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Iterative Technology in a Singular Fractional Boundary Value Problem with q-Difference

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Abstract

In this paper, we apply the iterative technology to establish the existence of solutions for a fractional boundary value problem with q-difference. Explicit iterative sequences are given to approxinate the solutions and the error estimations are also given.

Keywords

Fractional Boundary Value Problem with q-Difference, Iterative Sequence, Green's Function, Error Estimation

1. Introduction

This paper deals with the existence of solutions for the following fractional boundary value problem with q-difference

$$\begin{cases} D_q^\alpha x(t) + f(t, x(t)) = 0, \ t \in (0,1), \\ x(0) = (D_q x)(0) = 0, \ (D_q) x(1) = b, \end{cases} \tag{1.1}$$

where $0 < q < 1$, $2 < \alpha \le 3$ and $f(t, u)$ may be singular at $u = 0, t = 0$ (and/or $t = 1$).

Fractional differential equations have been of great interest recently because of their intensive applications in economics, financial mathematics and other applied science (see [1]-[13] and the references therein). The q-difference calculus or quantum calculus is an old subject and is rich in history and in applications. In recent years, there have been papers investigating the existence and uniqueness of the positive solution for the fractional boundary value problem with q-difference (see [1]-[4] and the references therein).

For problem (1.1), there have been paid attention to the existences of solutions. Rui [1] investigated the exi-

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stence of positive solutions by applying a fixed point theorem in cones. By fixed point theorem again, Li and Han [2] considered a similar fractional q-difference equations given as

$$D_q^\alpha x(t) + \lambda h(t) f(x(t)) = 0, \ t \in (0,1),$$

subject to the boundary conditions $x(0) = (D_q x)(0) = (D_q)x(1) = 0$. In this work, we will apply the iterative technology ([9] [11] [14]), and as far as we know, there are few papers to establish the existence of solutions by the iterative technology for the boundary value problem with q-difference.

Motivated by the work mentioned above, with the iterative technology and properties of $f(t, x)$, explicit iterative sequences are given to approximate the solutions and the error estimations are also given.

2. Preliminaries and Some Lemmas

In this section, we introduce some definitions and lemmas.

Definition 2.1 [1]. *Let* $\alpha \geq 0$, $q \in (0,1)$ *and f be a function defined on* $[0,1]$. *The fractional q-integral of the Riemann-Liouville type is defined by* $(I_q^0 f)(x) = f(x)$ *and*

$$\left(I_q^\alpha f\right)(x) = \Gamma_q(\alpha) \int_0^x (x - qt)^{(\alpha-1)} f(t) \mathrm{d}_q t, \alpha > 0, \ x \in [0,1].$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$\left(I_q f\right)(x) = \int_0^x f(t) \mathrm{d}_q t = x(1-q) \sum_{n=0}^\infty f(xq^n) q^n, x \in [0,b],$$

and q-integral of higher order I_q^n is defined by

$$\left(I_q^0 f\right)(x) = f(x), \left(I_q^n\right) f(x) = I_q\left(I_q^{n-1} f\right)(x), n \in N.$$

Remark 1: $\alpha \in R$, $(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^\infty \dfrac{a - bq^n}{a - bq^{\alpha+n}}$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in R \setminus \{0, -1, -2, \cdots\}, \quad \text{and} \quad \text{satisfies} \quad \Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \text{where} \quad [x]_q = \frac{1-q^x}{1-q},$$

$q \in (0,1)$.

Definition 2.2 [1]. *Let* $\alpha \geq 0$, $q \in (0,1)$. *The fractional q-derivative of the Riemann-Liouville type of order* α *is defined by* $(D_q^0 f)(x) = f(x)$ *and*

$$\left(D_q^\alpha f\right)(x) = \left(D_q^m I_q^{m-\alpha} f\right)(x), \alpha > 0,$$

where m is the smallest integer greater than or equal to α. The q-derivative of a function f is defined by

$$\left(D_q f\right)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \ \left(D_q f\right)(0) = \lim_{x \to 0} \left(D_q f\right)(x),$$

and q-derivatives of higher order by

$$\left(D_q^0 f\right)(x) = f(x), \left(D_q^n f\right)(x) = D_q\left(D_q^{n-1} f\right)(x), \ n \in N.$$

