.3. *Let* $D \in \Sigma_3(X, 8)$ *and* $Z_7 \neq \varnothing$. *Then any subsemilattices of the semilattice D having diagram* 17 - 30 *are never XI-semilattices.*

Proof: Remark, that the all subsemilattices of semilattice *D* which has diagrams of form 17 - 30 are never *XI*-semilattices. For example we consider the semilatticesuchis defined by the diagram of the form 30 of the **Figure 2**.

Let $Q'=\{T_6,T_5,T_4,T_3,T_2,T_1,T_0\}$ and $C(Q')=\{P_0,P_1,P_2,P_3,P_4,P_5,P_6\}$ is a family sets, where $P_0,P_1,P_2,P_3,P_4,P_5,P_6$ are pairwise disjoint subsets of the set X and $\varphi=\begin{pmatrix} T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 \\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{pmatrix}$ is a mapping of the semilattice Q' onto the family sets $C(Q')$. Then for the formal equalities of the semilattice Q' we have a form:

$$\begin{aligned} T_0 &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_6 \cup P_7, \\ T_1 &= P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_6 \cup P_7, \\ T_2 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_6 \cup P_7, \\ T_3 &= P_0 \cup P_2 \cup P_4 \cup P_6 \cup P_7, \\ T_4 &= P_0 \cup P_3 \cup P_6 \cup P_7, \\ T_5 &= P_0 \cup P_3 \cup P_7 \\ T_6 &= P_0 \end{aligned}$$

Here, the elements P_1,P_2,P_3,P_6 are basis sources, the element P_0,P_4,P_7 is sources of completenes of the semilattice Q'. Therefore $|X|\geq 3$ and $\delta=4$ Then of the formal equalities follows, that

$$Q'_t=\begin{cases} Q', \text{ if } t\in P_0, \\ \{T_2,T_0\}, \text{ if } t\in P_1 \\ \{T_3,T_1,T_0\}, \text{ if } t\in P_2, \\ \{T_6,T_4,T_2,T_1,T_0\}, \text{ if } t\in P_3, \\ \{T_3,T_2,T_1,T_0\}, \text{ if } t\in P_4, \\ \{T_4,T_3,T_2,T_1,T_0\}, \text{ if } t\in P_5, \\ \{T_6,T_4,T_3,T_2,T_1,T_0\}, \text{ if } t\in P_6, \end{cases} \qquad \Lambda(D',D'_t)=\begin{cases} Z_7, \text{ if } t\in P_0 \\ Z_2, \text{ if } t\in P_1 \\ Z_3, \text{ if } t\in P_2 \\ Z_6, \text{ if } t\in P_3 \\ Z_7, \text{ if } t\in P_4 \\ Z_7, \text{ if } t\in P_5 \\ Z_7, \text{ if } t\in P_6 \end{cases}$$

We have $Q'^{\wedge}=\{T_6,T_5,T_3,T_2\}$ and $\Lambda(Q',Q'_t)\in Q'$ for all $t\in T_0$. But element T_4 is not union of some elements of the set $Q'^{\wedge}$. So, from the Definition 1.2 follows that semilattice D' which has diagram 41 of the **Figure 3** never is *XI*-semilattice.

Lemma is proved.

Lemma 2.4. Let $D\in\Sigma_3(X,8)$ and $Z_7\neq\varnothing$. Then the following sets are all *XI*-subsemilattices of the given semilattice *D*:

1) $\{Z_7\},\{Z_6\},\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}$;

(see diagram 1 of the **Figure 4**);

2) $\{Z_7,Z_6\},\{Z_7,Z_5\},\{Z_7,Z_4\},\{Z_7,Z_3\},\{Z_7,Z_2\},\{Z_7,Z_1\},\{Z_7,\breve{D}\},\{Z_6,Z_4\},$
$\{Z_6,Z_2\},\{Z_6,Z_1\},\{Z_6,\breve{D}\},\{Z_5,Z_4\},\{Z_5,Z_3\},\{Z_5,Z_2\},\{Z_5,Z_1\},\{Z_5,\breve{D}\},$
$\{Z_4,Z_2\},\{Z_4,Z_1\},\{Z_4,\breve{D}\},\{Z_3,Z_1\},\{Z_3,\breve{D}\},\{Z_2,\breve{D}\},\{Z_1,\breve{D}\}$;

(see diagram 2 of the **Figure 4**);

3) $\{Z_7,Z_6,Z_4\},\{Z_7,Z_6,Z_2\},\{Z_7,Z_6,Z_1\},\{Z_7,Z_6,\breve{D}\},\{Z_7,Z_5,Z_4\},\{Z_7,Z_5,Z_3\},\{Z_7,Z_5,Z_2\},\{Z_7,Z_5,Z_1\},$
$\{Z_7,Z_5,\breve{D}\},\{Z_7,Z_4,Z_2\},\{Z_7,Z_4,Z_1\},\{Z_7,Z_4,D\},\{Z_7,Z_3,Z_1\},\{Z_7,Z_3,\breve{D}\},\{Z_7,Z_2,\breve{D}\},\{Z_7,Z_1,\breve{D}\},$
$\{Z_6,Z_4,Z_2\},\{Z_6,Z_4,Z_1\},\{Z_6,Z_4,\breve{D}\},\{Z_6,Z_2,\breve{D}\},\{Z_6,Z_1,\breve{D}\},\{Z_5,Z_4,Z_2\},\{Z_5,Z_4,Z_1\},\{Z_5,Z_4,\breve{D}\},$
$\{Z_5,Z_3,Z_1\},\{Z_5,Z_3,\breve{D}\},\{Z_5,Z_2,\breve{D}\},\{Z_5,Z_1,\breve{D}\},\{Z_4,Z_2,\breve{D}\},\{Z_4,Z_1,\breve{D}\},\{Z_3,Z_1,\breve{D}\}$;

(see diagram 3 of the **Figure 4**);

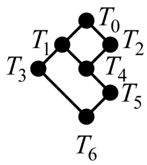

Figure 3. Diagram of Q'.

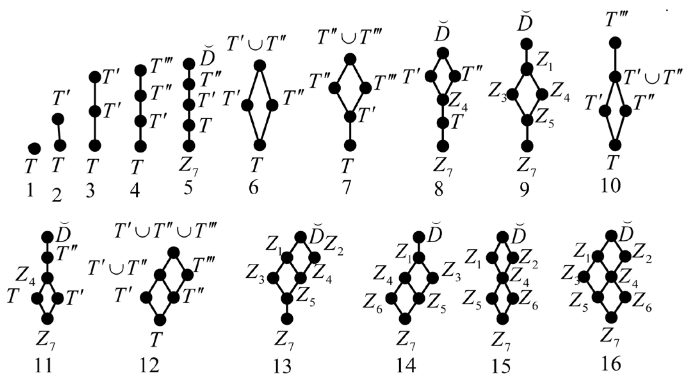

Figure 4. All diagrams *XI*-subsemilattices of thesemilattice *D*.

4) $\{Z_7,Z_6,Z_4,Z_2\},\{Z_7,Z_6,Z_4,Z_1\},\{Z_7,Z_6,Z_4,\breve{D}\},\{Z_7,Z_6,Z_2,D\},\{Z_7,Z_6,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_2\},$
$\{Z_7,Z_5,Z_4,Z_1\},\{Z_7,Z_5,Z_4,\breve{D}\}\ \{Z_7,Z_5,Z_3,Z_1\},\{Z_7,Z_5,Z_3,\breve{D}\},\{Z_7,Z_5,Z_2,\breve{D}\},\{Z_7,Z_5,Z_1,\breve{D}\},$
$\{Z_7,Z_4,Z_2,D\},\{Z_7,Z_4,Z_1,\breve{D}\},\{Z_7,Z_3,Z_1,\breve{D}\},\ \{Z_6,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_1,\breve{D}\},\{Z_5,Z_4,Z_2,\breve{D}\},$
$\{Z_5,Z_4,Z_1,\breve{D}\},\{Z_5,Z_3,Z_1,\breve{D}\};$

(see diagram 4 of the **Figure 4**);

5) $\{Z_7,Z_6,Z_4,Z_2,\breve{D}\},\{Z_7,Z_6,Z_4,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_2,\breve{D}\},\{Z_7,Z_5,Z_4,Z_1,\breve{D}\},\{Z_7,Z_5,Z_3,Z_1,\breve{D}\},$

(see diagram 5 of the **Figure 4**);

6) $\{Z_7,Z_6,Z_5,Z_4\},\{Z_7,Z_6,Z_3,Z_1\},\{Z_7,Z_4,Z_3,Z_1\},\{Z_7,Z_3,Z_2,\breve{D}\},\{Z_7,Z_2,Z_1,\breve{D}\},$
$\{Z_6,Z_2,Z_1,\breve{D}\},\{Z_5,Z_4,Z_3,Z_1\},\{Z_5,Z_3,Z_2,\breve{D}\},\{Z_5,Z_2,Z_1,\breve{D}\},\{Z_4,Z_2,Z_1,\breve{D}\};$

(see diagram 6 of the **Figure 4**);

7) $\{Z_7,Z_6,Z_2,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_3,Z_1,\},\{Z_7,Z_5,Z_3,Z_2,\breve{D}\},\{Z_7,Z_5,Z_2,Z_1,\breve{D}\},$
$\{Z_7,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2,Z_1,\breve{D}\}\{Z_5,Z_4,Z_2,Z_1,\breve{D}\};$

(see diagram 7 of the **Figure 4**);

8) $\{Z_7,Z_6,Z_4,Z_2,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\};$

(see diagram 8 of the **Figure 4**);

9) $\{Z_7,Z_5,Z_4,Z_3,Z_1,\breve{D}\};$

(see diagram 9 of the **Figure 4**);

10) $\{Z_7,Z_6,Z_5,Z_4,Z_2\},\{Z_7,Z_6,Z_5,Z_4,Z_1\},\{Z_7,Z_6,Z_5,Z_4,\breve{D}\},\{Z_7,Z_4,Z_3,Z_1,\breve{D}\},$
$\{Z_7,Z_6,Z_3,Z_1,\breve{D}\},\{Z_5,Z_4,Z_3,Z_1,\breve{D}\};$

(see diagram 10 of the **Figure 4**);

11) $\{Z_7,Z_6,Z_5,Z_4,Z_2,\breve{D}\},\{Z_7,Z_6,Z_5,Z_4,Z_1,\breve{D}\};$

(see diagram 11 of the **Figure 4**);

12) $\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_1\},\{Z_7,Z_6,Z_3,Z_2,Z_1,\breve{D}\},\{Z_7,Z_4,Z_3,Z_2,Z_1,\breve{D}\},\{Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$;

(see diagram 12 of the **Figure 4**);

13) $\{Z_7,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$;

(see diagram 13 of the **Figure 4**);

14) $\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_1,\breve{D}\}$;

(see diagram 14 of the **Figure 4**);

15) $\{Z_7,Z_6,Z_5,Z_4,Z_2,Z_1,\breve{D}\}$;

(see diagram 15 of the **Figure 4**);

16) $\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$;

(see diagram 16 of the **Figure 4**);

Proof: The statements 1), 2), 3), 4), 5) immediately follows from the Theorems 11.6.1 in [2], 11.6.1 in [3], the statements 6), 7), 8), 9), 10), 11) immediately follows from the Theorems 11.6.3in [2], 11.6.3 in [3]; the statement 12) immediately follows from the Theorems 11.7.2 in [2]; the statement 13) immediately follows from the Theorema 2.1 in [4], the statement 14) immediately follows from the lemma 2.1. in [5], the statements 15) immediately follows from the Theorems 13.11.1 in [2] and the statement 16) immediately follows from the theorem 2.1. in [6].

