Scientific
Research
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Generating Set of the Complete Semigroups of Binary Relations

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Abstract

Difficulties encountered in studying generators of semigroup $B_X(D)$ of binary relations defined by a complete *X*-semilattice of unions *D* arise because of the fact that they are not regular as a rule, which makes their investigation problematic. In this work, for special *D*, it has been seen that the semigroup $B_X(D)$, which are defined by semilattice *D*, can be generated by the set

$$B = \left\{ \alpha \in B_X(D) \middle| V(X^\bullet, \alpha) = D \right\}.$$

Keywords

1. Introduction

Theorem 1. *Let* $D = \left\{ \breve{D}, Z_1, Z_2, \cdots, Z_{m-1} \right\}$ *be some finite X-semilattice of unions and*

$$C(D) = \left\{ P_0, P_1, P_2, \cdots, P_{m-1} \right\}$$

be the family of sets of pairwise nonintersecting subsets of the set X.

If φ is a mapping of the semilattice *D* on the family of sets $C(D)$ which satisfies the condition $\varphi(\breve{D}) = P_0$ and $\varphi(Z_i) = P_i$ for any $i = 1, 2, \cdots, m-1$ and $\hat{D}_Z = D \setminus D_T$, then the following equalities are valid:

$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-1},$$
$$Z_i = P_0 \cup \bigcup_{T \in \hat{D}_{Z_i}} \varphi(T). \tag{1}$$

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice D are represented in the form 1, then among the parameters P_i $(i = 0,1,2,\cdots,m-1)$ there exist such parameters that cannot be empty sets for D. Such sets P_i $(0 < i \leq m-1)$ are called basis sources, whereas sets P_i $(0 \leq j \leq m-1)$ which can be empty sets too are called completeness sources.

It is proved that under the mapping φ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see [1], Chapter 11). Some positive results in this direction can be found in [2]-[6].

Let $P_0, P_1, P_2, \cdots, P_{m-1}$ be parameters in the formal equalities, $\beta \in B_X(D)$ and

$$\overline{\beta} = \bigcup_{i=0}^{m-1}\left(P_i \times \bigcup_{t \in P_i} t\beta \right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times t'\beta \right). \tag{2}$$

$$\tilde{\beta} = \bigcup_{i=0}^{m-1}\left(P_i \times \bigcup_{t \in P_i} t\beta \right) \cup \left(\left(X \setminus \breve{D} \right) \times \breve{D} \right) \tag{3}$$

The representation of the binary relation β of the form $\overline{\beta}$ and $\tilde{\beta}$ will be called subquasinormal and maximal subquasinormal.

If $\overline{\beta}$ and $\tilde{\beta}$ are the subquasinormal and maximal subquasinormal representations of the binary relation β, then for the binary relations $\overline{\beta}$ and $\tilde{\beta}$ the following statements are true:

a) $\overline{\beta}, \tilde{\beta} \in B_X(D)$;

b) $\beta \subseteq \overline{\beta} \subseteq \tilde{\beta}$;

c) the subquasinormal representation of the binary relation β is quasinormal;

d) if

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & \cdots & P_{m-1} \\ P_0\overline{\beta} & P_1\overline{\beta} & \cdots & P_{m-1}\overline{\beta} \end{pmatrix},$$

then $\overline{\beta}_1$ is a mapping of the family of sets $C(D) = \{P_0, P_1, P_2, \cdots, P_{m-1}\}$ in the X-semilattice of unions

$$D = \left\{ \breve{D}, Z_1, Z_2, \cdots, Z_{m-1} \right\}.$$

e) if $\overline{\beta}_2 : X \setminus \breve{D} \to D$ is a mapping satisfying the condition $\overline{\beta}_2(t') = t'\beta$ for all $t' \in X \setminus \breve{D}$, then

$$\overline{\beta} = \bigcup_{i=0}^{m-1}\left(P_i \times \bigcup_{t \in P_i} t\beta \right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \overline{\beta}_2(t') \right).$$

2. Results

Proposition 2. *Let* $\alpha, \beta \in B_X(D)$. *Then*

$$\alpha \circ \beta = \alpha \circ \overline{\beta} = \alpha \circ \tilde{\beta}.$$

Proof. It is easy to see the inclusion $\alpha \circ \beta \subseteq \alpha \circ \overline{\beta} \subseteq \alpha \circ \tilde{\beta}$ holds, since $\beta \subseteq \overline{\beta} \subseteq \tilde{\beta}$. If $x\left(\alpha \circ \tilde{\beta}\right) y$ $(x, y \in X)$, then $x\alpha z\tilde{\beta} y$ for some $z \in X$. So, $z \notin X \setminus \breve{D}$ since $z \in x\alpha \in D$ and $z\left(\bigcup_{i=0}^{m-1}\left(P_i \times \bigcup_{t \in P_i} t\beta \right) \right) y$.

Then $z\left(P_k \times \bigcup_{t \in P_k} t\beta \right) y$ for some k $(0 \leq k \leq m-1)$ *i.e.* $z \in P_k$ and $y \in z\beta \subseteq \bigcup_{t \in P_k} t\beta$. For the last condition follows that $z\beta y$. We have $x\alpha z\beta y$ and $x(\alpha \circ \beta) y$. Therefore, the inclusion $\alpha \circ \tilde{\beta} \subseteq \alpha \circ \beta$ is true. Of this and by inclusion $\alpha \circ \beta \subseteq \alpha \circ \overline{\beta} \subseteq \alpha \circ \tilde{\beta}$ follows that the equality $\alpha \circ \beta = \alpha \circ \overline{\beta} = \alpha \circ \tilde{\beta}$ holds. $\square$

Corollary 1. *If* $\alpha, \delta, \beta \in B_X(D)$ *and* $\beta \subseteq \delta \subseteq \tilde{\beta}$, *then* $\alpha \circ \beta = \alpha \circ \delta = \alpha \circ \tilde{\beta}$.