Lemma 2.1 [1]. *Suppose* $2 < \alpha \leq 3, 0 < q < 1, b \geq 0$ *and* $h(t)$ *is q-integrable on* $(0,1)$. *Then the boundary value problem*

$$\begin{cases} \left(D_q^\alpha x\right)(t) + h(t) = 0, \ t \in (0,1), \\ x(0) = (D_q x)(0) = 0, \ (D_q x)(1) = b, \end{cases}$$

has the unique solution

$$x(t) = g(t) + \int_0^1 G(t, qs) h(s) \mathrm{d}_q s,$$

where

$$g(t) = \frac{b}{[\alpha-1]_q} t^{\alpha-1}, \tag{2.1}$$

$$G(t,qs) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (1-qs)^{(\alpha-2)} t^{\alpha-1} - (t-qs)^{(\alpha-1)}, & 0 \le qs \le t \le 1, \\ (1-qs)^{(\alpha-2)} t^{\alpha-1}, & 0 \le t \le qs \le 1. \end{cases} \tag{2.2}$$

Lemma 2.2 [1]. *Function G defined as* (2.2). *Then G satisfies the following properties*:
(1) $G(t,qs) \ge 0$, and $G(t,qs) \le G(1,qs)$ for all $t,s \in [0,1]$.
(2) $t^{\alpha-1}G(1,qs) \le G(t,qs)$ for all $t,s \in [0,1]$.
Lemma 2.3. *Function G defined as* (2.2). *Then*

$$t^{\alpha-1}G(1,qs) \le G(t,qs) \le \frac{1}{\Gamma_q(\alpha)} t^{\alpha-1}.$$

Proof. Note that (2.2) and $0 \le (1-qs)^{(\alpha-2)} \le 1$, it follows that $G(t,qs) \le \frac{1}{\Gamma_q(\alpha)} t^{\alpha-1}$ for all $t,s \in [0,1]$.

This, with Lemma 2.2, implies that

$$t^{\alpha-1}G(1,qs) \le G(t,qs) \le \frac{1}{\Gamma_q(\alpha)} t^{\alpha-1}.$$

3. Main Result

First, for the existence results of problem (1.1), we need the following assumptions.
(A$_1$) $f(t,x):(0,1)\times(0,+\infty) \to [0,+\infty)$ is continuous.
(A$_2$) For $(t,x) \in (0,1)\times(0,+\infty)$, f is non-decreasing respect to x and for any $\xi \in (0,1)$, there exists a constant $\lambda \in (0,1)$ such that

$$f(t,\xi x) \ge \xi^\lambda f(t,x). \tag{3.1}$$

Then, we let the Banach space $E = C[0,1]$, $P = \{x \in E : x(t) \ge 0, t \in [0,1]\}$ and

$$Q = \{x \in P : \text{there exist a positive constant } m \in (0,1),$$

$$\text{such that } mt^{\alpha-1} \le x(t) \le m^{-1}t^{\alpha-1} \text{ for } t \in [0,1]\}.$$

Clearly P is a normal cone and Q is a subset of P in the Banach space E.
In what follows, we define the operator $T:Q \to E$

$$Tx(t) = g(t) + \int_0^1 G(t,qs) f(s,x(s)) d_q s, \tag{3.2}$$

where $g(t), G(t,qs)$ are given by (2.1) and (2.2).
Now, we are in the position to give the main results of this work.
Theorem 3.1. *Suppose* (A$_1$), (A$_2$) *hold. Then problem* (1.1) *has at least one positive solution* $x^*(t)$ *in Q if*

$$0 < \int_0^1 f(t,t^{\alpha-1}) d_q t < \infty. \tag{3.3}$$

Proof. We shall prove the existence of solution in three steps.
Step 1. The operator T defined in (3.2) is $T:Q \to Q$.
For any $x(t) \in Q$, there exists a positive constant $m \in (0,1)$ such that

$$mt^{\alpha-1} \le x(t) \le m^{-1}t^{\alpha-1}, t \in [0,1].$$

Then from (A$_2$): $f(t,x)$ is non-decreasing respect to x and (3.1), we can imply that for $s \in (0,1), \lambda \in (0,1)$,

$$m^\lambda f(s,s^{\alpha-1}) \le f(s,ms^{\alpha-1}) \le f(s,x(s)) \le f(s,m^{-1}s^{\alpha-1}) \le m^{-\lambda} f(s,s^{\alpha-1}), \tag{3.4}$$

where

$$f\left(s, m^{-1}s^{\alpha-1}\right) \le m^{-\lambda} f\left(s, s^{\alpha-1}\right)$$

is implied by the equivalent form to (3.1): if $M > 1$,

$$f\left(t, Mu\right) \le M^{\lambda} f\left(t, u\right), \text{ for all } \left(t, u\right) \in \left(0,1\right) \times \left(0, +\infty\right).$$