We denote the following semitattices Q_i $i=(1,2,\cdots,16)$ as follows:

1) $Q_1=\{T\}$, where $T\in D$;

2) $Q_2=\{T,T'\}$, where $T,T'\in D$, $T\subset T'$;

3) $Q_3=\{T,T',T''\}$, where $T,T',T''\in D$, $T\subset T'\subset T''$;

4) $Q_4=\{T,T',T'',T'''\}$, where $T,T',T'',T'''\in D$, $T\subset T'\subset T''\subset T'''$;

5) $Q_5=\{Z_7,T,T',T'',\breve{D}\}$, where $Z_7,T,T',T'',\breve{D}\in D$, $Z_7\subset T\subset T'\subset T''\subset\breve{D}$;

6) $Q_6=\{T,T',T'',T'\cup T''\}$, where $T,T',T''\in D$, $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$;

7) $Q_7=\{T,T',T'',T''',T''\cup T'''\}$, where, $T\subset T'\subset T''$, $T\subset T'\subset T'''$, $T''\setminus T'''\neq\varnothing$, $T'''\setminus T''\neq\varnothing$;

8) $Q_8=\{Z_7,T,Z_4,Z_2,Z_1,\breve{D}\}$, where $T\in\{Z_6,Z_5\}$;

9) $Q_9=\{Z_7,Z_5,Z_4,Z_3,Z_1,\breve{D}\}$, where $Z_7\subset Z_5\subset Z_3$, $Z_7\subset Z_5\subset Z_4$, $Z_3\setminus Z_4\neq\varnothing$, $Z_4\setminus Z_3\neq\varnothing$;

10) $Q_{10}=\{T,T',T'',T'\cup T'',T'''\}$, where $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $T'\cup T''\subset T'''$;

11) $Q_{11}=\{Z_7,Z_6,Z_5,Z_4,T,\breve{D}\}$, where $T\in\{Z_2,Z_1\}$;

12) $Q_{12}=\{T,T',T'',T'\cup T'',T''',T'\cup T''\cup T'''\}$, where, $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$; $T''\subset T'''$, $(T'\cup T'')\setminus T'''\neq\varnothing$, $T'''\setminus(T'\cup T'')\neq\varnothing$;

13) $Q_{13}=\{Z_7,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$, where $Z_5\subset Z_3$, $Z_5\subset Z_4$, $Z_3\setminus Z_4\neq\varnothing$, $Z_4\setminus Z_3\neq\varnothing$, $Z_4\subset Z_2$; $Z_1\setminus Z_2\neq\varnothing$, $Z_2\setminus Z_1\neq\varnothing$;

14) $Q_{14}=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_1,\breve{D}\}$, where $Z_6\subset Z_4, Z_6\subset Z_3$;

15) $Q_{15}=\{Z_7,Z_6,Z_5,Z_4,Z_2,Z_1,\breve{D}\}$;

16) $Q_{16}=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$.

Theorem 2.1. Let $D\in\Sigma_3(X,8)$, $Z_7\neq\varnothing$ and $\alpha\in B_X(D)$. Binary relation α is an idempotent relation of the semigroup $B_X(D)$ iff binary relation α satisfies only one conditions of the following conditions:

1) $\alpha=X\times T$, where $T\in D$;

2) $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')$, where $T,T'\in D$, $T\subset T'$, $Y_T^\alpha,Y_{T'}^\alpha\notin\{\varnothing\}$, and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_{T'}^\alpha\cap T'\neq\varnothing$;

3) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)$, where $T,T',T''\in D$, $T\subset T'\subset T''$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$, and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$;

4) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'''}^\alpha\times T'''\right)$, where $T,T',T'',T'''\in D$, $T\subset T'\subset T''\subset T'''$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha,Y_{T'''}^\alpha\notin\{\varnothing\}$, and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_{T'''}^\alpha\cap T'''\neq\varnothing$;

5) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Z_7\subset T\subset T'\subset T''\subset\breve{D}$, $Y_7^\alpha,Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha,Y_0^\alpha\notin\{\varnothing\}$, and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_7^\alpha\cup Y_T^\alpha\supseteq T$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_T^\alpha\cap T\neq\varnothing$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

6) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right)$, where $T,T',T''\in D$, $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$;

7) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'''}^\alpha\times T'''\right)\cup\left(Y_{T''\cup T'''}^\alpha\times\left(T''\cup T'''\right)\right)$, where, $T\subset T'\subset T''$, $T\subset T'\subset T'''$, $T''\setminus T'''\neq\varnothing$, $T'''\setminus T''\neq\varnothing$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha,Y_{T'''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T'''}^\alpha\supseteq T'''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_{T'''}^\alpha\cap T'''\neq\varnothing$;

8) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_T^\alpha\times T\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_2^\alpha\times Z_2\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T\in\{Z_6,Z_5\}$, $Y_7^\alpha,Y_T^\alpha,Y_4^\alpha,Y_2^\alpha,Y_1^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_7^\alpha\cup Y_T^\alpha\supseteq T$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq Z_2$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_4^\alpha\cup Y_1^\alpha\supseteq Z_1$, $Y_T^\alpha\cap T\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$, $Y_2^\alpha\cap Z_2\neq\varnothing$, $Y_1^\alpha\cap Z_1\neq\varnothing$;

9) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Z_7\subset Z_5\subset Z_3$, $Z_7\subset Z_5\subset Z_4$, $Z_3\setminus Z_4\neq\varnothing$, $Z_4\setminus Z_3\neq\varnothing$, $Y_7^\alpha,Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_3^\alpha\cap Z_3\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

10) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right)\cup\left(Y_{T'''}^\alpha\times T'''\right)$, where $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $T'\cup T''\subset T'''$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha,Y_{T'''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_{T'''}^\alpha\cap T'''\neq\varnothing$;

11) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_T^\alpha\times T\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T\in\{Z_2,Z_1\}$, $Y_7^\alpha,Y_6^\alpha,Y_5^\alpha,Y_T^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$ $Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_T^\alpha\supseteq T$, $Y_6^\alpha\cap Z_6\neq\varnothing$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_T^\alpha\cap T\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

12) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right)\cup\left(Y_{T'''}^\alpha\times T'''\right)\cup\left(Y_{T'\cup T''\cup T'''}^\alpha\times\left(T'\cup T''\cup T'''\right)\right)$, where $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $T''\subset T'''$, $\left(T'\cup T''\right)\setminus T'''\neq\varnothing$, $T'''\setminus\left(T'\cup T''\right)\neq\varnothing$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha,Y_{T'''}^\alpha,Y_{T'\cup T''\cup T'''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions:, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_T^\alpha\cup Y_{T''}^\alpha\cup Y_{T'''}^\alpha\supseteq T'''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_{T'''}^\alpha\cap T'''\neq\varnothing$;

13) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_2^\alpha\times Z_2\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Z_5\subset Z_3$, $Z_5\subset Z_4$, $Z_3\setminus Z_4\neq\varnothing$, $Z_4\setminus Z_3\neq\varnothing$, $Z_4\subset Z_2$, $Z_1\setminus Z_2\neq\varnothing$, $Z_2\setminus Z_1\neq\varnothing$, $Y_7^\alpha,Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_2^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$ $Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\supseteq Z_4$,

$Y_7^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_1^\alpha \supseteq Z_1$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_3^\alpha \cap Z_3 \neq \varnothing$, $Y_4^\alpha \cap Z_4 \neq \varnothing$, $Y_1^\alpha \cap Z_1 \neq \varnothing$;

14) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where, $Z_6 \subset Z_4$, $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_7^\alpha \cup Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq Z_3$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_3^\alpha \cap Z_3 \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

15) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_4^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \cup Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq Z_2$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_1^\alpha \supseteq Z_1$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_2^\alpha \cap Z_2 \neq \varnothing$, $Y_1^\alpha \cap Z_1 \neq \varnothing$;

16) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where, $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \supseteq Z_7$, $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_7^\alpha \cup Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq Z_3$, $Y_7^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq Z_2$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_3^\alpha \cap Z_3 \neq \varnothing$, $Y_2^\alpha \cap Z_2 \neq \varnothing$.

Proof. The statements 1), 2), 3), 4) and 5) immediately follows from the Corollary 13.1.1 in [2], 13.1.1 in [3], the statements 6) - 11) immediately follows from the Corollary 13.3.1 in [2], 13.3.1 in [3]; the statement 12) immediately follows from the Theorems 13.7.2 in [2]; the statement 13) immediately follows from the corollary 2.1 in [4], the statement 14) immediately follows from the lemma 2.1. in [5], the statements 15) immediately follows from the Theorems 13.11.1 in [2] and the statement 16) immediately follows from the theorem 2.1. in [6].