Proof. We have $\beta \subseteq \delta \subseteq \tilde{\beta}$ and $\alpha \circ \beta \subseteq \alpha \circ \delta \subseteq \alpha \circ \tilde{\beta}$. Of this follows that $\alpha \circ \beta = \alpha \circ \delta = \alpha \circ \tilde{\beta}$ since $\alpha \circ \beta = \alpha \circ \tilde{\beta}$. $\square$

Let the X-semilattice $D = \{Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ of unions given by the diagram of **Figure 1**. Formal equalities of the given semilattice have a form:
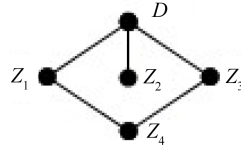
Figure 1. Diagram of D.

$$\check{D} = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4,$$
$$Z_1 = P_0 \cup P_2 \cup P_3 \cup P_4,$$
$$Z_2 = P_0 \cup P_1 \cup P_3 \cup P_4, \qquad\qquad (4)$$
$$Z_3 = P_0 \cup P_1 \cup P_2 \cup P_4,$$
$$Z_4 = P_0 \cup P_2.$$

The parameters P_1, P_2, P_3 are basis sources and the parameters P_0, P_4 are completeness sources, *i.e.* $|X| \geq 3$.

Example 3. *Let* $X = \{1,2,3,4,5,6,7\}$, $P_1 = \{1,4\}$, $P_2 = \{2,5\}$, $P_3 = \{3,6\}$, $P_0 = P_4 = \varnothing$. *Then for the formal equalities of the semilattice D follows that* $Z_4 = \{2,5\}$, $Z_3 = \{1,2,4,5\}$, $Z_2 = \{1,3,4,6\}$, $Z_1 = \{2,3,5,6\}$, $\check{D} = \{1,2,3,4,5,6\}$, $D = \{\{2\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$, *and*

$$\beta = \big(\{1\} \times \{2,3,5,6\}\big) \cup \big(\{2\} \times \{1,3,4,6\}\big) \cup \big(\{3\} \times \{1,2,4,5\}\big) \cup \big(\{4,5,6,7\} \times \{2,5\}\big).$$

Then we have:

$$\bigcup_{t \in P_1} t\beta = \{2,3,5,6\}, \bigcup_{t \in P_2} t\beta = \{1,2,3,4,5,6\}, \bigcup_{t \in P_3} t\beta = \{1,2,4,5\},$$

$$\bigcup_{t \in P_0} t\beta = \bigcup_{t \in P_4} t\beta = \varnothing;$$

$$\overline{\beta} = \big(\{1,4\} \times \{2,3,5,6\}\big) \cup \big(\{2,5\} \times \{1,2,3,4,5,6\}\big) \cup \big(\{3,6\} \times \{1,2,4,5\}\big) \cup \big(\{7\} \times \{2,5\}\big);$$

$$\tilde{\beta} = \big(\{1,4\} \times \{2,3,5,6\}\big) \cup \big(\{2,5\} \times \{1,2,3,4,5,6\}\big) \cup \big(\{3,6\} \times \{1,2,4,5\}\big) \cup \big(\{7\} \times \{1,2,3,4,5,6,7\}\big).$$

Theorem 4. *Let the X-semilattice* $D = \{Z_4, Z_3, Z_2, Z_1, \check{D}\}$ *of unions given by the diagram of* **Figure 1**, $B = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = D\}$ *and* $|X \setminus \check{D}| \geq 3$. *Then the set B is generating set of the semigroup* $B_X(D)$.

Proof. It is easy to see that $|X| \geq 6$ since $P_1, P_2, P_3 \notin \{\varnothing\}$ and $|X \setminus \check{D}| \geq 3$. Now, let α be any binary relation of the semigroup $B_X(D)$; $\alpha_1, \alpha_2, \cdots, \alpha_n \in B$, $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n$ and $\beta = \alpha_2 \circ \alpha_3 \circ \cdots \circ \alpha_n$. Then the equality $\alpha = \alpha_1 \circ \beta = \alpha_1 \circ \check{\beta}$ ($\check{\beta}$ is subquasinormal representation of a binary relation β) is true. By assumption $\alpha_1 \in B_X(D)$, *i.e.* the quasinormal representation of a binary relation α_1 have a form

$$\alpha_1 = \big(Y_4^{\alpha_1} \times Z_4\big) \cup \big(Y_3^{\alpha_1} \times Z_3\big) \cup \big(Y_2^{\alpha_1} \times Z_2\big) \cup \big(Y_1^{\alpha_1} \times Z_1\big) \cup \big(Y_0^{\alpha_1} \times \check{D}\big).$$

