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# Some Sum Formulas of (*s*,*t*)-Jacobsthal and (*s*,*t*)-Jacobsthal Lucas Matrix Sequences

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## Abstract

**In this study, we first give the definitions of (*s*,*t*)-Jacobsthal and (*s*,*t*)-Jacobsthal Lucas sequence. By using these formulas we define (*s*,*t*)-Jacobsthal and (*s*,*t*)-Jacobsthal Lucas matrix sequences. After that we establish some sum formulas for these matrix sequences.**

## Keywords

**Jacobsthal Numbers, Jacobsthal Lucas Numbers, Matrix Sequences**

## 1. Introduction

There are so many studies in the literature that are concern about special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, and Padovan in [1] [2]. They are widely used in many research areas as Engineering, Architecture, Nature and Art in [3]-[6]. For example, microcontrollers (and other computers) use conditional instructions to change the flow of execution of a program. In addition to branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction. This winds up being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 21 on 6 bits, 43 on 7 fits, 85 on 8 fits, ⋯, which are exactly the Jacosthal numbers [7]. Jacobsthal and Jacobsthal Lucas numbers are given by the recurrence relations $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 0$, $j_1 = 1$ and $c_n = c_{n-1} + 2c_{n-2}$, $c_0 = 2$, $c_1 = 1$ for $n \geq 2$, respectively in [7]-[9]. Generalization of number sequences is studied in many articles. For example the generalization of Jacobsthal sequences is defined in [10]. We can see any properties of these numbers in [7]-[9] [11] [12]. Some properties of these sequences were deduced directly from elementary matrix algebra in [13] [14]. By using matrix algebra H. Civciv and R. Turkmen defined $(s,t)$ Fibonacci and $(s,t)$ Lucas matrix sequences in [15] [16]. Similarly K. Uslu and Ş. Uygun defined $(s,t)$ Jacosthal and $(s,t)$ Jacosthal Lucas matrix sequences and by using them found some properties of Jacobsthal numbers in [17].

**Definition 1.** *The* (*s*,*t*)*-Jacobsthal sequence* $\left\{\hat{j}_n(s,t)\right\}_{n\in\mathbb{N}}$ *and* (*s*,*t*)*-Jacobsthal Lucas sequence* $\left\{\hat{c}_n(s,t)\right\}_{n\in N}$

*are defined by the recurrence relations*

$$\hat{j}_{n+1}(s,t) = s\hat{j}_n(s,t) + 2t\hat{j}_{n-1}(s,t),\ \ \hat{j}_0(s,t) = 0,\ \hat{j}_1(s,t) = 1 \quad (1)$$

$$\hat{c}_{n+1}(s,t) = s\hat{c}_n(s,t) + 2t\hat{c}_{n-1}(s,t),\ \ \hat{c}_0(s,t) = 2,\ \hat{c}_1(s,t) = s \quad (2)$$

respectively, where $n \geq 1$, $s > 0, t \neq 0$ and $s^2 + 8t > 0$ [10].

Some basic properties of these sequences are given in the following:

$$\hat{c}_n(s,t) = s\hat{j}_n(s,t) + 4t\hat{j}_{n-1}(s,t),$$

$$\hat{j}_n = \frac{r_1^n - r_2^n}{r_1 - r_2},\ \ \hat{c}_n = r_1^n + r_2^n,$$

$$r_1 = \frac{s + \sqrt{s^2 + 8t}}{2},\ \ r_2 = \frac{s - \sqrt{s^2 + 8t}}{2},$$

$$r_1^2 = sr_1 + 2t,\ \ \ r_2^2 = sr_2 + 2t,\ \ r_1 \cdot r_2 = -2t,\ \ r_1 + r_2 = s.$$

In the following definition, (*s*,*t*)-Jacosthal $\{J_n(s,t)\}_{n\in\mathbb{N}}$ and (*s*,*t*)-Jacosthal Lucas $\{C_n(s,t)\}_{n\in\mathbb{N}}$ matrix sequences are defined by carrying to matrix theory (*s*,*t*)-Jacosthal and (*s*,*t*)-Jacosthal Lucas sequences.