Lemma 2.6. If X be a finite set, then the following equalities are true:

a) $|I(Q_1)| = 8$;

b) $|I(Q_2)| = \left(2^{|T'\setminus T|} - 1\right) \cdot 2^{|X\setminus T'|}$;

c) $|I(Q_3)| = \left(2^{|T'\setminus T|} - 1\right) \cdot \left(3^{|T''\setminus T'|} - 2^{|T''\setminus T'|}\right) \cdot 3^{|X\setminus T''|}$;

d) $|I(Q_4)| = \left(2^{|T'\setminus T|} - 1\right) \cdot \left(3^{|T''\setminus T'|} - 2^{|T''\setminus T'|}\right) \cdot \left(4^{|T'''\setminus T''|} - 3^{|T'''\setminus T''|}\right) \cdot 4^{|X\setminus T'''|}$;

e) $|I(Q_5)| = \left(2^{|T\setminus Z_7|} - 1\right) \cdot \left(3^{|T'\setminus T|} - 2^{|T'\setminus T|}\right) \cdot \left(4^{|T''\setminus T'|} - 3^{|T''\setminus T'|}\right) \cdot \left(5^{|\breve{D}\setminus T''|} - 4^{|\breve{D}\setminus T''|}\right) \cdot 5^{|X\setminus \breve{D}|}$;

f) $|I(Q_6)| = \left(2^{|T'\setminus T''|} - 1\right) \cdot \left(2^{|T''\setminus T'|} - 1\right) \cdot 4^{|X\setminus (T'\cup T'')|}$;

g) $|I(Q_7)| = \left(2^{|T'\setminus T|} - 1\right) \cdot 2^{|(T''\cap T''')\setminus T'|} \cdot \left(3^{|T''\setminus T'''|} - 2^{|T''\setminus T'''|}\right) \cdot \left(3^{|T'''\setminus T''|} - 2^{|T'''\setminus T''|}\right) \cdot 5^{|X\setminus (T''\cup T''')|}$;

h) $|I(Q_8)| = \left(2^{|T\setminus Z_7|} - 1\right) \cdot \left(3^{|Z_4\setminus T|} - 2^{|Z_4\setminus T|}\right) \cdot 3^{|(Z_2\cap Z_1)\setminus Z_4|} \cdot \left(4^{|Z_1\setminus Z_2|} - 3^{|Z_1\setminus Z_2|}\right) \cdot \left(4^{|Z_2\setminus Z_1|} - 3^{|Z_2\setminus Z_1|}\right) \cdot 6^{|X\setminus \breve{D}|}$;

i) $|I(Q_9)| = \left(2^{|Z_5\setminus Z_7|} - 1\right) \cdot 2^{|(Z_3\cap Z_4)\setminus Z_5|} \cdot \left(3^{|Z_3\setminus Z_4|} - 2^{|Z_3\setminus Z_4|}\right) \cdot \left(3^{|Z_4\setminus Z_3|} - 2^{|Z_4\setminus Z_3|}\right) \cdot \left(6^{|\breve{D}\setminus (Z_3\cup Z_4)|} - 5^{|\breve{D}\setminus (Z_3\cup Z_4)|}\right) \cdot 6^{|X\setminus \breve{D}|}$;

j) $|I(Q_{10})| = \left(2^{|T'\setminus T''|} - 1\right) \cdot \left(2^{|T''\setminus T'|} - 1\right) \cdot \left(5^{|T'''\setminus (T'\cup T'')|} - 4^{|T'''\setminus (T'\cup T'')|}\right) \cdot 5^{|X\setminus T'''|}$;

k) $|I(Q_{11})| = \left(2^{|T\setminus T'|} - 1\right) \cdot \left(2^{|T'\setminus T|} - 1\right) \cdot \left(5^{|T''\setminus Z_4|} - 4^{|T''\setminus Z_4|}\right) \cdot \left(6^{|\breve{D}\setminus T''|} - 5^{|\breve{D}\setminus T|''}\right) \cdot 6^{|X\setminus \breve{D}|}$;

l) $|I(Q_{12})| = \left(2^{|T'\setminus T'''|} - 1\right) \cdot \left(2^{|T''\setminus T|} - 1\right) \cdot \left(3^{|T'''\setminus (T'\cup T'')|} - 2^{|T'''\setminus (T'\cup T'')|}\right) \cdot 6^{|X\setminus (T'\cup T''\cup T''')|}$;

m) $|I(Q_{13})| = \left(2^{|Z_5\setminus Z_7|} - 1\right) \cdot 2^{|(Z_3\cap Z_2)\setminus Z_5|} \cdot \left(3^{|Z_3\setminus Z_4|} - 2^{|Z_3\setminus Z_4|}\right) \cdot \left(3^{|Z_4\setminus Z_2|} - 2^{|Z_4\setminus Z_2|}\right) \cdot \left(4^{|Z_2\setminus Z_1|} - 3^{|Z_2\setminus Z_1|}\right) \cdot 7^{|X\setminus \breve{D}|}$;

n) $|I(Q_{14})| = \left(2^{|Z_5\setminus Z_4|} - 1\right) \cdot \left(2^{|Z_6\setminus Z_5|} - 1\right) \cdot \left(3^{|Z_4\setminus Z_3|} - 2^{|Z_4\setminus Z_3|}\right) \cdot \left(7^{|\breve{D}\setminus Z_1|} - 6^{|\breve{D}\setminus Z_1|}\right) \cdot 7^{|X\setminus \breve{D}|}$;

o) $\left|\mathrm{I}(Q_{15})\right| = \left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot 4^{|(Z_2\cap Z_1)\setminus Z_4|}\cdot \left(5^{|Z_2\setminus Z_1|}-4^{|Z_2\setminus Z_1|}\right)\cdot\left(5^{|Z_1\setminus Z_2|}-4^{|Z_1\setminus Z_2|}\right)\cdot 7^{|X\setminus\breve{D}|}$;

p) $\left|\mathrm{I}(Q_{16})\right| = \left(2^{|Z_6\setminus Z_3|}-1\right)\cdot 2^{|Z_5\setminus Z_4|}\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot\left(5^{|Z_2\setminus Z_1|}-4^{|Z_2\setminus Z_1|}\right)\cdot 8^{|X\setminus\breve{D}|}$.

Proof. The statements 1), 2), 3), 4), 5) immediately follows from the Corollary 13.1.5 in [2], 13.1.5 in [3], the statements 6)-12) immediately follows from the Corollary 13.3.3 in [2], 13.3.3 in [3], the statement 13 immediately follows corollary 1.5 in [4] and corollary 6.3.6 in [3], the statement 14 immediately follows from corollary 2.1 in [5] and corollary 6.3.6 in [3], the statement 15) immediately follows from the Corollary 13.11.1 in [2] and the statement 16 immediately follows from the Corollary 2.1 in [6].

Theorem is proved.

Lemma 2.7. Let $D\in\Sigma_3(X,8)$ and $Z_7\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_1)\right|$ may be calculated by the formula $\left|I^*(Q_1)\right| = 8$.

Proof. By definition of the given semilattice D we have

$$Q_1\vartheta_{XI} = \left\{\{Z_7\},\{Z_6\},\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}\right\}.$$

If the following equalities are hold

$$D_1' = \{Z_7\}, D_2' = \{Z_6\}, D_3' = \{Z_5\}, D_4' = \{Z_4\}, D_5' = \{Z_3\}, D_6' = \{Z_2\}, D_7' = \{Z_1\}, D_8' = \{\breve{D}\},$$

then

$$\left|I^*(Q_1)\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right| + \left|I(D_4')\right| + \left|I(D_5')\right| + \left|I(D_6')\right| + \left|I(D_7')\right| + \left|I(D_8')\right|$$

(see Theorem 1.4). Of this equality we have:

$$\left|I^*(Q_1)\right| = 1+1+1+1+1+1+1+1 = 8$$

(see statement a) of the Lemma 2.6).

Lemma 2.8. Let $D\in\Sigma_3(X,8)$ and $Z_7\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_2)\right|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_2)\right| = {}& \left(2^{|\breve{D}\setminus Z_7|}-1\right)\cdot 2^{|X\setminus\breve{D}|} + \left(2^{|\breve{D}\setminus Z_6|}-1\right)\cdot 2^{|X\setminus\breve{D}|} + \left(2^{|\breve{D}\setminus Z_5|}-1\right)\cdot 2^{|X\setminus\breve{D}|} + \left(2^{|\breve{D}\setminus Z_4|}-1\right)\cdot 2^{|X\setminus\breve{D}|}\\ &+ \left(2^{|\breve{D}\setminus Z_3|}-1\right)\cdot 2^{|X\setminus\breve{D}|} + \left(2^{|\breve{D}\setminus Z_2|}-1\right)\cdot 2^{|X\setminus\breve{D}|} + \left(2^{|\breve{D}\setminus Z_1|}-1\right)\cdot 2^{|X\setminus\breve{D}|} + \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 2^{|X\setminus Z_6|}\\ &+ \left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 2^{|X\setminus Z_5|} + \left(2^{|Z_4\setminus Z_7|}-1\right)\cdot 2^{|X\setminus Z_4|} + \left(2^{|Z_3\setminus Z_7|}-1\right)\cdot 2^{|X\setminus Z_3|} + \left(2^{|Z_2\setminus Z_7|}-1\right)\cdot 2^{|X\setminus Z_2|}\\ &+ \left(2^{|Z_1\setminus Z_7|}-1\right)\cdot 2^{|X\setminus Z_1|} + \left(2^{|Z_4\setminus Z_6|}-1\right)\cdot 2^{|X\setminus Z_4|} + \left(2^{|Z_2\setminus Z_6|}-1\right)\cdot 2^{|X\setminus Z_2|} + \left(2^{|Z_1\setminus Z_6|}-1\right)\cdot 2^{|X\setminus Z_1|}\\ &+ \left(2^{|Z_4\setminus Z_5|}-1\right)\cdot 2^{|X\setminus Z_4|} + \left(2^{|Z_3\setminus Z_5|}-1\right)\cdot 2^{|X\setminus Z_3|} + \left(2^{|Z_2\setminus Z_5|}-1\right)\cdot 2^{|X\setminus Z_2|} + \left(2^{|Z_1\setminus Z_5|}-1\right)\cdot 2^{|X\setminus Z_1|}\\ &+ \left(2^{|Z_2\setminus Z_4|}-1\right)\cdot 2^{|X\setminus Z_2|} + \left(2^{|Z_1\setminus Z_4|}-1\right)\cdot 2^{|X\setminus Z_1|} + \left(2^{|Z_1\setminus Z_3|}-1\right)\cdot 2^{|X\setminus Z_1|}\end{aligned}$$