Of this follows that

$$\alpha = \alpha_1 \circ \overline{\beta} = \big(Y_4^{\alpha_1} \times Z_4\big) \cup \big(Y_3^{\alpha_1} \times Z_3\big) \cup \big(Y_2^{\alpha_1} \times Z_2\big) \cup \big(Y_1^{\alpha_1} \times Z_1\big) \cup \big(Y_0^{\alpha_1} \times \check{D}\big) \circ \overline{\beta}$$
$$= \big(Y_4^{\alpha_1} \times Z_4 \overline{\beta}\big) \cup \big(Y_3^{\alpha_1} \times Z_3 \overline{\beta}\big) \cup \big(Y_2^{\alpha_1} \times Z_2 \overline{\beta}\big) \cup \big(Y_1^{\alpha_1} \times Z_1 \overline{\beta}\big) \cup \big(Y_0^{\alpha_1} \times \check{D} \overline{\beta}\big). \qquad (5)$$

For the binary relation α we consider the following case.

a) Let $|V(X^*, \alpha)| = 1$. Then $\alpha = X \times T$, where $T \in D$. By element *T* we consider the following cases:

1. $T \neq \check{D}$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & T & T & T & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \check{D}$ on the set $\{Z_4, Z_3, Z_2, Z_1\} \setminus \{T\}$. Then

$$\bar{\beta} = \left(P_0 \times \varnothing\right) \cup \left(P_1 \times T\right) \cup \left(P_2 \times T\right) \cup \left(P_3 \times T\right) \cup \left(P_4 \times \varnothing\right) \cup \bigcup_{t' \in X \setminus \check{D}} \left(\{t'\} \times \bar{\beta}_2\left(t'\right)\right),$$

$$\alpha_1 = \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \check{D}\right), \tag{6}$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, then it is easy to see, that $\alpha_1, \bar{\beta} \in B$ since $V\left(D, \alpha_1\right) = V\left(D, \bar{\beta}\right) = D$. From the formal equality and equalities (6) and (5) we have:

$$Z_4\bar{\beta} = \left(P_0 \cup P_2\right)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} = \varnothing \cup T = T;$$

$$Z_3\bar{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup T \cup T \cup \varnothing = T;$$

$$Z_2\bar{\beta} = \left(P_0 \cup P_1 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup T \cup T \cup \varnothing = T;$$

$$Z_1\bar{\beta} = \left(P_0 \cup P_2 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup T \cup T \cup \varnothing = T;$$

$$\check{D}\bar{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta}$$
$$= \varnothing \cup T \cup T \cup T \cup \varnothing = T;$$

$$\alpha = \alpha_1 \circ \bar{\beta}$$
$$= \left(Y_4^{\alpha_1} \times Z_4\bar{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\bar{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\bar{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\bar{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \check{D}\bar{\beta}\right)$$
$$= \left(Y_4^{\alpha_1} \times T\right) \cup \left(Y_3^{\alpha_1} \times T\right) \cup \left(Y_2^{\alpha_1} \times T\right) \cup \left(Y_1^{\alpha_1} \times T\right) \cup \left(Y_0^{\alpha_1} \times T\right) = X \times T$$

since $Y_4^{\alpha_1} \cup Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = X$.

2. $T = \check{D}$. In this case suppose that

$$\bar{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \check{D} & Z_1 & Z_2 & Z_3 & Z_4 \end{pmatrix}$$

and $\bar{\beta}_2$ are mapping of the set $X \setminus \check{D}$ in the set D. Then

$$\bar{\beta} = \left(P_0 \times \check{D}\right) \cup \left(P_1 \times Z_1\right) \cup \left(P_2 \times Z_2\right) \cup \left(P_3 \times Z_3\right) \cup \left(P_4 \times Z_4\right) \cup \bigcup_{t' \in X \setminus \check{D}} \left(\{t'\} \times \bar{\beta}_2\left(t'\right)\right),$$

$$\alpha_1 = \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \check{D}\right), \tag{7}$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, then it is easy to see, that $\alpha_1, \bar{\beta} \in B$ since $V\left(D, \alpha_1\right) = V\left(D, \bar{\beta}\right) = D$. From the formal equality and equalities (7) and (5) we have:

$$Z_4\bar{\beta} = \left(P_0 \cup P_2\right)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} = \check{D};$$

$$Z_3\bar{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_4\bar{\beta} = \check{D};$$

$$Z_2\bar{\beta} = \left(P_0 \cup P_1 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \check{D};$$

$$Z_1\bar{\beta} = \left(P_0 \cup P_2 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \check{D};$$

$$\check{D}\bar{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \check{D};$$

$$\alpha = \alpha_1 \circ \bar{\beta}$$
$$= \left(Y_4^{\alpha_1} \times Z_4\bar{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\bar{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\bar{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\bar{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \check{D}\bar{\beta}\right)$$
$$= \left(Y_4^{\alpha_1} \times \check{D}\right) \cup \left(Y_3^{\alpha_1} \times \check{D}\right) \cup \left(Y_2^{\alpha_1} \times \check{D}\right) \cup \left(Y_1^{\alpha_1} \times \check{D}\right) \cup \left(Y_0^{\alpha_1} \times \check{D}\right)$$
$$= X \times \check{D}.$$

b) $\left|V\left(X^*, \alpha\right)\right| = 2$. Then

$$V\left(X^*, \alpha\right) \in \left\{\{Z_4, Z_1\}, \{Z_4, Z_3\}, \{Z_4, \check{D}\}, \{Z_3, \check{D}\}, \{Z_2, \check{D}\}, \{Z_1, \check{D}\}\right\}$$

since $V\left(X^*, \alpha\right)$ is X-semilattice of unions. For the semilattice of unions $V\left(X^*, \alpha\right)$ consider the following cases.