**Definition 2.** *The* (*s*,*t*)*-Jacobsthal matrix sequence* $\{J_n(s,t)\}_{n\in\mathbb{N}}$ *and* (*s*,*t*)*-Jacobsthal Lucas matrix sequence* $\{C_n(s,t)\}_{n\in N}$ *are defined by the recurrence relations*

$$J_{n+1}(s,t) = sJ_n(s,t) + 2tJ_{n-1}(s,t),\ J_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ J_1 = \begin{bmatrix} s & 2 \\ t & 0 \end{bmatrix} \quad (3)$$

$$C_{n+1}(s,t) = sC_n(s,t) + 2tC_{n-1}(s,t), C_0 = \begin{bmatrix} s & 4 \\ 2t & -s \end{bmatrix}, C_1 = \begin{bmatrix} s^2 + 4t & 2s \\ st & 4t \end{bmatrix} \quad (4)$$

respectively, where $n \geq 1$, $s > 0, t \neq 0$ and $s^2 + 8t > 0$.

Throughout this paper, for convenience we will use the symbol $\hat{j}_n$ instead of $\hat{j}_n(s,t)$ and the symbol $\hat{c}_n$ instead of $\hat{c}_n(s,t)$. Similarly we will use the symbol $J_n$ instead of $J_n(s,t)$ and $C_n$ instead of $C_n(s,t)$.

**Proposition 3.** *Let us consider* $s > 0, t \neq 0$ *and* $s^2 + 8t > 0$. *The following properties are hold*:

1) $J_n = \begin{bmatrix} \hat{j}_{n+1} & 2\hat{j}_n \\ t\hat{j}_n & 2t\hat{j}_{n-1} \end{bmatrix}$ and $C_n = \begin{bmatrix} \hat{c}_{n+1} & 2\hat{c}_n \\ t\hat{c}_n & 2t\hat{c}_{n-1} \end{bmatrix}$.

2) For $m, n \in \mathbb{Z}^+$,

$$J_{n+m} = J_n J_m,$$
$$J_n = J_1^n,$$
$$J_m C_{n+1} = C_{n+1} J_m,$$
$$C_n = sJ_n + 4tJ_{n-1},$$
$$C_{n+1} = C_1 J_n.$$

3) For $n \in \mathbb{Z}^+$, $J_n = \left(\frac{J_1 - r_2 J_0}{r_1 - r_2}\right) r_1^n - \left(\frac{J_1 - r_1 J_0}{r_1 - r_2}\right) r_2^n$.

4) For $n \in \mathbb{Z}^+$, $C_{n+1} = \left(\frac{C_2 - C_1 r_2}{r_1 - r_2}\right) r_1^n - \left(\frac{C_2 - C_1 r_1}{r_1 - r_2}\right) r_2^n$.

For their proofs you can look at the Ref. [17].

## 2. The Generating Functions of Jacobsthal and Jacobsthal-Lucas Matrix Sequences

**Theorem 4.** *For* $n \in \mathbb{Z}^+$, $x \in \mathbb{R}$, *we have the generating function of Jacobsthal matrix sequence in the following*:

$$\sum_{k=1}^{n} J_k x^{-k} = -\frac{1}{x^n\left(x^2 - sx - 2t\right)}\left[xJ_{n+1} + 2tJ_n\right] + \frac{1}{\left(x^2 - sx - 2t\right)}\left[xJ_1 + \left(x^2 - sx\right)J_0\right]. \quad (5)$$