Proof. By definitionof the given semilattice D we have

$$\begin{aligned}Q_2\vartheta_{XI} = {}&\left\{\{Z_7,\breve{D}\},\{Z_6,\breve{D}\},\{Z_5,\breve{D}\},\{Z_4,\breve{D}\},\{Z_3,\breve{D}\},\{Z_2,\breve{D}\},\{Z_1,\breve{D}\},\{Z_7,Z_6\},\{Z_7,Z_5\},\right.\\ &\{Z_7,Z_4\},\{Z_7,Z_3\},\{Z_7,Z_2\},\{Z_7,Z_1\},\{Z_6,Z_4\},\{Z_6,Z_2\},\{Z_6,Z_1\},\{Z_5,Z_4\},\{Z_5,Z_3\},\\ &\left.\{Z_5,Z_2\},\{Z_5,Z_1\},\{Z_4,Z_2\},\{Z_4,Z_1\},\{Z_3,Z_1\}\right\}\end{aligned}$$

if

$$\begin{aligned}&D_1' = \{Z_7,\breve{D}\},\ D_2' = \{Z_6,\breve{D}\},\ D_3' = \{Z_5,\breve{D}\},\ D_4' = \{Z_4,\breve{D}\},\ D_5' = \{Z_3,\breve{D}\},\ D_6' = \{Z_2,\breve{D}\},\\ &D_7' = \{Z_1,\breve{D}\},\ D_8' = \{Z_7,Z_6\},\ D_9' = \{Z_7,Z_5\},\ D_{10}' = \{Z_7,Z_4\}, D_{11}' = \{Z_7,Z_3\}, D_{12}' = \{Z_7,Z_2\},\\ &D_{13}' = \{Z_7,Z_1\}, D_{14}' = \{Z_6,Z_4\}, D_{15}' = \{Z_6,Z_2\}, D_{16}' = \{Z_6,Z_1\}, D_{17}' = \{Z_5,Z_4\}, D_{18}' = \{Z_5,Z_3\},\\ &D_{19}' = \{Z_5,Z_2\}, D_{20}' = \{Z_5,Z_1\}, D_{21}' = \{Z_4,Z_2\}, D_{22}' = \{Z_4,Z_1\}, D_{23}' = \{Z_3,Z_1\},\end{aligned}$$

Then

$$\begin{aligned}\left|I^*(Q_2)\right| = {} & \left|I(D'_1)\right| + \left|I(D'_2)\right| + \left|I(D'_3)\right| + \left|I(D'_4)\right| + \left|I(D'_5)\right| + \left|I(D'_6)\right| + \left|I(D'_7)\right| + \left|I(D'_8)\right| \\ & + \left|I(D'_9)\right| + \left|I(D'_{10})\right| + \left|I(D'_{11})\right| + \left|I(D'_{12})\right| + \left|I(D'_{13})\right| + \left|I(D'_{14})\right| + \left|I(D'_{15})\right| + \left|I(D'_{16})\right| \\ & + \left|I(D'_{17})\right| + \left|I(D'_{18})\right| + \left|I(D'_{19})\right| + \left|I(D'_{20})\right| + \left|I(D'_{21})\right| + \left|I(D'_{22})\right| + \left|I(D'_{23})\right|\end{aligned}$$

(see Theorem 1.4). Of this equality we have:

$$\begin{aligned}\left|I^*(Q_2)\right| = {} & \left(2^{|\tilde{D}\setminus Z_7|}-1\right)\cdot 2^{|X\setminus\tilde{D}|} + \left(2^{|\tilde{D}\setminus Z_6|}-1\right)\cdot 2^{|X\setminus\tilde{D}|} + \left(2^{|\tilde{D}\setminus Z_5|}-1\right)\cdot 2^{|X\setminus\tilde{D}|} + \left(2^{|\tilde{D}\setminus Z_4|}-1\right)\cdot 2^{|X\setminus\tilde{D}|} \\ & + \left(2^{|\tilde{D}\setminus Z_3|}-1\right)\cdot 2^{|X\setminus\tilde{D}|} + \left(2^{|\tilde{D}\setminus Z_2|}-1\right)\cdot 2^{|X\setminus\tilde{D}|} + \left(2^{|\tilde{D}\setminus Z_1|}-1\right)\cdot 2^{|X\setminus\tilde{D}|} + \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 2^{|X\setminus Z_6|} \\ & + \left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 2^{|X\setminus Z_5|} + \left(2^{|Z_4\setminus Z_7|}-1\right)\cdot 2^{|X\setminus Z_4|} + \left(2^{|Z_3\setminus Z_7|}-1\right)\cdot 2^{|X\setminus Z_3|} + \left(2^{|Z_2\setminus Z_7|}-1\right)\cdot 2^{|X\setminus Z_2|} \\ & + \left(2^{|Z_1\setminus Z_7|}-1\right)\cdot 2^{|X\setminus Z_1|} + \left(2^{|Z_4\setminus Z_6|}-1\right)\cdot 2^{|X\setminus Z_4|} + \left(2^{|Z_2\setminus Z_6|}-1\right)\cdot 2^{|X\setminus Z_2|} + \left(2^{|Z_1\setminus Z_6|}-1\right)\cdot 2^{|X\setminus Z_1|} \\ & + \left(2^{|Z_4\setminus Z_5|}-1\right)\cdot 2^{|X\setminus Z_4|} + \left(2^{|Z_3\setminus Z_5|}-1\right)\cdot 2^{|X\setminus Z_3|} + \left(2^{|Z_2\setminus Z_5|}-1\right)\cdot 2^{|X\setminus Z_2|} + \left(2^{|Z_1\setminus Z_5|}-1\right)\cdot 2^{|X\setminus Z_1|} \\ & + \left(2^{|Z_2\setminus Z_4|}-1\right)\cdot 2^{|X\setminus Z_2|} + \left(2^{|Z_1\setminus Z_4|}-1\right)\cdot 2^{|X\setminus Z_1|} + \left(2^{|Z_1\setminus Z_3|}-1\right)\cdot 2^{|X\setminus Z_1|}\end{aligned}$$

(see statement b) of the Lemma 2.6).

Lemma is proved.

Lemma 2.9. Let $D \in \Sigma_3(X,8)$ and $Z_7 \neq \varnothing$. If X is a finite set, then the number $\left|I^*(Q_3)\right|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_3)\right| = {} & \left(2^{|Z_1\setminus Z_7|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_1|}-2^{|\tilde{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\tilde{D}|} + \left(2^{|Z_2\setminus Z_7|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_2|}-2^{|\tilde{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\tilde{D}|} \\ & + \left(2^{|Z_3\setminus Z_7|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_3|}-2^{|\tilde{D}\setminus Z_3|}\right)\cdot 3^{|X\setminus\tilde{D}|} + \left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_4|}-2^{|\tilde{D}\setminus Z_4|}\right)\cdot 3^{|X\setminus\tilde{D}|} \\ & + \left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_5|}-2^{|\tilde{D}\setminus Z_5|}\right)\cdot 3^{|X\setminus\tilde{D}|} + \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_6|}-2^{|\tilde{D}\setminus Z_6|}\right)\cdot 3^{|X\setminus\tilde{D}|} \\ & + \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_4|} + \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_2|} \\ & + \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot 3^{|X\setminus Z_1|} + \left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_4|} \\ & + \left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_3|} + \left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_2|} \\ & + \left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot 3^{|X\setminus Z_1|} + \left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\cdot 3^{|X\setminus Z_2|} \\ & + \left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot 3^{|X\setminus Z_1|} + \left(2^{|Z_3\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 3^{|X\setminus Z_1|} \\ & + \left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\cdot 3^{|X\setminus Z_2|} + \left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot 3^{|X\setminus Z_1|} \\ & + \left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_4|}-2^{|\tilde{D}\setminus Z_4|}\right)\cdot 3^{|X\setminus\tilde{D}|} + \left(2^{|Z_2\setminus Z_6|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_2|}-2^{|\tilde{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\tilde{D}|} \\ & + \left(2^{|Z_1\setminus Z_6|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_1|}-2^{|\tilde{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\tilde{D}|} + \left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\cdot 3^{|X\setminus Z_2|} \\ & + \left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot 3^{|X\setminus Z_1|} + \left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_4|}-2^{|\tilde{D}\setminus Z_4|}\right)\cdot 3^{|X\setminus\tilde{D}|} \\ & + \left(2^{|Z_3\setminus Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot 3^{|X\setminus Z_1|} + \left(2^{|Z_3\setminus Z_5|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_3|}-2^{|\tilde{D}\setminus Z_3|}\right)\cdot 3^{|X\setminus\tilde{D}|} \\ & + \left(2^{|Z_2\setminus Z_5|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_2|}-2^{|\tilde{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\tilde{D}|} + \left(2^{|Z_1\setminus Z_5|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_1|}-2^{|\tilde{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\tilde{D}|} \\ & + \left(2^{|Z_2\setminus Z_4|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_2|}-2^{|\tilde{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\tilde{D}|} + \left(2^{|Z_1\setminus Z_4|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_1|}-2^{|\tilde{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\tilde{D}|} \\ & + \left(2^{|Z_1\setminus Z_3|}-1\right)\cdot\left(3^{|\tilde{D}\setminus Z_1|}-2^{|\tilde{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\tilde{D}|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$
\begin{aligned}
Q_3\vartheta_{XI} = \Big\{ & \{Z_7, Z_1, \breve{D}\}, \{Z_7, Z_2, \breve{D}\}, \{Z_7, Z_3, \breve{D}\}, \{Z_7, Z_4, D\}, \{Z_7, Z_5, \breve{D}\}, \{Z_7, Z_6, \breve{D}\}, \{Z_7, Z_6, Z_4\}, \\
& \{Z_7, Z_6, Z_2\}, \{Z_7, Z_6, Z_1\}, \{Z_7, Z_5, Z_4\}, \{Z_7, Z_5, Z_3\}, \{Z_7, Z_5, Z_2\}, \{Z_7, Z_5, Z_1\}, \{Z_7, Z_4, Z_2\}, \\
& \{Z_7, Z_4, Z_1\}, \{Z_7, Z_3, Z_1\}, \{Z_6, Z_4, Z_2\}, \{Z_6, Z_4, Z_1\}, \{Z_6, Z_4, \breve{D}\}, \{Z_6, Z_2, \breve{D}\}, \{Z_6, Z_1, \breve{D}\}, \\
& \{Z_5, Z_4, Z_2\}, \{Z_5, Z_4, Z_1\}, \{Z_5, Z_4, \breve{D}\}, \{Z_5, Z_3, Z_1\}, \{Z_5, Z_3, \breve{D}\}, \{Z_5, Z_2, \breve{D}\}, \{Z_5, Z_1, \breve{D}\}, \\
& \{Z_4, Z_2, \breve{D}\}, \{Z_4, Z_1, \breve{D}\}, \{Z_3, Z_1, \breve{D}\} \Big\}
\end{aligned}
$$