1. Let $V\left(X^*, \alpha\right) = \{Z_4, T\}$, where, $T \in \{Z_1, Z_3\}$. Then binary relation α has representation of the form

$\alpha = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_T^\alpha \times T\right)$. In this case suppose that

$$\bar{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & T & Z_4 & T & \varnothing \end{pmatrix}$$

and $\bar{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_4, T\}$. Then

$$\bar{\beta} = (P_0 \times \varnothing) \cup (P_1 \times T) \cup (P_2 \times Z_4) \cup (P_3 \times T) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \bar{\beta}_2(t')\right),$$

$$\alpha_1 = \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right),$$

\hfill (8)

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_4^\alpha$ and $Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^\alpha$, then it is easy to see, that $\alpha_1, \bar{\beta} \in B$ since $V(D, \alpha_1) = V(D, \bar{\beta}) = D$. From the formal equality and equalities (8) and (5) we have:

$$Z_4 \bar{\beta} = (P_0 \cup P_2)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} = \varnothing \cup Z_4 = Z_4;$$

$$Z_3 \bar{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_4)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_4\bar{\beta}$$
$$= \varnothing \cup T \cup Z_4 \cup \varnothing = T;$$

$$Z_2 \bar{\beta} = (P_0 \cup P_1 \cup P_3 \cup P_4)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta}$$
$$= \varnothing \cup T \cup T \cup \varnothing = T;$$

$$Z_1 \bar{\beta} = (P_0 \cup P_2 \cup P_3 \cup P_4)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta}$$
$$= \varnothing \cup Z_4 \cup T \cup \varnothing = T;$$

$$\breve{D} \bar{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta}$$
$$= \varnothing \cup T \cup Z_4 \cup T \cup \varnothing = T;$$

$$\alpha = \alpha_1 \circ \bar{\beta}$$
$$= \left(Y_4^{\alpha_1} \times Z_4\bar{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\bar{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\bar{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\bar{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\bar{\beta}\right)$$
$$= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times T\right) \cup \left(Y_2^{\alpha_1} \times T\right) \cup \left(Y_1^{\alpha_1} \times T\right) \cup \left(Y_0^{\alpha_1} \times T\right)$$
$$= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(\left(Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}\right) \times T\right) = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_0^\alpha \times T\right).$$

2. Let $V(X^*, \alpha) = \{T, \breve{D}\}$, where, $T \in \{Z_4, Z_3, Z_2, Z_1\}$. Then binary relation α has representation of the form $\alpha = \left(Y_T^\alpha \times T\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$. In this case suppose that

$$\bar{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & \breve{D} & T & \breve{D} & \varnothing \end{pmatrix}$$

and $\bar{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{T, \breve{D}\}$. Then

$$\bar{\beta} = (P_0 \times \varnothing) \cup (P_1 \times \breve{D}) \cup (P_2 \times T) \cup (P_3 \times \breve{D}) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times f(t')\right),$$

$$\alpha_1 = \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right),$$

\hfill (9)

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_T^\alpha$ and $Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^\alpha$, then it is easy to see, that $\alpha_1, \bar{\beta} \in B$ since $V(D, \alpha_1) = V(D, \bar{\beta}) = D$. From the formal equality and equalities (9) and (5) we have:

$$Z_4\bar{\beta} = \left(P_0 \cup P_2\right)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} = \varnothing \cup T = T;$$

$$Z_3\bar{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup \breve{D} \cup T \cup \varnothing = \breve{D};$$

$$Z_2\bar{\beta} = \left(P_0 \cup P_1 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup \breve{D} \cup \breve{D} \cup \varnothing = \breve{D};$$

$$Z_1\bar{\beta} = \left(P_0 \cup P_2 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup T \cup \breve{D} \cup \varnothing = \breve{D};$$

$$\breve{D}\bar{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta}$$

$$= \varnothing \cup \breve{D} \cup T \cup \breve{D} \cup \varnothing = \breve{D};$$

$$\alpha = \alpha_1 \circ \bar{\beta}$$

$$= \left(Y_4^{\alpha_1} \times Z_4\bar{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\bar{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\bar{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\bar{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\bar{\beta}\right)$$

$$= \left(Y_4^{\alpha_1} \times T\right) \cup \left(Y_3^{\alpha_1} \times \breve{D}\right) \cup \left(Y_2^{\alpha_1} \times \breve{D}\right) \cup \left(Y_1^{\alpha_1} \times \breve{D}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right)$$

$$= \left(Y_4^{\alpha_1} \times T\right) \cup \left(\left(Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}\right) \times \breve{D}\right) = \left(Y_T^{\alpha} \times T\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right).$$

c) $\left|V\left(X^*, \alpha\right)\right| = 3$. Then

$$V\left(X^*, \alpha\right) \in \left\{\left\{Z_4, Z_1, \breve{D}\right\}, \left\{Z_4, Z_3, \breve{D}\right\}, \left\{Z_4, Z_2, \breve{D}\right\}, \left\{Z_1, Z_2, \breve{D}\right\}, \left\{Z_1, Z_3, \breve{D}\right\}, \left\{Z_2, Z_3, \breve{D}\right\}\right\}$$

since $V\left(X^*, \alpha\right)$ is X-semilattice of unions. For the semilattice of unions $V\left(X^*, \alpha\right)$ consider the following cases.