**Proof.** By using the expansion of geometric series and proposition 3, we can write

$$\begin{aligned}
\sum_{k=1}^{n} \frac{J_k}{x^k} &= \sum_{k=1}^{\infty} \frac{1}{x^k}\left[\left(\frac{J_1 - r_2 J_0}{r_1 - r_2}\right) r_1^k - \left(\frac{J_1 - r_1 J_0}{r_1 - r_2}\right) r_2^k\right] \\
&= \left(\frac{J_1 - r_2 J_0}{r_1 - r_2}\right)\sum_{k=1}^{n}\left(\frac{r_1}{x}\right)^k - \left(\frac{J_1 - r_1 J_0}{r_1 - r_2}\right)\sum_{k=1}^{n}\left(\frac{r_2}{x}\right)^k \\
&= \left(\frac{J_1 - r_2 J_0}{r_1 - r_2}\right)\left(\frac{1 - \left(\frac{r_1}{x}\right)^{n+1}}{1 - \left(\frac{r_1}{x}\right)}\right) - \left(\frac{J_1 - r_1 J_0}{r_1 - r_2}\right)\left(\frac{1 - \left(\frac{r_2}{x}\right)^{n+1}}{1 - \left(\frac{r_2}{x}\right)}\right) \\
&= \frac{1}{x^n}\left[\left(\frac{J_1 - r_2 J_0}{r_1 - r_2}\right)\left(\frac{x^{n+1} - r_1^{n+1}}{x - r_1}\right) - \left(\frac{J_1 - r_1 J_0}{r_1 - r_2}\right)\left(\frac{x^{n+1} - r_2^{n+1}}{x - r_2}\right)\right] \\
&= \frac{1}{x^n\left(x^2 - sx - 2t\right)}\left[\left(\frac{J_1 - r_2 J_0}{r_1 - r_2}\right)\left(x^{n+1} - r_1^{n+1}\right)\left(x - r_2\right)\right. \\
&\qquad \left. - \left(\frac{J_1 - r_1 J_0}{r_1 - r_2}\right)\left(x^{n+1} - r_2^{n+1}\right)\left(x - r_1\right)\right] \\
&= \frac{1}{x^n\left(x^2 - sx - 2t\right)}\left\{-x\left[\left(\frac{J_1 - r_2 J_0}{r_1 - r_2}\right) r_1^{n+1} - \left(\frac{J_1 - r_1 J_0}{r_1 - r_2}\right) r_2^{n+1}\right]\right\} \\
&\quad + \frac{1}{x^n\left(x^2 - sx - 2t\right)}\left\{r_1 \cdot r_2\left[\left(\frac{J_1 - r_2 J_0}{r_1 - r_2}\right) r_1^n - \left(\frac{J_1 - J_0 r_1}{r_1 - r_2}\right) r_2^n\right]\right. \\
&\quad \left. + \frac{x^{n+1}\left(r_1 - r_2\right)J_1 + x^{n+2}\left(r_1 - r_2\right)J_0 + \left(-r_1^2 + r_2^2\right)x^n J_0}{x^n\left(r_1 - r_2\right)\left(x^2 - sx - 2t\right)}\right\} \\
&= -\frac{1}{x^n\left(x^2 - sx - 2t\right)}\left[xJ_{n+1} + 2tJ_n\right] + \frac{1}{\left(x^2 - sx - 2t\right)}\left[xJ_1 + \left(x^2 - s\right)J_0\right]
\end{aligned}$$

■

**Corollary 5.** *Let* $x \in R, x > \frac{s + \sqrt{s^2 + 8t}}{2}$. *Then for* (*s*,*t*)*-Jacobsthal sequence we have*

$$\sum_{k=1}^{\infty} \hat{j}_{k+1} x^{-k} = \frac{x^2 + sx - s}{\left(x^2 - sx - 2t\right)}.$$

and

$$\sum_{k=1}^{\infty} \hat{j}_k x^{-k} = \frac{x}{\left(x^2 - sx - 2t\right)}.$$

**Corollary 6.** *Let* $x \in R, x > \frac{s + \sqrt{s^2 + 8t}}{2}$. *Then we have*

$$\sum_{k=1}^{\infty} J_k x^{-k} = \frac{1}{\left(x^2 - sx - 2t\right)}\left[xJ_1 + \left(x^2 - s\right)J_0\right].$$