If

$$
\begin{aligned}
& D'_1 = \{Z_7, Z_1, \breve{D}\}, \quad D'_2 = \{Z_7, Z_2, \breve{D}\}, \quad D'_3 = \{Z_7, Z_3, \breve{D}\}, \quad D'_4 = \{Z_7, Z_4, D\}, \quad D'_5 = \{Z_7, Z_5, \breve{D}\}, \\
& D'_6 = \{Z_7, Z_6, \breve{D}\}, \quad D'_7 = \{Z_7, Z_6, Z_4\}, \quad D'_8 = \{Z_7, Z_6, Z_2\}, \quad D'_9 = \{Z_7, Z_6, Z_1\}, \quad D'_{10} = \{Z_7, Z_5, Z_4\}, \\
& D'_{11} = \{Z_7, Z_5, Z_3\}, \; D'_{12} = \{Z_7, Z_5, Z_2\}, \; D'_{13} = \{Z_7, Z_5, Z_1\}, \; D'_{14} = \{Z_7, Z_4, Z_2\}, \; D'_{15} = \{Z_7, Z_4, Z_1\}, \\
& D'_{16} = \{Z_7, Z_3, Z_1\}, \; D'_{17} = \{Z_6, Z_4, Z_2\}, \; D'_{18} = \{Z_6, Z_4, Z_1\}, \; D'_{19} = \{Z_6, Z_4, \breve{D}\}, \; D'_{20} = \{Z_6, Z_2, \breve{D}\}, \\
& D'_{21} = \{Z_6, Z_1, \breve{D}\}, \; D'_{22} = \{Z_5, Z_4, Z_2\}, \; D'_{23} = \{Z_5, Z_4, Z_1\}, \; D'_{24} = \{Z_5, Z_4, \breve{D}\}, \; D_{25} = \{Z_5, Z_3, Z_1\}, \\
& D'_{26} = \{Z_5, Z_3, \breve{D}\}, \; D'_{27} = \{Z_5, Z_2, \breve{D}\}, \; D'_{28} = \{Z_5, Z_1, \breve{D}\}, \; D'_{29} = \{Z_4, Z_2, \breve{D}\}, \; D'_{30} = \{Z_4, Z_1, \breve{D}\}, \\
& D'_{31} = \{Z_3, Z_1, \breve{D}\}
\end{aligned}
$$

Then

$$
\begin{aligned}
\left|I^*(Q_3)\right| = & \left|I(D'_1)\right| + \left|I(D'_2)\right| + \left|I(D'_3)\right| + \left|I(D'_4)\right| + \left|I(D'_5)\right| + \left|I(D'_6)\right| + \left|I(D'_7)\right| + \left|I(D'_8)\right| \\
& + \left|I(D'_9)\right| + \left|I(D'_{10})\right| + \left|I(D'_{11})\right| + \left|I(D'_{12})\right| + \left|I(D'_{13})\right| + \left|I(D'_{14})\right| + \left|I(D'_{15})\right| + \left|I(D'_{16})\right| \\
& + \left|I(D'_{17})\right| + \left|I(D'_{18})\right| + \left|I(D'_{19})\right| + \left|I(D'_{20})\right| + \left|I(D'_{21})\right| + \left|I(D'_{22})\right| + \left|I(D'_{23})\right| + \left|I(D'_{24})\right| \\
& + \left|I(D'_{25})\right| + \left|I(D'_{26})\right| + \left|I(D'_{27})\right| + \left|I(D'_{28})\right| + \left|I(D'_{29})\right| + \left|I(D'_{30})\right| + \left|I(D'_{31})\right|
\end{aligned}
$$

(see Theorem 1.4). Of this equality we have:

$$
\begin{aligned}
\left|I^*(Q_3)\right| = & \left(2^{|Z_1 \setminus Z_7|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}\right) \cdot 3^{|X \setminus \breve{D}|} + \left(2^{|Z_2 \setminus Z_7|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_2|} - 2^{|\breve{D} \setminus Z_2|}\right) \cdot 3^{|X \setminus \breve{D}|} \\
& + \left(2^{|Z_3 \setminus Z_7|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_3|} - 2^{|\breve{D} \setminus Z_3|}\right) \cdot 3^{|X \setminus \breve{D}|} + \left(2^{|Z_4 \setminus Z_7|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_4|} - 2^{|\breve{D} \setminus Z_4|}\right) \cdot 3^{|X \setminus \breve{D}|} \\
& + \left(2^{|Z_5 \setminus Z_7|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_5|} - 2^{|\breve{D} \setminus Z_5|}\right) \cdot 3^{|X \setminus \breve{D}|} + \left(2^{|Z_6 \setminus Z_7|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_6|} - 2^{|\breve{D} \setminus Z_6|}\right) \cdot 3^{|X \setminus \breve{D}|} \\
& + \left(2^{|Z_6 \setminus Z_7|} - 1\right) \cdot \left(3^{|Z_4 \setminus Z_6|} - 2^{|Z_4 \setminus Z_6|}\right) \cdot 3^{|X \setminus Z_4|} + \left(2^{|Z_6 \setminus Z_7|} - 1\right) \cdot \left(3^{|Z_2 \setminus Z_6|} - 2^{|Z_2 \setminus Z_6|}\right) \cdot 3^{|X \setminus Z_2|} \\
& + \left(2^{|Z_6 \setminus Z_7|} - 1\right) \cdot \left(3^{|Z_1 \setminus Z_6|} - 2^{|Z_1 \setminus Z_6|}\right) \cdot 3^{|X \setminus Z_1|} + \left(2^{|Z_5 \setminus Z_7|} - 1\right) \cdot \left(3^{|Z_4 \setminus Z_5|} - 2^{|Z_4 \setminus Z_5|}\right) \cdot 3^{|X \setminus Z_4|} \\
& + \left(2^{|Z_5 \setminus Z_7|} - 1\right) \cdot \left(3^{|Z_3 \setminus Z_5|} - 2^{|Z_3 \setminus Z_5|}\right) \cdot 3^{|X \setminus Z_3|} + \left(2^{|Z_5 \setminus Z_7|} - 1\right) \cdot \left(3^{|Z_2 \setminus Z_5|} - 2^{|Z_2 \setminus Z_5|}\right) \cdot 3^{|X \setminus Z_2|} \\
& + \left(2^{|Z_5 \setminus Z_7|} - 1\right) \cdot \left(3^{|Z_1 \setminus Z_5|} - 2^{|Z_1 \setminus Z_5|}\right) \cdot 3^{|X \setminus Z_1|} + \left(2^{|Z_4 \setminus Z_7|} - 1\right) \cdot \left(3^{|Z_2 \setminus Z_4|} - 2^{|Z_2 \setminus Z_4|}\right) \cdot 3^{|X \setminus Z_2|} \\
& + \left(2^{|Z_4 \setminus Z_7|} - 1\right) \cdot \left(3^{|Z_1 \setminus Z_4|} - 2^{|Z_1 \setminus Z_4|}\right) \cdot 3^{|X \setminus Z_1|} + \left(2^{|Z_3 \setminus Z_7|} - 1\right) \cdot \left(3^{|Z_1 \setminus Z_3|} - 2^{|Z_1 \setminus Z_3|}\right) \cdot 3^{|X \setminus Z_1|} \\
& + \left(2^{|Z_4 \setminus Z_6|} - 1\right) \cdot \left(3^{|Z_2 \setminus Z_4|} - 2^{|Z_2 \setminus Z_4|}\right) \cdot 3^{|X \setminus Z_2|} + \left(2^{|Z_4 \setminus Z_6|} - 1\right) \cdot \left(3^{|Z_1 \setminus Z_4|} - 2^{|Z_1 \setminus Z_4|}\right) \cdot 3^{|X \setminus Z_1|} \\
& + \left(2^{|Z_4 \setminus Z_6|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_4|} - 2^{|\breve{D} \setminus Z_4|}\right) \cdot 3^{|X \setminus \breve{D}|} + \left(2^{|Z_2 \setminus Z_6|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_2|} - 2^{|\breve{D} \setminus Z_2|}\right) \cdot 3^{|X \setminus \breve{D}|} \\
& + \left(2^{|Z_1 \setminus Z_6|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}\right) \cdot 3^{|X \setminus \breve{D}|} + \left(2^{|Z_4 \setminus Z_5|} - 1\right) \cdot \left(3^{|Z_2 \setminus Z_4|} - 2^{|Z_2 \setminus Z_4|}\right) \cdot 3^{|X \setminus Z_2|} \\
& + \left(2^{|Z_4 \setminus Z_5|} - 1\right) \cdot \left(3^{|Z_1 \setminus Z_4|} - 2^{|Z_1 \setminus Z_4|}\right) \cdot 3^{|X \setminus Z_1|} + \left(2^{|Z_4 \setminus Z_5|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_4|} - 2^{|\breve{D} \setminus Z_4|}\right) \cdot 3^{|X \setminus \breve{D}|} \\
& + \left(2^{|Z_3 \setminus Z_5|} - 1\right) \cdot \left(3^{|Z_1 \setminus Z_3|} - 2^{|Z_1 \setminus Z_3|}\right) \cdot 3^{|X \setminus Z_1|} + \left(2^{|Z_3 \setminus Z_5|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_3|} - 2^{|\breve{D} \setminus Z_3|}\right) \cdot 3^{|X \setminus \breve{D}|} \\
& + \left(2^{|Z_2 \setminus Z_5|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_2|} - 2^{|\breve{D} \setminus Z_2|}\right) \cdot 3^{|X \setminus \breve{D}|} + \left(2^{|Z_1 \setminus Z_5|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}\right) \cdot 3^{|X \setminus \breve{D}|} \\
& + \left(2^{|Z_2 \setminus Z_4|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_2|} - 2^{|\breve{D} \setminus Z_2|}\right) \cdot 3^{|X \setminus \breve{D}|} + \left(2^{|Z_1 \setminus Z_4|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}\right) \cdot 3^{|X \setminus \breve{D}|} \\
& + \left(2^{|Z_1 \setminus Z_3|} - 1\right) \cdot \left(3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}\right) \cdot 3^{|X \setminus \breve{D}|}
\end{aligned}
$$

(see statement c) of the Lemma 2.6).

Lemma is proved.