1. Let $V\left(X^*, \alpha\right) = \left\{Z_4, Z_1, \breve{D}\right\}$. Then binary relation α has representation of the form $\alpha = \left(Y_4^{\alpha} \times Z_4\right) \cup \left(Y_1^{\alpha} \times Z_1\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right)$. In this case suppose that

$$\bar{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_2 & Z_4 & Z_1 & \varnothing \end{pmatrix}$$

and $\bar{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \left\{Z_4, Z_2, Z_1\right\}$. Then

$$\bar{\beta} = \left(P_0 \times \varnothing\right) \cup \left(P_1 \times Z_2\right) \cup \left(P_2 \times Z_4\right) \cup \left(P_3 \times Z_1\right) \cup \left(P_4 \times \varnothing\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\left\{t'\right\} \times f\left(t'\right)\right),$$

$$\alpha_1 = \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right), \quad (10)$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \left\{\varnothing\right\}$, $Y_4^{\alpha_1} = Y_4^{\alpha}$, $Y_1^{\alpha_1} = Y_1^{\alpha}$ and $Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^{\alpha}$, then it is easy to see, that $\alpha_1, \bar{\beta} \in B$ since $V\left(D, \alpha_1\right) = V\left(D, \bar{\beta}\right) = D$. From the formal equality and equalities (10) and (5) we have:

$$Z_4\bar{\beta} = \left(P_0 \cup P_2\right)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} = \varnothing \cup Z_4 = Z_4;$$

$$Z_3\bar{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup Z_2 \cup Z_4 \cup \varnothing = \breve{D};$$

$$Z_2\bar{\beta} = \left(P_0 \cup P_1 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup Z_2 \cup Z_1 \cup \varnothing = \breve{D};$$

$$Z_1\bar{\beta} = \left(P_0 \cup P_2 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup Z_4 \cup Z_1 \cup \varnothing = Z_1;$$

$$\breve{D}\bar{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta}$$

$$= \varnothing \cup Z_2 \cup Z_4 \cup Z_1 \cup \varnothing = \breve{D};$$

$$\alpha = \alpha_1 \circ \bar{\beta}$$

$$= \left(Y_4^{\alpha_1} \times Z_4\bar{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\bar{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\bar{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\bar{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\bar{\beta}\right)$$

$$= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times \breve{D}\right) \cup \left(Y_2^{\alpha_1} \times \breve{D}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right)$$

$$= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(\left(Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_0^{\alpha_1}\right) \times \breve{D}\right)$$

$$= \left(Y_4^{\alpha} \times Z_4\right) \cup \left(Y_1^{\alpha} \times Z_1\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right).$$

2. Let $V\left(X^*, \alpha\right) = \left\{Z_4, Z_3, \breve{D}\right\}$. Then binary relation α has representation of the form

$\alpha = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_3 & Z_4 & Z_2 & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_4, Z_3, Z_2\}$. Then

$$\overline{\beta} = \left(P_0 \times \varnothing\right) \cup \left(P_1 \times Z_3\right) \cup \left(P_2 \times Z_4\right) \cup \left(P_3 \times Z_2\right) \cup \left(P_4 \times \varnothing\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times f\left(t'\right)\right),$$

$$\alpha_1 = \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right),$$

(11)

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_4^\alpha$, $Y_3^{\alpha_1} = Y_3^\alpha$ and $Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^\alpha$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V\left(D, \alpha_1\right) = V\left(D, \overline{\beta}\right) = D$. From the formal equality and equalities (11) and (5) we have:

$$Z_4 \overline{\beta} = \left(P_0 \cup P_2\right)\overline{\beta} = P_0 \overline{\beta} \cup P_2 \overline{\beta} = \varnothing \cup Z_4 = Z_4;$$

$$Z_3 \overline{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_4\right)\overline{\beta} = P_0 \overline{\beta} \cup P_1 \overline{\beta} \cup P_2 \overline{\beta} \cup P_4 \overline{\beta}$$

$$= \varnothing \cup Z_3 \cup Z_4 \cup \varnothing = Z_3;$$

$$Z_2 \overline{\beta} = \left(P_0 \cup P_1 \cup P_3 \cup P_4\right)\overline{\beta} = P_0 \overline{\beta} \cup P_1 \overline{\beta} \cup P_3 \overline{\beta} \cup P_4 \overline{\beta}$$

$$= \varnothing \cup Z_3 \cup Z_2 \cup \varnothing = \breve{D};$$

$$Z_1 \overline{\beta} = \left(P_0 \cup P_2 \cup P_3 \cup P_4\right)\overline{\beta} = P_0 \overline{\beta} \cup P_2 \overline{\beta} \cup P_3 \overline{\beta} \cup P_4 \overline{\beta}$$

$$= \varnothing \cup Z_4 \cup Z_2 \cup \varnothing = \breve{D};$$

$$\breve{D}\overline{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4\right)\overline{\beta} = P_0 \overline{\beta} \cup P_1 \overline{\beta} \cup P_2 \overline{\beta} \cup P_3 \overline{\beta} \cup P_4 \overline{\beta}$$

$$= \varnothing \cup Z_3 \cup Z_4 \cup Z_2 \cup \varnothing = \breve{D};$$

$$\alpha = \alpha_1 \circ \overline{\beta}$$

$$= \left(Y_4^{\alpha_1} \times Z_4 \overline{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3 \overline{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2 \overline{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1 \overline{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\overline{\beta}\right)$$

$$= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times \breve{D}\right) \cup \left(Y_1^{\alpha_1} \times \breve{D}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right)$$

$$= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(\left(Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}\right) \times \breve{D}\right)$$

$$= \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_0^\alpha \times \breve{D}\right).$$