**Corollary 7.** *Let* $n \in Z^{+}, x \in R.$ *Then we have we have the generating function of Jacobsthal-Lucas matrix sequence in the following*:

$$\sum_{k=1}^{n} C_{k+1} x^{-k} = -\frac{1}{x^{n}\left(x^{2}-sx-2t\right)}\left[xC_{n+2}+2tC_{n+1}\right]+\frac{1}{\left(x^{2}-sx-2t\right)}\left[xC_{2}+\left(x^{2}-s\right)C_{1}\right]. \tag{6}$$

**Proof.** It can be seen easily by using theorem 4 and the property of $C_{n+1} = C_1 J_n$. ■

**Corollary 8.** *Let* $x \in R, x > \frac{s+\sqrt{s^{2}+8t}}{2}$. *Then for (s,t)-Jacobsthal Lucas matrix sequence we have*

$$\sum_{k=1}^{\infty} C_{k+1} x^{-k} = \frac{1}{\left(x^{2}-sx-2t\right)}\left[xC_{2}+\left(x^{2}-s\right)C_{1}\right].$$

**Corollary 9.** *Let* $x \in R, x > \frac{s+\sqrt{s^{2}+8t}}{2}$. *Then for (s,t)-Jacobsthal Lucas sequence we have*

$$\sum_{k=1}^{\infty} \hat{c}_{k+1} x^{-k} = \frac{s^{2}x+sx^{2}+4xt-s^{2}}{x^{2}-sx-2t}$$

and

$$\sum_{k=1}^{\infty} \hat{c}_{k} x^{-k} = \frac{2x^{2}+sx-2st}{x^{2}-sx-2t}.$$

**Theorem 10.** *For* $\left|r_1^k r_2^{r-k} x\right| < 1$, *let be r is odd positive integer and* $X = \left(\frac{J_1 - J_0 r_2}{r_1 - r_2}\right)$, $Y = \left(\frac{J_1 - J_0 r_1}{r_1 - r_2}\right)$.

Then we have

$$\sum_{i=0}^{\infty} J_i^r x^i = \sum_{i=0}^{\frac{r-1}{2}}\left[(-1)^k \binom{r}{k} X^k Y^k \frac{X^{r-2k} - Y^{r-2k} + (-2t)^k \left(Y^{r-2k} r_1^{r-2k} - X^{r-2k} r_2^{r-2k}\right)x}{1-(-2t)^k \hat{c}_{r-2k} x + (-2t)^r x^2}\right],$$

and for *r* is even positive integer

$$\sum_{i=0}^{\infty} J_i^r x^i = \sum_{i=0}^{\frac{r}{2}-1}\left[(-1)^k \binom{r}{k} X^k Y^k \frac{X^{r-2k} + Y^{r-2k} - (-2t)^k \left(Y^{r-2k} r_1^{r-2k} + X^{r-2k} r_2^{r-2k}\right)x}{1-(-2t)^k \hat{c}_{r-2k} x + (-2t)^r x^2}\right]$$
$$+\binom{r}{r/2} Y^k \frac{X^{r/2}(-Y)^{r/2}}{1-(-2t)^{r/2} x}.$$

**Proof.** By using proposition 3 (iv), the *n*th element of (*s*,*t*)-Jacobsthal matrix sequence can be written in the following:

$$J_n = X r_1^n - Y r_2^n.$$

From this equality we have

$$\sum_{i=0}^{\infty}\left(X r_1^i - Y r_2^i\right)^r x^i = \sum_{i=0}^{\infty}\left(\sum_{k=0}^{r}\binom{r}{k}\left(X r_1^i\right)^k \left(-Y r_2^i\right)^{r-k}\right)x^i$$
$$= \sum_{k=0}^{r}\binom{r}{k}(X)^k (-Y)^{r-k} \sum_{i=0}^{\infty}\left(r_1^k r_2^{r-k} x\right)^i$$

$$U(r,x) = \sum_{k=0}^{r}\binom{r}{k}(X)^k (-Y)^{r-k} \frac{1}{1-r_1^k r_2^{r-k} x}.$$