From (3.4) and Lemma 2.3, we can have

$$Tx(t) = g(t) + \int_0^1 G(t, qs) f\left(s, x(s)\right) d_q s$$

$$\le \frac{b}{\left[\alpha-1\right]_q} t^{\alpha-1} + \frac{1}{\Gamma_q(\alpha)} \int_0^1 t^{\alpha-1} f\left(s, x(s)\right) d_q s$$

$$\le t^{\alpha-1} \left(\frac{b}{\left[\alpha-1\right]_q} + \frac{1}{\Gamma_q(\alpha)} m^{-\lambda} \int_0^1 f\left(s, s^{\alpha-1}\right) d_q s\right) \le c_x^{-1} t^{\alpha-1},$$

and

$$Tx(t) = g(t) + \int_0^1 G(t, qs) f\left(s, x(s)\right) d_q s$$

$$\ge \frac{b}{\left[\alpha-1\right]_q} t^{\alpha-1} + t^{\alpha-1} \int_0^1 G(1, qs) f\left(s, x(s)\right) d_q s$$

$$\ge \frac{b}{\left[\alpha-1\right]_q} t^{\alpha-1} \ge c_x t^{\alpha-1},$$

where c_x:

$$0 < c_x < \min\left\{1, \frac{b}{\left[\alpha-1\right]_q}, \frac{1}{\frac{b}{\left[\alpha-1\right]_q} + \frac{1}{\Gamma_q(\alpha)} m^{-\lambda} \int_0^1 f\left(s, s^{\alpha-1}\right) d_q s}\right\} \le 1.$$

This implies T is $Q \to Q$.

Step 2. There exist iterative sequences $\{x_n\}$, $\{y_n\}$ satisfying

$$x_0(t) \le x_1(t) \le \cdots \le x_n(t) \le \cdots \le y_n(t) \le \cdots \le y_1(t) \le y_0(t), \forall t \in \left[0,1\right].$$

Since $T\varphi \in Q$ for $\varphi(t) = t^{\alpha-1}, t \in \left[0,1\right]$, there exists a constant $c_\varphi \in (0,1)$ such that

$$c_\varphi \varphi(t) \le T\varphi(t) \le c_\varphi^{-1} \varphi(t). \tag{3.5}$$

For c_φ defined in (3.5), there exist constants δ, γ satisfying

$$0 < \delta < c_\varphi^{\frac{1}{1-\lambda}}, \gamma \ge c_\varphi^{-\frac{1}{1-\lambda}}. \tag{3.6}$$

Let

$$x_0(t) = \delta\varphi(t), \ y_0(t) = \gamma\varphi(t), \tag{3.7}$$

$$x_n(t) = Tx_{n-1}(t), \ y_n(t) = Ty_{n-1}(t). \tag{3.8}$$

Then it follows that

$$x_0(t) \le x_1(t) \le \cdots \le x_n(t) \le \cdots \le y_n(t) \le \cdots \le y_1(t) \le y_0(t), \forall t \in \left[0,1\right].$$

In fact, from (3.6)-(3.8), we have

$$x_0(t) \le y_0(t), \tag{3.9}$$

$$x_1(t) = Tx_0(t) = g(t) + \int_0^1 G(t, qs) f(s, \delta\varphi(s)) \mathrm{d}_q s$$

$$\geq g(t) + \delta^\lambda \int_0^1 G(t, qs) f(s, \varphi(s)) \mathrm{d}_q s \tag{3.10}$$

$$\geq \delta^\lambda T\varphi(t) \geq \delta^\lambda c_\varphi \varphi(t) \geq \delta\varphi(t) = x_0(t),$$

$$y_1(t) = Ty_0(t) = g(t) + \int_0^1 G(t, qs) f(s, \gamma\varphi(s)) \mathrm{d}_q s$$

$$\leq g(t) + \gamma^\lambda \int_0^1 G(t, qs) f(s, \varphi(s)) \mathrm{d}_q s \tag{3.11}$$

$$\leq \gamma^\lambda T\varphi(t) \geq \gamma^\lambda c_\varphi^{-1} \varphi(t) \geq \gamma\varphi(t) = y_0(t).$$