Lemma 2.10. Let $D \in \Sigma_3(X,8)$ and $Z_7 \neq \varnothing$. If X is a finite set, then the number $\left|I^*(Q_4)\right|$ may be calculated by the formula

$$\begin{aligned}
\left|I^*(Q_4)\right| = &\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)\\
&\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\\
&\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\\
&\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_3\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\\
&\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_2|}+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\times\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\times 4^{|X\setminus Z_1|}\\
&+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_3|}-3^{|Z_1\setminus Z_3|}\right)\cdot 4^{|X\setminus Z_1|}+\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\\
&\times\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\times 4^{|X\setminus Z_2|}+\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_1|}\\
&+\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\times\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\times 4^{|X\setminus\breve{D}|}+\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\\
&\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\times\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\times 4^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}\\
&+\left(2^{|Z_3\setminus Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}
\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$\begin{aligned}
Q_4\vartheta_{XI} = \Big\{&\{Z_7,Z_6,Z_4,\breve{D}\},\ \{Z_7,Z_6,Z_2,D\},\ \{Z_7,Z_6,Z_1,\breve{D}\},\ \{Z_7,Z_5,Z_4,\breve{D}\},\ \{Z_7,Z_5,Z_3,\breve{D}\},\\
&\{Z_7,Z_5,Z_2,\breve{D}\},\ \{Z_7,Z_5,Z_1,\breve{D}\},\ \{Z_7,Z_4,Z_2,D\},\ \{Z_7,Z_4,Z_1,\breve{D}\},\ \{Z_7,Z_3,Z_1,\breve{D}\},\\
&\{Z_7,Z_5,Z_4,Z_2\},\ \{Z_7,Z_5,Z_4,Z_1\},\ \{Z_7,Z_5,Z_3,Z_1\},\ \{Z_7,Z_6,Z_4,Z_2\},\ \{Z_7,Z_6,Z_4,Z_1\},\\
&\{Z_6,Z_4,Z_2,\breve{D}\},\ \{Z_6,Z_4,Z_1,\breve{D}\},\ \{Z_5,Z_4,Z_2,\breve{D}\},\ \{Z_5,Z_4,Z_1,\breve{D}\},\ \{Z_5,Z_3,Z_1,\breve{D}\}\Big\}
\end{aligned}$$

If

$$\begin{aligned}
&D_1'=\{Z_7,Z_6,Z_4,\breve{D}\},D_2'=\{Z_7,Z_6,Z_2,D\},D_3'=\{Z_7,Z_6,Z_1,\breve{D}\},D_4'=\{Z_7,Z_5,Z_4,\breve{D}\},\\
&D_5'=\{Z_7,Z_5,Z_3,\breve{D}\},D_6'=\{Z_7,Z_5,Z_2,\breve{D}\},D_7'=\{Z_7,Z_5,Z_1,\breve{D}\},D_8'=\{Z_7,Z_4,Z_2,D\},\\
&D_9'=\{Z_7,Z_4,Z_1,\breve{D}\},D_{10}'=\{Z_7,Z_3,Z_1,\breve{D}\},D_{11}'=\{Z_7,Z_6,Z_4,Z_2\},D_{12}'=\{Z_7,Z_6,Z_4,Z_1\}\\
&D_{13}'=\{Z_7,Z_5,Z_4,Z_2\},D_{14}'=\{Z_7,Z_5,Z_4,Z_1\},D_{15}'=\{Z_7,Z_5,Z_3,Z_1\},D_{16}'=\{Z_5,Z_3,Z_1,\breve{D}\}\\
&D_{17}'=\{Z_6,Z_4,Z_2,\breve{D}\},D_{19}'=\{Z_6,Z_4,Z_1,\breve{D}\},D_{18}'=\{Z_5,Z_4,Z_2,\breve{D}\},D_{20}'=\{Z_5,Z_4,Z_1,\breve{D}\},
\end{aligned}$$

Then

$$\left|I^*(Q_4)\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right| + \left|I(D_4')\right| + \left|I(D_5')\right| + \left|I(D_6')\right| + \left|I(D_7')\right| + \left|I(D_8')\right|$$
$$+ \left|I(D_9')\right| + \left|I(D_{10}')\right| + \left|I(D_{11}')\right| + \left|I(D_{12}')\right| + \left|I(D_{13}')\right| + \left|I(D_{14}')\right| + \left|I(D_{15}')\right|$$
$$+ \left|I(D_{16}')\right| + \left|I(D_{17}')\right| + \left|I(D_{18}')\right| + \left|I(D_{19}')\right| + \left|I(D_{20}')\right|$$

(see Theorem 1.4). Of this equality we have:

$$\left|I^*(Q_4)\right| = \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|}\right)$$
$$\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)$$
$$\cdot\left(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)$$
$$\cdot\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_3\setminus Z_7|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)$$
$$\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_2|} + \left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\times\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\times 4^{|X\setminus Z_1|}$$
$$+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_3|}-3^{|Z_1\setminus Z_3|}\right)\cdot 4^{|X\setminus Z_1|} + \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)$$
$$\times\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\times 4^{|X\setminus Z_2|} + \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot 4^{|X\setminus Z_1|}$$
$$+\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\times\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\times 4^{|X\setminus\breve{D}|} + \left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)$$
$$\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|} + \left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|}\right)\times\left(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|}\right)\times 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_3\setminus Z_5|}-1\right)\cdot\left(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|}\right)\cdot\left(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|}\right)\cdot 4^{|X\setminus\breve{D}|}$$

(see statement d) of the Lemma 2.6).

Lemma is proved.

Lemma 2.11. Let $D \in \Sigma_3(X, 8)$ and $Z_7 \neq \varnothing$. If X is a finite set, then the number $\left|I^*(Q_5)\right|$ may be calculated by the formula

$$\left|I^*(Q_5)\right| = \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_3|}-3^{|Z_1\setminus Z_3|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}$$

Proof. By definition of the given semilattice D we have

$$Q_5\vartheta_{XI}=\left\{\left\{Z_7,Z_6,Z_4,Z_2,\breve{D}\right\},\left\{Z_7,Z_6,Z_4,Z_1,\breve{D}\right\},\left\{Z_7,Z_5,Z_4,Z_2,\breve{D}\right\},\right.$$
$$\left.\left\{Z_7,Z_5,Z_4,Z_1,\breve{D}\right\},\left\{Z_7,Z_5,Z_3,Z_1,\breve{D}\right\}\right\}.$$

If

$$D_1'=\left\{Z_7,Z_6,Z_4,Z_2,\breve{D}\right\},\ D_2'=\left\{Z_7,Z_6,Z_4,Z_1,\breve{D}\right\},\ D_3'=\left\{Z_7,Z_5,Z_4,Z_2,\breve{D}\right\},$$
$$D_4'=\left\{Z_7,Z_5,Z_4,Z_1,\breve{D}\right\},\ D_5'=\left\{Z_7,Z_5,Z_3,Z_1,\breve{D}\right\}.$$

Then

$$\left|I^*(Q_5)\right|=\left|I(D_1')\right|+\left|I(D_2')\right|+\left|I(D_3')\right|+\left|I(D_4')\right|+\left|I(D_5')\right|$$

(see Theorem 1.4). Of this equality we have:

$$\begin{aligned}\left|I^*(Q_5)\right|=&\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}\\&+\left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\&+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}\\&+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\&+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|}\right)\cdot\left(4^{|Z_1\setminus Z_3|}-3^{|Z_1\setminus Z_3|}\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\end{aligned}$$

(see statement e) of the Lemma 2.6).

Lemma is proved.

Lemma 2.12. Let $D\in\Sigma_3(X,8)$ and $Z_7\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_6)\right|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_6)\right|=&\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot 4^{|X\setminus Z_4|}+\left(2^{|Z_3\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|}\\&+\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|}+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\&+\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\&+\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|}+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\&+\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$Q_6\vartheta_{XI}=\left\{\left\{Z_7,Z_2,Z_1,\breve{D}\right\},\ \left\{Z_7,Z_6,Z_5,Z_4\right\},\left\{Z_7,Z_6,Z_3,Z_1\right\},\ \left\{Z_7,Z_4,Z_3,Z_1\right\},\ \left\{Z_7,Z_3,Z_2,\breve{D}\right\},\right.$$
$$\left.\left\{Z_6,Z_2,Z_1,\breve{D}\right\},\ \left\{Z_5,Z_2,Z_1,\breve{D}\right\},\ \left\{Z_4,Z_2,Z_1,\breve{D}\right\},\ \left\{Z_5,Z_4,Z_3,Z_1\right\},\ \left\{Z_5,Z_3,Z_2,\breve{D}\right\}\right\}$$

$$D_1'=\left\{Z_7,Z_2,Z_1,\breve{D}\right\},\ D_2'=\left\{Z_7,Z_6,Z_5,Z_4\right\},\ D_3'=\left\{Z_7,Z_6,Z_3,Z_1\right\},\ D_4'=\left\{Z_7,Z_4,Z_3,Z_1\right\},$$
$$D_5'=\left\{Z_7,Z_3,Z_2,\breve{D}\right\},\ D_6'=\left\{Z_6,Z_2,Z_1,\breve{D}\right\},\ D_7'=\left\{Z_5,Z_2,Z_1,\breve{D}\right\},\ D_8'=\left\{Z_4,Z_2,Z_1,\breve{D}\right\},$$
$$D_9'=\left\{Z_5,Z_4,Z_3,Z_1\right\},\ D_{10}'=\left\{Z_5,Z_3,Z_2,\breve{D}\right\},$$

$$\left|I^*(Q_6)\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right| + \left|I(D_4')\right| + \left|I(D_5')\right| + \left|I(D_6')\right| + \left|I(D_7')\right| + \left|I(D_8')\right| + \left|I(D_9')\right| + \left|I(D_{10}')\right|$$

(see Theorem 1.4). Of this equality we have:

$$\begin{aligned}\left|I^*(Q_6)\right| &= \left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot 4^{|X\setminus Z_4|} + \left(2^{|Z_3\setminus Z_6|}-1\right)\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|}\\ &+\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|} + \left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus \breve{D}|} + \left(2^{|Z_1\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|} + \left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot 4^{|X\setminus \breve{D}|} + \left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus \breve{D}|}\end{aligned}$$

(see statement f) of the Lemma 2.6).

Lemma is proved.