3. Let $V\left(X^*, \alpha\right) = \{Z_4, Z_2, \breve{D}\}$. Then binary relation α has representation of the form

$\alpha = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_2^\alpha \times Z_2\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_2 & Z_4 & Z_2 & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_4, Z_2\}$. Then

$$\overline{\beta} = \left(P_0 \times \varnothing\right) \cup \left(P_1 \times Z_2\right) \cup \left(P_2 \times Z_4\right) \cup \left(P_3 \times Z_2\right) \cup \left(P_4 \times \varnothing\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times f\left(t'\right)\right),$$

$$\alpha_1 = \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right),$$

(12)

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_4^\alpha$, $Y_2^{\alpha_1} = Y_2^\alpha$ and $Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^\alpha$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V\left(D, \alpha_1\right) = V\left(D, \overline{\beta}\right) = D$. From the formal equality and equalities (12) and (5) we have:

$$Z_4\bar{\beta} = \left(P_0 \cup P_2\right)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} = \varnothing \cup Z_4 = Z_4;$$

$$Z_3\bar{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup Z_2 \cup Z_4 \cup \varnothing = \breve{D};$$

$$Z_2\bar{\beta} = \left(P_0 \cup P_1 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup Z_2 \cup Z_2 \cup \varnothing = Z_2;$$

$$Z_1\bar{\beta} = \left(P_0 \cup P_2 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup Z_4 \cup Z_2 \cup \varnothing = \breve{D};$$

$$\breve{D}\bar{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta}$$

$$= \varnothing \cup Z_2 \cup Z_4 \cup Z_2 \cup \varnothing = \breve{D};$$

$$\alpha = \alpha_1 \circ \bar{\beta}$$

$$= \left(Y_4^{\alpha_1} \times Z_4\bar{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\bar{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\bar{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\bar{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\bar{\beta}\right)$$

$$= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times \breve{D}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times \breve{D}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right)$$

$$= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(\left(Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}\right) \times \breve{D}\right)$$

$$= \left(Y_4^{\alpha} \times Z_4\right) \cup \left(Y_2^{\alpha} \times Z_2\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right).$$

4. Let $V\left(X^*, \alpha\right) = \left\{Z_1, Z_2, \breve{D}\right\}$. Then binary relation α has representation of the form $\alpha = \left(Y_1^{\alpha} \times Z_1\right) \cup \left(Y_2^{\alpha} \times Z_2\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right)$. In this case suppose that

$$\bar{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_1 & Z_2 & Z_1 & \varnothing \end{pmatrix}$$

and $\bar{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_2, Z_1\}$. Then

$$\bar{\beta} = \left(P_0 \times \varnothing\right) \cup \left(P_1 \times Z_1\right) \cup \left(P_2 \times Z_2\right) \cup \left(P_3 \times Z_1\right) \cup \left(P_4 \times \varnothing\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times f\left(t'\right)\right),$$

$$\alpha_1 = \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right),$$

(13)

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_2^{\alpha_1} = Y_1^{\alpha}$, $Y_4^{\alpha_1} = Y_2^{\alpha}$ and $Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^{\alpha}$, then it is easy to see, that $\alpha_1, \bar{\beta} \in B$ since $V\left(D, \alpha_1\right) = V\left(D, \bar{\beta}\right) = D$. From the formal equality and equalities (13) and (5) we have:

$$Z_4\bar{\beta} = \left(P_0 \cup P_2\right)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} = \varnothing \cup Z_2 = Z_2;$$

$$Z_3\bar{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup Z_1 \cup Z_2 \cup \varnothing = \breve{D};$$

$$Z_2\bar{\beta} = \left(P_0 \cup P_1 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup Z_1 \cup Z_1 \cup \varnothing = Z_1;$$

$$Z_1\bar{\beta} = \left(P_0 \cup P_2 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta} = \varnothing \cup Z_2 \cup Z_1 \cup \varnothing = \breve{D};$$

$$\breve{D}\bar{\beta} = \left(P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4\right)\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} \cup P_2\bar{\beta} \cup P_3\bar{\beta} \cup P_4\bar{\beta}$$

$$= \varnothing \cup Z_1 \cup Z_2 \cup Z_1 \cup \varnothing = \breve{D};$$

$$\alpha = \alpha_1 \circ \bar{\beta}$$

$$= \left(Y_4^{\alpha_1} \times Z_4\bar{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\bar{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\bar{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\bar{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\bar{\beta}\right)$$

$$= \left(Y_4^{\alpha_1} \times Z_2\right) \cup \left(Y_3^{\alpha_1} \times \breve{D}\right) \cup \left(Y_2^{\alpha_1} \times Z_1\right) \cup \left(Y_1^{\alpha_1} \times \breve{D}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right)$$

$$= \left(Y_2^{\alpha_1} \times Z_1\right) \cup \left(Y_4^{\alpha_1} \times Z_2\right) \cup \left(\left(Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}\right) \times \breve{D}\right)$$

$$= \left(Y_1^{\alpha} \times Z_1\right) \cup \left(Y_2^{\alpha} \times Z_2\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right).$$