If *r* is an odd positive integer, then we have

$$U(r,x)=\sum_{k=0}^{\frac{r-1}{2}}\binom{r}{k}\left(\frac{(X)^k(-Y)^{r-k}}{1-r_1^k r_2^{r-k}x}+\frac{(X)^{r-k}(-Y)^k}{1-r_1^{r-k}r_2^k x}\right)$$

$$=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\left(\frac{X^{r-k}Y^k}{1-r_1^{r-k}r_2^k x}-\frac{X^kY^{r-k}}{1-r_1^k r_2^{r-k}x}\right)$$

$$=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\frac{X^{r-k}Y^k-X^kY^{r-k}+\left(X^kY^{r-k}r_1^{r-k}r_2^k-X^{r-k}Y^k r_1^k r_2^{r-k}\right)x}{1-\left(r_1^k r_2^{r-k}+r_1^{r-k}r_2^k\right)x+r_1^r r_2^r x^2}$$

$$=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\frac{X^{r-k}Y^k-X^kY^{r-k}+(-2t)^k\left(X^kY^{r-k}r_1^{r-2k}-X^{r-k}Y^k r_2^{r-2k}\right)x}{1-(-2t)^k\left(r_2^{r-2k}+r_1^{r-2k}\right)x^r+(-2t)^r x^2}$$

$$=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k X^kY^k\binom{r}{k}\frac{X^{r-2k}-Y^{r-2k}+(-2t)^k\left(Y^{r-2k}r_1^{r-2k}-X^{r-2k}r_2^{r-2k}\right)x}{1-(-2t)^k\hat{c}_{r-2k}x+(-2t)^r x^2}.$$

If $r$ is an even positive integer, then we have

$$U(r,x)=\sum_{k=0}^{\frac{r}{2}-1}\binom{r}{k}\left(\frac{(X)^k(-Y)^{r-k}}{1-r_1^k r_2^{r-k}x}+\frac{(X)^{r-k}(-Y)^k}{1-r_1^{r-k}r_2^k x}\right)+\binom{r}{r/2}\frac{(X)^{r/2}(-Y)^{r/2}}{1-(-2t)^{r/2}x}$$

$$=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\left(\frac{X^{r-k}Y^k}{1-r_1^{r-k}r_2^k x}+\frac{X^kY^{r-k}}{1-r_1^k r_2^{r-k}x}\right)+\binom{r}{r/2}\frac{(X)^{r/2}(-Y)^{r/2}}{1-(-2t)^{r/2}x}$$

$$=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\left(\frac{X^{r-k}Y^k+X^kY^{r-k}-\left(X^kY^{r-k}r_1^{r-k}r_2^k+X^{r-k}Y^k r_1^k r_2^{r-k}\right)x}{1-\left(r_1^k r_2^{r-k}+r_1^{r-k}r_2^k\right)x+r_1^r r_2^r x^2}\right)+\binom{r}{r/2}\frac{(X)^{r/2}(-Y)^{r/2}}{1-(-2t)^{r/2}x}$$

$$=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\left(\frac{X^{r-k}Y^k+X^kY^{r-k}-(-2t)^k\left(X^kY^{r-k}r_1^{r-2k}+X^{r-k}Y^k r_1^k r_2^{r-2k}\right)x}{1-(-2t)^k(r_2^{r-2k}+r_1^{r-2k})x+t^r x^2}\right)+\binom{r}{r/2}\frac{(X)^{r/2}(-Y)^{r/2}}{1-(-2t)^{r/2}x}$$

$$=\sum_{k=0}^{\frac{r-1}{2}}(-1)^k X^kY^k\binom{r}{k}\left(\frac{X^{r-2k}+Y^{r-2k}-(-2t)^k\left(Y^{r-2k}r_1^{r-2k}+X^{r-2k}r_2^{r-2k}\right)x}{1-(-2t)^k C_{r-2k}x+t^r x^2}\right)+\binom{r}{r/2}\frac{(X)^{r/2}(-Y)^{r/2}}{1-(-2t)^{r/2}x}.$$