Then, by (3.9)-(3.11), (A$_2$) and induction, the iterative sequences $\{x_n\}$, $\{y_n\}$ satisfy

$$x_0(t) \leq x_1(t) \leq \cdots \leq x_n(t) \leq \cdots \leq y_n(t) \leq \cdots \leq y_1(t) \leq y_0(t), \forall t \in [0,1].$$

Step 3. There exists $x^* \in Q$ such that

$$x_n(t) \to x^*(t), \ y_n(t) \to x^*(t), \text{ uniformly on } [0,1].$$

Note that $x_0(t) = \dfrac{\delta}{\gamma} y_0(t)$. By induction it is easy to obtain

$$x_n(t) \geq \left(\frac{\delta}{\gamma}\right)^{\lambda^n} y_n(t).$$

Thus, for $\forall n, p \in N$, we have

$$0 \leq x_{n+p}(t) - x_n(t) \leq y_n(t) - x_n(t) \leq \left(1 - \left(\frac{\delta}{\gamma}\right)^{\lambda^n}\right) y_n(t)$$

$$\leq \left(1 - \left(\frac{\delta}{\gamma}\right)^{\lambda^n}\right) y_0(t) \to 0, \text{ as } n \to \infty. \tag{3.12}$$

This yields that there exists $x^* \in Q$ such that

$$x_n(t) \to x^*(t) \text{ uniformly on } [0,1].$$

Moreover, from (3.12) and

$$0 \leq y_n(t) - x^*(t) = y_n(t) - x_n(t) + x_n(t) - x^*(t),$$

we have

$$y_n(t) \to x^*(t) \text{ uniformly on } [0,1].$$

Letting $n \to \infty$ in (3.8), $x^* \in Q$ is a fixed point of T. That is, $x^*(t)$ is a positive solution of problem (1.1).

Theorem 3.2. *Suppose the conditions hold in Theorem 3.1. Then for any initial $x_0 \in Q$, there exists a sequence $\{x_n\}$ such that $\{x_n(t) \to x^*(t)\}$ uniformly on $[0,1]$ as $n \to \infty$, where $x^*(t)$ is the positive solution of problem* (1.1). *And the error estimation for the sequence $\{x_n(t)\}$ is*

$$\max_{t \in [0,1]} |x_n(t) - x^*(t)| = O\left(1 - k^{\lambda^n}\right), \tag{3.13}$$

where k is a constant with $0 < k < 1$ and determined by x_0.

Proof. Let $x_0 \in Q$ be given. Then there exists a constant $c_\varphi \in (0,1)$ such that

$$c_\varphi \varphi(t) \leq Tx_0(t) \leq c_\varphi^{-1} \varphi(t). \tag{3.14}$$

For c_φ defined in (3.14), choose constants δ, γ such that

$$0 < \delta < c_\varphi^{\frac{1}{1-\lambda}}, \ \gamma \geq c_\varphi^{\frac{1}{1-\lambda}}.$$

Then define $x_0(t), x_n(t)$ as (3.7), (3.8), and we can have $\{x_n(t)\}$ converges uniformly to the positive solution $x^*(t)$ of problem (1.1) on $[0,1]$ as $n \to \infty$.

For the error estimation (3.13), it can be obtained by letting $p \to \infty$ in (3.12).

Example 3.3. Consider the function

$$f(t,x) = t^{-\frac{1}{2}} x^{\frac{2}{3} - \frac{1}{6}\sin t}, \ (t,x) \in (0,1) \times (0,\infty),$$

$f(t,x)$ satisfies (A$_2$) and is singular at $t = 0$. Let $q = \frac{1}{2}$, $\alpha = 2.5$. Then

$$\int_0^1 f(t, t^{\alpha-1}) \mathrm{d}_q t \leq \int_0^1 t^{\frac{1}{4}} d_q t = \frac{1}{1 - 2^{-\frac{5}{4}}} < \infty.$$

By Theorem 3.1, the following problem

$$\begin{cases} D_{0.5}^{2.5} x(t) + t^{-\frac{1}{2}} x^{\frac{2}{3} - \frac{1}{6}\sin t} = 0, \ t \in (0,1), \\ x(0) = (D_{0.5} x)(0) = 0, \ (D_{0.5}) x(1) = 1, \end{cases}$$

has at least one positive solution.

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