5. Let $V\left(X^*, \alpha\right) = \left\{Z_1, Z_3, \breve{D}\right\}$. Then binary relation α has representation of the form $\alpha = \left(Y_1^{\alpha} \times Z_1\right) \cup \left(Y_3^{\alpha} \times Z_3\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right)$. In this case suppose that

$$\bar{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_3 & Z_1 & Z_3 & \varnothing \end{pmatrix}$$

and $\bar{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_3, Z_1\}$. Then

$$\bar{\beta} = (P_0 \times \varnothing) \cup (P_1 \times Z_3) \cup (P_2 \times Z_1) \cup (P_3 \times Z_3) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} (\{t'\} \times f(t')),$$

$$\alpha_1 = (Y_4^{\alpha_1} \times Z_4) \cup (Y_3^{\alpha_1} \times Z_3) \cup (Y_2^{\alpha_1} \times Z_2) \cup (Y_1^{\alpha_1} \times Z_1) \cup (Y_0^{\alpha_1} \times \breve{D}),$$

(14)

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_1^{\alpha}$, $Y_2^{\alpha_1} = Y_3^{\alpha}$ and $Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^{\alpha}$, then it is easy to see, that $\alpha_1, \bar{\beta} \in B$ since $V(D, \alpha_1) = V(D, \bar{\beta}) = D$. From the formal equality and equalities (14) and (5) we have:

$$Z_4 \bar{\beta} = (P_0 \cup P_2) \bar{\beta} = P_0 \bar{\beta} \cup P_2 \bar{\beta} = \varnothing \cap Z_1 = Z_1;$$

$$Z_3 \bar{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_4) \bar{\beta} = P_0 \bar{\beta} \cup P_1 \bar{\beta} \cup P_2 \bar{\beta} \cup P_4 \bar{\beta} = \varnothing \cup Z_3 \cup Z_1 \cup \varnothing = \breve{D};$$

$$Z_2 \bar{\beta} = (P_0 \cup P_1 \cup P_3 \cup P_4) \bar{\beta} = P_0 \bar{\beta} \cup P_1 \bar{\beta} \cup P_3 \bar{\beta} \cup P_4 \bar{\beta} = \varnothing \cup Z_3 \cup Z_3 \cup \varnothing = Z_3;$$

$$Z_1 \bar{\beta} = (P_0 \cup P_2 \cup P_3 \cup P_4) \bar{\beta} = P_0 \bar{\beta} \cup P_2 \bar{\beta} \cup P_3 \bar{\beta} \cup P_4 \bar{\beta} = \varnothing \cup Z_1 \cup Z_3 \cup \varnothing = \breve{D};$$

$$\breve{D} \bar{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4) \bar{\beta} = P_0 \bar{\beta} \cup P_1 \bar{\beta} \cup P_2 \bar{\beta} \cup P_3 \bar{\beta} \cup P_4 \bar{\beta}$$
$$= \varnothing \cup Z_3 \cup Z_1 \cup Z_3 \cup \varnothing = \breve{D};$$

$$\alpha = \alpha_1 \circ \bar{\beta}$$
$$= (Y_4^{\alpha_1} \times Z_4 \bar{\beta}) \cup (Y_3^{\alpha_1} \times Z_3 \bar{\beta}) \cup (Y_2^{\alpha_1} \times Z_2 \bar{\beta}) \cup (Y_1^{\alpha_1} \times Z_1 \bar{\beta}) \cup (Y_0^{\alpha_1} \times \breve{D} \bar{\beta})$$
$$= (Y_4^{\alpha_1} \times Z_1) \cup (Y_3^{\alpha_1} \times \breve{D}) \cup (Y_2^{\alpha_1} \times Z_3) \cup (Y_1^{\alpha_1} \times \breve{D}) \cup (Y_0^{\alpha_1} \times \breve{D})$$
$$= (Y_4^{\alpha_1} \times Z_1) \cup (Y_2^{\alpha_1} \times Z_3) \cup ((Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}) \times \breve{D})$$
$$= (Y_1^{\alpha} \times Z_1) \cup (Y_3^{\alpha} \times Z_3) \cup (Y_0^{\alpha} \times \breve{D}).$$

6. Let $V(X^*, \alpha) = \{Z_2, Z_3, \breve{D}\}$. Then binary relation α has representation of the form $\alpha = (Y_2^{\alpha} \times Z_2) \cup (Y_3^{\alpha} \times Z_3) \cup (Y_0^{\alpha} \times \breve{D})$. In this case suppose that

$$\bar{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_3 & Z_2 & Z_3 & \varnothing \end{pmatrix}$$

and $\bar{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_3, Z_2\}$. Then

$$\bar{\beta} = (P_0 \times \varnothing) \cup (P_1 \times Z_3) \cup (P_2 \times Z_2) \cup (P_3 \times Z_3) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} (\{t'\} \times f(t'))$$

$$\alpha_1 = (Y_4^{\alpha_1} \times Z_4) \cup (Y_3^{\alpha_1} \times Z_3) \cup (Y_2^{\alpha_1} \times Z_2) \cup (Y_1^{\alpha_1} \times Z_1) \cup (Y_0^{\alpha_1} \times \breve{D}),$$

(15)