■

## 3. Partial Sums of Jacobsthal and Jacobsthal-Lucas Matrix Sequences

**Theorem 11.** *The partial sum of* $(s,t)$*-Jacobsthal matrix sequence for* $s+2t\neq 1$ *is given in the following*

$$\sum_{k=1}^{n}J_k=\begin{bmatrix}\hat{j}_{n+2}-s+2t\hat{j}_{n+1}-2t & 2\left(\hat{j}_{n+1}+2t\hat{j}_n-1\right)\\ t\left(\hat{j}_{n+1}+2t\hat{j}_n-1\right) & 2t\left(\hat{j}_n+2t\hat{j}_{n-1}-1\right)\end{bmatrix}.$$

**Proof.** Let $S_n=\sum_{k=1}^{n}J_k$. By multiplying $J_1$ two sides of the equality, we get

$$S_nJ_1=J_2+J_3+\cdots+J_{n+1}.$$

By adding $J_1$ two sides of the equality, we get

$$S_nJ_1+J_1=J_1+J_2+J_3+\cdots+J_{n+1}$$

$$S_nJ_1-S_n=J_{n+1}-J_1$$

$$S_n\left(J_1-J_0\right)=J_{n+1}-J_1.$$

The inverse of $J_1 - J_0$ is available for $\det(J_1 - J_0) = 1 - s - 2t \neq 0$. Then we get

$$S_n = (J_{n+1} - J_1)(J_1 - J_0)^{-1}.$$

By using following equalities $J_{n+1} - J_1 = \begin{bmatrix} \hat{j}_{n+2} - s & 2\hat{j}_{n+1} - 2 \\ t\hat{j}_{n+1} - t & 2t\hat{j}_n \end{bmatrix}$, $J_1 - J_0 = \begin{bmatrix} s-1 & 2 \\ t & -1 \end{bmatrix}$ and

$(J_1 - J_0)^{-1} = \frac{1}{s+2t-1}\begin{bmatrix} 1 & 2 \\ t & 1-s \end{bmatrix}$, we get

$$S_n = \frac{1}{s+2t-1}\begin{bmatrix} \hat{j}_{n+2} - s & 2\hat{j}_{n+1} - 2 \\ t\hat{j}_{n+1} - t & 2t\hat{j}_n \end{bmatrix}\begin{bmatrix} 1 & 2 \\ t & 1-s \end{bmatrix}$$
$$= \frac{1}{s+2t-1}\begin{bmatrix} \hat{j}_{n+2} + 2t\hat{j}_{n+1} - s - 2t & 2\left(\hat{j}_{n+1} + 2t\hat{j}_n - 1\right) \\ t\left(\hat{j}_{n+1} + 2t\hat{j}_n - 1\right) & 2t\left(\hat{j}_n + 2t\hat{j}_{n-1} - 1\right) \end{bmatrix}.$$

■

**Corollary 12.** *The partial sums of* (*s*,*t*)-*Jacobsthal sequence for* $s + 2t \neq 1$ *are given in the following*:

$$\sum_{k=1}^{n} \hat{j}_{k+1} = \frac{\hat{j}_{n+2} - s + 2t\hat{j}_{n+1} - 2t}{s+2t-1}$$

and

$$\sum_{k=1}^{n} \hat{j}_k = \frac{\hat{j}_{n+1} + 2t\hat{j}_n - 1}{s+2t-1}.$$

**Proof.** It is proved by the equality of matrix sequences and from Theorem 11. ■

**Theorem 13.** *The partial sum of* (*s*,*t*)-*Jacobsthal Lucas matrix sequence for* $s + 2t \neq 1$ *is given in the following* $\sum_{k=1}^{n} C_{k+1} = (a_{ij})$.