Lemma 2.13. Let $D \in \Sigma_3(X,8)$ and $Z_7 \neq \varnothing$. If X is a finite set, then the number $\left|I^*(Q_7)\right|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_7)\right| &= \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_6|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 2^{|(Z_3\cap Z_4)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot 5^{|X\setminus Z_1|}\\ &+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 5^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_5|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$\begin{aligned}Q_7\vartheta_{XI} = \Big\{&\{Z_7,Z_4,Z_2,Z_1,\breve{D}\},\{Z_7,Z_6,Z_2,Z_1,\breve{D}\},\ \{Z_7,Z_5,Z_2,Z_1,\breve{D}\},\{Z_7,Z_5,Z_3,Z_2,\breve{D}\},\\ &\{Z_7,Z_5,Z_4,Z_3,Z_1\},\ \{Z_6,Z_4,Z_2,Z_1,\breve{D}\},\{Z_5,Z_4,Z_2,Z_1,\breve{D}\}\Big\}\end{aligned}$$

If

$$\begin{aligned}&D_1' = \{Z_7,Z_4,Z_2,Z_1,\breve{D}\},\ D_2' = \{Z_7,Z_6,Z_2,Z_1,\breve{D}\},\ D_3' = \{Z_7,Z_5,Z_2,Z_1,\breve{D}\},\\ &D_5' = \{Z_7,Z_5,Z_3,Z_2,\breve{D}\},\ D_6' = \{Z_7,Z_5,Z_4,Z_3,Z_1\},\ D_6' = \{Z_6,Z_4,Z_2,Z_1,\breve{D}\},\\ &D_7' = \{Z_5,Z_4,Z_2,Z_1,\breve{D}\},\end{aligned}$$

$$\left|I^*(Q_7)\right| = \left|I(D_1')\right| + \left|I(D_2')\right| + \left|I(D_3')\right| + \left|I(D_4')\right| + \left|I(D_5')\right| + \left|I(D_6')\right| + \left|I(D_7')\right|$$

(see Theorem 1.4). Of this equality we have:

$$\begin{aligned}\left|I^*(Q_7)\right| &= \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_6|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 2^{|(Z_3\cap Z_4)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot 5^{|X\setminus Z_1|}\\ &+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|}\right)\cdot 5^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_5|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}\end{aligned}$$

(see statement g) of the Lemma 2.6).

Lemma is proved.

Lemma 2.14. Let $D\in\Sigma_3(X,8)$ and $Z_7\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_8)\right|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_8)\right| &= \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot 3^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot 3^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$Q_8\vartheta_{XI}=\left\{\left\{Z_7,Z_6,Z_4,Z_2,Z_1,\breve{D}\right\},\left\{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\right\}\right\}$$

If

$$D_1'=\left\{Z_7,Z_6,Z_4,Z_2,Z_1,\breve{D}\right\},\ D_2'=\left\{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\right\}$$

$$\left|I^*(Q_8)\right|=\left|I(D_1')\right|+\left|I(D_2')\right|$$

(see Theorem 1.4). Of this equality we have:

$$\begin{aligned}\left|I^*(Q_8)\right| &= \left(2^{|Z_6\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|}\right)\cdot 3^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\\ &+\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|}\right)\cdot 3^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|}\end{aligned}$$

(see statement h) of the Lemma 2.6).

Lemma is proved.

Lemma 2.15. Let $D\in\Sigma_3(X,8)$ and $Z_7\neq\varnothing$. If X is a finite set, then the number $\left|I^*(Q_9)\right|$ may be calculated by the formula

$$\left|I^*(Q_9)\right|=\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 2^{|(Z_3\cap Z_4)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot\left(6^{|\breve{D}\setminus Z_1|}-5^{|\breve{D}\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|};$$

Proof. By definition of the given semilattice D we have $Q_9\vartheta_{XI}=\left\{\left\{Z_7,Z_6,Z_5,Z_4,Z_2,Z_1,\breve{D}\right\}\right\}$.

If the following equality is hold $D_1'=\left\{Z_7,Z_6,Z_5,Z_4,Z_2,Z_1,\breve{D}\right\}$ then $\left|I^*(Q_9)\right|=\left|I(D_1')\right|$

(see Theorem 1.4). Of this equality we have:

$$\left|I^*(Q_9)\right|=\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot 2^{|(Z_3\cap Z_4)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|}\right)\cdot\left(6^{|\breve{D}\setminus Z_1|}-5^{|\breve{D}\setminus Z_1|}\right)\cdot 6^{|X\setminus \breve{D}|};$$

(see statement i) of the Lemma 2.6).

Lemma is proved.

Lemma 2.16. Let $D \in \Sigma_3(X,8)$ and $Z_7 \neq \varnothing$. If X is a finite set, then the number $\left|I^*(Q_{10})\right|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_{10})\right| &= \left(2^{|Z_6 \setminus Z_5|}-1\right)\cdot\left(2^{|Z_5 \setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_4|}-4^{|\breve{D}\setminus Z_4|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_6 \setminus Z_3|}-1\right)\cdot\left(2^{|Z_3 \setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_4 \setminus Z_3|}-1\right)\cdot\left(2^{|Z_3 \setminus Z_4|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_6 \setminus Z_5|}-1\right)\cdot\left(2^{|Z_5 \setminus Z_6|}-1\right)\cdot\left(5^{|Z_2\setminus Z_4|}-4^{|Z_2\setminus Z_4|}\right)\cdot 5^{|X\setminus Z_2|}\\ &+\left(2^{|Z_6 \setminus Z_5|}-1\right)\cdot\left(2^{|Z_5 \setminus Z_6|}-1\right)\cdot\left(5^{|Z_1\setminus Z_4|}-4^{|Z_1\setminus Z_4|}\right)\cdot 5^{|X\setminus Z_1|}\\ &+\left(2^{|Z_4 \setminus Z_3|}-1\right)\cdot\left(2^{|Z_3 \setminus Z_4|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$\begin{aligned}Q_{10}\vartheta_{XI} = \Big\{&\left\{Z_7, Z_6, Z_5, Z_4, \breve{D}\right\}, \left\{Z_7, Z_6, Z_3, Z_1, \breve{D}\right\}, \left\{Z_7, Z_4, Z_3, Z_1, \breve{D}\right\},\\ &\left\{Z_7, Z_6, Z_5, Z_4, Z_2\right\}, \left\{Z_7, Z_6, Z_5, Z_4, Z_1\right\}, \left\{Z_5, Z_4, Z_3, Z_1, \breve{D}\right\}\Big\}\end{aligned}$$

If

$$\begin{aligned}&D_1' = \left\{Z_7, Z_6, Z_5, Z_4, \breve{D}\right\},\ D_2' = \left\{Z_7, Z_6, Z_3, Z_1, \breve{D}\right\}, D_3' = \left\{Z_7, Z_4, Z_3, Z_1, \breve{D}\right\},\\ &D_4' = \left\{Z_7, Z_6, Z_5, Z_4, Z_2\right\},\ D_5' = \left\{Z_7, Z_6, Z_5, Z_4, Z_1\right\},\ D_6' = \left\{Z_5, Z_4, Z_3, Z_1, \breve{D}\right\},\end{aligned}$$

$$\left|I^*(Q_{10})\right| = \left|I(D_1')\right|+\left|I(D_2')\right|+\left|I(D_3')\right|+\left|I(D_4')\right|+\left|I(D_5')\right|+\left|I(D_6')\right|$$

(see Theorem 1.4). Of this equality we have:

$$\begin{aligned}\left|I^*(Q_{10})\right| &= \left(2^{|Z_6 \setminus Z_5|}-1\right)\cdot\left(2^{|Z_5 \setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_4|}-4^{|\breve{D}\setminus Z_4|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_6 \setminus Z_3|}-1\right)\cdot\left(2^{|Z_3 \setminus Z_6|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_4 \setminus Z_3|}-1\right)\cdot\left(2^{|Z_3 \setminus Z_4|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_6 \setminus Z_5|}-1\right)\cdot\left(2^{|Z_5 \setminus Z_6|}-1\right)\cdot\left(5^{|Z_2\setminus Z_4|}-4^{|Z_2\setminus Z_4|}\right)\cdot 5^{|X\setminus Z_2|}\\ &+\left(2^{|Z_6 \setminus Z_5|}-1\right)\cdot\left(2^{|Z_5 \setminus Z_6|}-1\right)\cdot\left(5^{|Z_1\setminus Z_4|}-4^{|Z_1\setminus Z_4|}\right)\cdot 5^{|X\setminus Z_1|}\\ &+\left(2^{|Z_4 \setminus Z_3|}-1\right)\cdot\left(2^{|Z_3 \setminus Z_4|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\end{aligned}$$

(see statement j) of the Lemma 2.6).

Lemma is proved.

Lemma 2.17. Let $D \in \Sigma_3(X,8)$ and $Z_7 \neq \varnothing$. If X is a finite set, then the number $\left|I^*(Q_{11})\right|$ may be calculated by the formula

$$\begin{aligned}\left|I^*(Q_{11})\right| &= \left(2^{|Z_6 \setminus Z_5|}-1\right)\cdot\left(2^{|Z_5 \setminus Z_6|}-1\right)\cdot\left(5^{|Z_2\setminus Z_4|}-4^{|Z_2\setminus Z_4|}\right)\cdot\left(6^{|\breve{D}\setminus Z_2|}-5^{|\breve{D}\setminus Z_2|}\right)\cdot 6^{|X\setminus\breve{D}|}\\ &+\left(2^{|Z_6 \setminus Z_5|}-1\right)\cdot\left(2^{|Z_5 \setminus Z_6|}-1\right)\cdot\left(5^{|Z_1\setminus Z_4|}-4^{|Z_1\setminus Z_4|}\right)\cdot\left(6^{|\breve{D}\setminus Z_1|}-5^{|\breve{D}\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}\end{aligned}$$

Proof. By definition of the given semilattice D we have

$$Q_{11}\vartheta_{XI}=\left\{\left\{Z_7,Z_6,Z_5,Z_4,Z_2,\breve{D}\right\},\left\{Z_7,Z_6,Z_5,Z_4,Z_1,\breve{D}\right\}\right\}$$

If

$$D_1'=\left\{Z_7,Z_6,Z_5,Z_4,Z_2,\breve{D}\right\},D_2'=\left\{Z_7,Z_6,Z_5,Z_4,Z_1,\breve{D}\right\},$$

$$\left|I^*\left(Q_{11}\right)\right|=\left|I\left(D_1'\right)\right|+\left|I\left(D_2'\right)\right|$$

(see Theorem 1.4). Of this equality we have:

$$\left|I^*\left(Q_{11}\right)\right|=\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|Z_2\setminus Z_4|}-4^{|Z_2\setminus Z_4|}\right)\cdot\left(6^{|\breve{D}\setminus Z_2|}-5^{|\breve{D}\setminus Z_2|}\right)\cdot6^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_6\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(5^{|Z_1\setminus Z_4|}-4^{|Z_1\setminus Z_4|}\right)\cdot\left(6^{|\breve{D}\setminus Z_1|}-5^{|\breve{D}\setminus Z_1|}\right)\cdot6^{|X\setminus\breve{D}|}$$

(see statement k) of the Lemma 2.6).

Lemma is proved.