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_2^{\alpha}$, $Y_2^{\alpha_1} = Y_3^{\alpha}$ and $Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^{\alpha}$, then it is easy to see, that $\alpha_1, \bar{\beta} \in B$ since $V(D, \alpha_1) = V(D, \bar{\beta}) = D$. From the formal equality and equalities (15) and (5) we have:

$$Z_4 \bar{\beta} = (P_0 \cup P_2) \bar{\beta} = P_0 \bar{\beta} \cup P_2 \bar{\beta} = \varnothing \cup Z_2 = Z_2;$$

$$Z_3 \bar{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_4) \bar{\beta} = P_0 \bar{\beta} \cup P_1 \bar{\beta} \cup P_2 \bar{\beta} \cup P_4 \bar{\beta} = \varnothing \cup Z_3 \cup Z_2 \cup \varnothing = \breve{D};$$

$$Z_2 \bar{\beta} = (P_0 \cup P_1 \cup P_3 \cup P_4) \bar{\beta} = P_0 \bar{\beta} \cup P_1 \bar{\beta} \cup P_3 \bar{\beta} \cup P_4 \bar{\beta} = \varnothing \cup Z_3 \cup Z_3 \cup \varnothing = Z_3;$$

$$Z_1 \bar{\beta} = (P_0 \cup P_2 \cup P_3 \cup P_4) \bar{\beta} = P_0 \bar{\beta} \cup P_2 \bar{\beta} \cup P_3 \bar{\beta} \cup P_4 \bar{\beta} = \varnothing \cup Z_2 \cup Z_3 \cup \varnothing = \breve{D};$$

$$\breve{D} \bar{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4) \bar{\beta} = P_0 \bar{\beta} \cup P_1 \bar{\beta} \cup P_2 \bar{\beta} \cup P_3 \bar{\beta} \cup P_4 \bar{\beta}$$
$$= \varnothing \cup Z_3 \cup Z_2 \cup Z_3 \cup \varnothing = \breve{D};$$

$$\alpha = \alpha_1 \circ \bar{\beta}$$
$$= (Y_4^{\alpha_1} \times Z_4 \bar{\beta}) \cup (Y_3^{\alpha_1} \times Z_3 \bar{\beta}) \cup (Y_2^{\alpha_1} \times Z_2 \bar{\beta}) \cup (Y_1^{\alpha_1} \times Z_1 \bar{\beta}) \cup (Y_0^{\alpha_1} \times \breve{D} \bar{\beta})$$
$$= (Y_4^{\alpha_1} \times Z_2) \cup (Y_3^{\alpha_1} \times \breve{D}) \cup (Y_2^{\alpha_1} \times Z_3) \cup (Y_1^{\alpha_1} \times \breve{D}) \cup (Y_0^{\alpha_1} \times \breve{D})$$
$$= (Y_4^{\alpha_1} \times Z_2) \cup (Y_2^{\alpha_1} \times Z_3) \cup ((Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}) \times \breve{D})$$
$$= (Y_2^{\alpha} \times Z_2) \cup (Y_3^{\alpha} \times Z_3) \cup (Y_0^{\alpha} \times \breve{D}).$$

7. Let $V(X^*, \alpha) = \{Z_4, Z_3, Z_1, \breve{D}\}$. Then binary relation α has representation of the form $\alpha = (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_3 & Z_4 & Z_1 & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_4, Z_3, Z_1\}$. Then

$$\overline{\beta} = (P_0 \times \varnothing) \cup (P_1 \times Z_3) \cup (P_2 \times Z_4) \cup (P_3 \times Z_1) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} (\{t'\} \times f(t')),$$

$$\alpha_1 = (Y_4^{\alpha_1} \times Z_4) \cup (Y_3^{\alpha_1} \times Z_3) \cup (Y_2^{\alpha_1} \times Z_2) \cup (Y_1^{\alpha_1} \times Z_1) \cup (Y_0^{\alpha_1} \times \breve{D}),$$

(16)

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_4^\alpha$, $Y_3^{\alpha_1} = Y_3^\alpha$, $Y_1^{\alpha_1} = Y_1^\alpha$ and $Y_2^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^\alpha$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V(D, \alpha_1) = V(D, \overline{\beta}) = D$. From the formal equality and equalities (16) and (5) we have:

$$Z_4 \overline{\beta} = (P_0 \cup P_2)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} = \varnothing \cup Z_4 = Z_4;$$

$$Z_3 \overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_3 \cup Z_4 \cup \varnothing = Z_3;$$

$$Z_2 \overline{\beta} = (P_0 \cup P_1 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_3 \cup Z_1 \cup \varnothing = \breve{D};$$

$$Z_1 \overline{\beta} = (P_0 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_4 \cup Z_1 \cup \varnothing = Z_1;$$

$$\breve{D} \overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta}$$

$$= \varnothing \cup Z_3 \cup Z_4 \cup Z_1 \cup \varnothing = \breve{D};$$

$$\alpha = \alpha_1 \circ \overline{\beta}$$

$$= (Y_4^{\alpha_1} \times Z_4\overline{\beta}) \cup (Y_3^{\alpha_1} \times Z_3\overline{\beta}) \cup (Y_2^{\alpha_1} \times Z_2\overline{\beta}) \cup (Y_1^{\alpha_1} \times Z_1\overline{\beta}) \cup (Y_0^{\alpha_1} \times \breve{D}\overline{\beta})$$

$$= (Y_4^{\alpha_1} \times Z_4) \cup (Y_3^{\alpha_1} \times Z_3) \cup (Y_2^{\alpha_1} \times \breve{D}) \cup (Y_1^{\alpha_1} \times Z_1) \cup (Y_0^{\alpha_1} \times \breve{D})$$

$$= (Y_4^{\alpha_1} \times Z_4) \cup (Y_3^{\alpha_1} \times Z_3) \cup (Y_1^{\alpha_1} \times Z_1) \cup ((Y_2^{\alpha_1} \cup Y_0^{\alpha_1}) \times \breve{D})$$

$$= (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}).$$

$\square$

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