$$a_{11} = \frac{1}{s+2t-1}\left(\hat{j}_{n+4} + 2t\hat{j}_{n+3} + 2t\hat{j}_{n+2} + 4t^2\hat{j}_{n+1} - s^2(s+2t) - 2t(3s+4t)\right)$$

$$a_{12} = \frac{2}{s+2t-1}\left(\hat{j}_{n+3} + 2t\hat{j}_{n+2} + 2t\hat{j}_{n+1} + 4t^2\hat{j}_n - s^2 - 4t - 2st\right)$$

$$a_{21} = \frac{t}{s+2t-1}\left(\hat{j}_{n+3} + 2t\hat{j}_{n+2} + 2t\hat{j}_{n+1} + 4t^2\hat{j}_n - 2st - s^2 - 4t\right)$$

$$a_{22} = \frac{2t}{s+2t-1}\left(\hat{j}_{n+2} + 2t\hat{j}_{n+1} + 2t\hat{j}_n + 4t^2\hat{j}_{n-1} - s - 4t\right).$$

**Proof.** By using $C_{k+1} = C_1 J_k$ and Theorem 11 we get

$$\sum_{k=1}^{n} C_{k+1} = \sum_{k=1}^{n} C_1 J_k = C_1 \sum_{k=1}^{n} J_k$$
$$= \frac{1}{s+2t-1}\begin{bmatrix} s^2+4t & 2s \\ st & 4t \end{bmatrix}\begin{bmatrix} \hat{j}_{n+2} + 2t\hat{j}_{n+1} - s - 2t & 2\left(\hat{j}_{n+1} + 2t\hat{j}_n - 1\right) \\ t\left(\hat{j}_{n+1} + 2t\hat{j}_n - 1\right) & 2t\left(\hat{j}_n + 2t\hat{j}_{n-1} - 1\right) \end{bmatrix}.$$

If the product of matrices is made the desired result is found. ■

**Corollary 14.** *The partial sums of* (*s*,*t*)-*Jacobsthal Lucas sequence for* $s + 2t \neq 1$ *are given in the following*:

$$\sum_{k=1}^{n} c_k = \frac{1}{s+2t-1}\left(\hat{j}_{n+2} + 2t\hat{j}_{n+1} + 2t\hat{j}_n + 4t^2\hat{j}_{n-1} - s - 4t\right)$$

and

$$\sum_{k=1}^{n} c_{k+1} = \frac{1}{s+2t-1}\left(\hat{j}_{n+3} + 2t\hat{j}_{n+2} + 2t\hat{j}_{n+1} + 4t^2\hat{j}_n - s^2 - 4t - 2st\right).$$

**Proof.** It is proved by the equality of matrix sequences and from Theorem 11. ■

**Theorem 15.** *Let* $s+2t \neq 1$, *and* $s-2t \neq -1$. *Then for* $S_{2n} = \sum_{k=1}^{n} J_{2k} = \left(a_{ij}\right)$ *we get*

$$a_{11} = \frac{1}{(s+2t-1)(s-2t+1)}\left(\hat{j}_{2n+3} - 4t^2\hat{j}_{2n+1} + 4t^2 - s^2 - 2t\right)$$

$$a_{12} = \frac{2}{(s+2t-1)(s-2t+1)}\left(\hat{j}_{2n+2} - 4t^2\hat{j}_{2n} - s\right)$$

$$a_{21} = \frac{t}{(s+2t-1)(s-2t+1)}\left(\hat{j}_{2n+2} - 4t^2\hat{j}_{2n} - s\right)$$

$$a_{22} = \frac{2t}{(s+2t-1)(s-2t+1)}\left(\hat{j}_{2n+1} - 4t^2\hat{j}_{2n-1} - 1 + 2t\right).$$