Lemma 2.18. Let $D\in\Sigma_3(X,8)$ and $Z_7\neq\varnothing$. If X is a finite set, then the number $\left|I^*\left(Q_{12}\right)\right|$ may be calculated by the formula

$$\left|I^*\left(Q_{12}\right)\right|=\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot6^{|X\setminus Z_1|}$$
$$+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot6^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot6^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot6^{|X\setminus\breve{D}|}$$

Proof. By definition of the given semilattice D we have

$$Q_{12}\vartheta_{XI}=\left\{\left\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_1\right\},\left\{Z_7,Z_6,Z_3,Z_2,Z_1,\breve{D}\right\},\left\{Z_7,Z_4,Z_3,Z_2,Z_1,\breve{D}\right\},\left\{Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\right\}\right\}$$

$$D_1'=\left\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_1\right\},D_2'=\left\{Z_7,Z_6,Z_3,Z_2,Z_1,\breve{D}\right\},$$
$$D_3'=\left\{Z_7,Z_4,Z_3,Z_2,Z_1,\breve{D}\right\},D_4'=\left\{Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\right\}.$$

$$\left|I^*\left(Q_{12}\right)\right|=\left|I\left(D_1'\right)\right|+\left|I\left(D_2'\right)\right|+\left|I\left(D_3'\right)\right|+\left|I\left(D_4'\right)\right|$$

(see Theorem 1.4). Of this equality we have:

$$\left|I^*\left(Q_{12}\right)\right|=\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(2^{|Z_5\setminus Z_6|}-1\right)\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot6^{|X\setminus Z_1|}$$
$$+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_6\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot6^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot6^{|X\setminus\breve{D}|}$$
$$+\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot6^{|X\setminus\breve{D}|}$$

(see statement l) of the Lemma 2.6).

Lemma is proved.

Lemma 2.19. Let $D\in\Sigma_3(X,8)$ and $Z_7\neq\varnothing$. If X is a finite set, then the number $\left|I^*\left(Q_{13}\right)\right|$ may be calculated by the formula

$$\left|I^*\left(Q_{13}\right)\right|=\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot2^{|(Z_3\cap Z_2)\setminus Z_5|}\cdot\left(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|}\right)\cdot\left(3^{|Z_4\setminus Z_2|}-2^{|Z_4\setminus Z_2|}\right)\cdot\left(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|}\right)\cdot7^{|X\setminus\breve{D}|};$$

Proof. By definition of the given semilattice D we have $Q_{13}\vartheta_{XI} = \left\{\left\{Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}\right\}$. If the following equality is hold $D_1' = \left\{Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}$, then $\left|I^*\left(Q_{13}\right)\right| = \left|I\left(D_1'\right)\right|$ (see Theorem 1.4). Of this equality we have:

$$\left|I^*\left(Q_{13}\right)\right| = \left(2^{|Z_5 \setminus Z_7|} - 1\right) \cdot 2^{|(Z_3 \cap Z_2) \setminus Z_5|} \cdot \left(3^{|Z_3 \setminus Z_4|} - 2^{|Z_3 \setminus Z_4|}\right) \cdot \left(3^{|Z_4 \setminus Z_2|} - 2^{|Z_4 \setminus Z_2|}\right) \cdot \left(4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}\right) \cdot 7^{|X \setminus \breve{D}|};$$

(see statement m) of the Lemma 2.6).

Lemma is proved.

Lemma 2.20. Let $D \in \Sigma_3(X, 8)$ and $Z_7 \neq \varnothing$. If X is a finite set, then the number $\left|I^*\left(Q_{14}\right)\right|$ may be calculated by the formula

$$\left|I^*\left(Q_{14}\right)\right| = \left(2^{|Z_5 \setminus Z_4|} - 1\right) \cdot \left(2^{|Z_6 \setminus Z_5|} - 1\right) \cdot \left(3^{|Z_4 \setminus Z_3|} - 2^{|Z_4 \setminus Z_3|}\right) \cdot \left(7^{|\breve{D} \setminus Z_1|} - 6^{|\breve{D} \setminus Z_1|}\right) \cdot 7^{|X \setminus \breve{D}|};$$

Proof. Bydefinitionof the given semilattice D we have $Q_{14}\vartheta_{XI} = \left\{\left\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\right\}\right\}$. If the following equality is hold $D_1' = \left\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\right\}$, then $\left|I^*\left(Q_{14}\right)\right| = \left|I\left(D_1'\right)\right|$ (see Theorem 1.4). Of this equality we have:

$$\left|I^*\left(Q_{14}\right)\right| = \left(2^{|Z_5 \setminus Z_4|} - 1\right) \cdot \left(2^{|Z_6 \setminus Z_5|} - 1\right) \cdot \left(3^{|Z_4 \setminus Z_3|} - 2^{|Z_4 \setminus Z_3|}\right) \cdot \left(7^{|\breve{D} \setminus Z_1|} - 6^{|\breve{D} \setminus Z_1|}\right) \cdot 7^{|X \setminus \breve{D}|};$$

(see statement n) of the Lemma 2.6).

Lemma is proved.

Lemma 2.21. Let $D \in \Sigma_3(X, 8)$ and $Z_7 \neq \varnothing$. If X is a finite set, then the number $\left|I^*\left(Q_{15}\right)\right|$ may be calculated by the formula

$$\left|I^*\left(Q_{15}\right)\right| = \left(2^{|Z_5 \setminus Z_6|} - 1\right) \cdot \left(2^{|Z_6 \setminus Z_5|} - 1\right) \cdot 4^{|(Z_1 \cap Z_2) \setminus Z_4|} \cdot \left(5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}\right) \cdot \left(5^{|Z_1 \setminus Z_2|} - 4^{|Z_1 \setminus Z_2|}\right) \cdot 7^{|X \setminus \breve{D}|};$$

Proof. By definition of the given semilattice D we have $Q_{15}\vartheta_{XI} = \left\{\left\{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\right\}\right\}$. If the following equality is hold $D_1' = \left\{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\right\}$, then $\left|I^*\left(Q_{15}\right)\right| = \left|I\left(D_1'\right)\right|$ (see Theorem 1.4). Of this equality we have:

$$\left|I^*\left(Q_{15}\right)\right| = \left(2^{|Z_5 \setminus Z_6|} - 1\right) \cdot \left(2^{|Z_6 \setminus Z_5|} - 1\right) \cdot 4^{|(Z_1 \cap Z_2) \setminus Z_4|} \cdot \left(5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}\right) \cdot \left(5^{|Z_1 \setminus Z_2|} - 4^{|Z_1 \setminus Z_2|}\right) \cdot 7^{|X \setminus \breve{D}|};$$

(see statement o) of the Lemma 2.6).

Lemma is proved.

Lemma 2.22. Let $D \in \Sigma_3(X, 8)$ and $Z_7 \neq \varnothing$. If X is a finite set, then the number $\left|I^*\left(Q_{16}\right)\right|$ may be calculated by the formula

$$\left|I^*\left(Q_{16}\right)\right| = \left(2^{|Z_6 \setminus Z_3|} - 1\right) \cdot 2^{|(Z_3 \cap Z_2) \setminus Z_4|} \cdot \left(2^{|Z_5 \setminus Z_6|} - 1\right) \cdot \left(3^{|Z_3 \setminus Z_2|} - 2^{|Z_3 \setminus Z_2|}\right) \cdot \left(5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}\right) \cdot 8^{|X \setminus \breve{D}|}$$

Proof. By definition of the given semilattice D we have $Q_{16}\vartheta_{XI} = \left\{\left\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}\right\}$. If the following equality is hold $D_1' = \left\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}$, then $\left|I^*\left(Q_{16}\right)\right| = \left|I\left(D_1'\right)\right|$ (see Theorem 1.4). Of this equality we have:

$$\left|I^*\left(Q_{16}\right)\right| = \left(2^{|Z_6 \setminus Z_3|} - 1\right) \cdot 2^{|(Z_3 \cap Z_2) \setminus Z_4|} \cdot \left(2^{|Z_5 \setminus Z_6|} - 1\right) \cdot \left(3^{|Z_3 \setminus Z_2|} - 2^{|Z_3 \setminus Z_2|}\right) \cdot \left(5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}\right) \cdot 8^{|X \setminus \breve{D}|}$$

(see statement p) of the Lemma 2.6).

Lemma is proved

Theorem 2.2. Let $D \in \Sigma_3(X, 8)$ and $Z_7 \neq \varnothing$. If X is a finite set, then the number $\left|I(D)\right|$ may be calculated by the formula

$$\begin{aligned}\left|I(D)\right| = {} & \left|I^*(Q_1)\right| + \left|I^*(Q_2)\right| + \left|I^*(Q_3)\right| + \left|I^*(Q_4)\right| + \left|I^*(Q_5)\right| + \left|I^*(Q_6)\right| + \left|I^*(Q_7)\right| + \left|I^*(Q_8)\right| \\ & + \left|I^*(Q_9)\right| + \left|I^*(Q_{10})\right| + \left|I^*(Q_{11})\right| + \left|I^*(Q_{11})\right| + \left|I^*(Q_{13})\right| + \left|I^*(Q_{14})\right| + \left|I^*(Q_{15})\right| + \left|I^*(Q_{16})\right|\end{aligned}$$

Proof. This Theorem immediately follows from the Theorem 2.1.

Theorem is proved.

Example 2.1. *Let* $X = \{1,2,3,4,5\}$,

$$P_0 = \{1\},\ P_1 = \{2\},\ P_2 = \{3\},\ P_3 = \{4\},\ P_6 = \{5\},\ P_4 = P_5 = P_7 = \varnothing.$$

Then $\breve{D} = \{1,2,3,4,5\}$, $Z_1 = \{1,3,4,5\}$, $Z_2 = \{1,2,4,5\}$, $Z_3 = \{1,3,5\}$, $Z_4 = \{1,4,5\}$, $Z_5 = \{1,5\}$, $Z_6 = \{1,4\}$, $Z_7 = \{1\}$, *and*

$$D = \{\{1\},\{1,4\},\{1,5\},\{1,4,5\},\{1,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{1,2,3,4,5\}\}.$$

Then we have that following equality are hold:

$|I^*(Q_1)| = 8$, $|I^*(Q_2)| = 147$, $|I^*(Q_3)| = 241$, $|I^*(Q_4)| = 75$, $|I^*(Q_5)| = 5$, $|I^*(Q_6)| = 46$, $|I^*(Q_7)| = 19$, $|I^*(Q_8)| = 2$, $|I^*(Q_9)| = 1$, $|I^*(Q_{10})| = 24$, $|I^*(Q_{11})| = 2$, $|I^*(Q_{12})| = 9$, $|I^*(Q_{13})| = 1$, $|I^*(Q_{14})| = 1$,

$|I^*(Q_{15})| = 1$, $|I^*(Q_{16})| = 1$, $|I_D| = 583$.

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