**Proof.** By multiplying $J_2$ two sides of the equality $S_{2n}$, we get

$$S_{2n}J_2 = J_4 + J_6 + \cdots + J_{2n+2}.$$

By adding $J_2$ two sides of the equality, we get

$$J_{2n}\left(J_2 - J_0\right) = J_{2n+2} - J_2$$

$$S_{2n} = \left(J_{2n+2} - J_2\right)\left(J_2 - J_0\right)^{-1}$$

$$J_{2n+2} - J_2 = \begin{bmatrix} \hat{j}_{2n+3} - s^2 - 2t & 2\left(\hat{j}_{2n+2} - s\right) \\ t\left(\hat{j}_{2n+2} - s\right) & 2t\left(\hat{j}_{2n+1} - 1\right) \end{bmatrix}$$

$$J_2 - J_0 = \begin{bmatrix} s^2 + 2t - 1 & 2s \\ st & 2t - 1 \end{bmatrix}$$

$$\left(J_2 - J_0\right)^{-1} = \frac{1}{(s+2t-1)(s-2t+1)}\begin{bmatrix} 1-2t & 2s \\ st & 1-2t-s^2 \end{bmatrix}$$

$$S_{2n} = \frac{1}{(s+2t-1)(s-2t+1)}\begin{bmatrix} \hat{j}_{2n+3} - s^2 - 2t & 2\left(\hat{j}_{2n+2} - s\right) \\ t\left(\hat{j}_{2n+2} - s\right) & 2t\left(\hat{j}_{2n+1} - 1\right) \end{bmatrix} \cdot \begin{bmatrix} 1-2t & 2s \\ st & 1-2t-s^2 \end{bmatrix}.$$

■

**Corollary 16.** *The odd and even elements sums of* (*s*,*t*)*-Jacobsthal sequence for* $s+2t \neq 1$ *and* $s-2t \neq -1$ *are given in the following*:

$$\sum_{k=1}^{n} \hat{j}_{2k+1} = \frac{1}{(s+2t-1)(s-2t+1)}\left(\hat{j}_{2n+3} - 4t^2\hat{j}_{2n+1} + 4t^2 - s^2 - 2t\right)$$

$$\sum_{k=1}^{n} \hat{j}_{2k} = \frac{1}{(s+2t-1)(s-2t+1)}\left(\hat{j}_{2n+2} - 2t\hat{j}_{2n+2} + 2st\hat{j}_{2n+1} - s\right).$$

In the following theorem we will show the partial sum of Jacobsthal Lucas matrix sequence of the elements of power of *n*.

**Theorem 17.** *For* (*s*,*t*)-*Jacobsthal matrix sequence the equality is hold.*

$$\sum_{i=0}^{n}\binom{n}{i}J_i^r x^i=\sum_{k=0}^{r}\binom{r}{k}X^k(-Y)^{r-k}\left(1+\alpha^k\beta^{r-k}x\right)^n$$

**Proof.** By using the equality of $X=\left(\frac{J_1-\beta J_0}{\alpha-\beta}\right), Y=\left(\frac{J_1-\alpha J_0}{\alpha-\beta}\right)$ we can write $J_n=X\alpha^n-Y\beta^n$. By using it

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}J_i^r x^i&=\sum_{i=0}^{n}\binom{n}{i}\left(X\alpha^i-Y\beta^i\right)^r x^i\\&=\sum_{i=0}^{n}\binom{n}{i}\sum_{k=0}^{r}\binom{r}{k}\left(X\alpha^i\right)^k\left(-Y\beta^i\right)^{r-k}x^i\\&=\sum_{k=0}^{r}\binom{r}{k}(X)^k(-Y)^{r-k}\sum_{i=0}^{n}\binom{n}{i}\left(\alpha^k\beta^{r-k}x\right)^i\\&=\sum_{k=0}^{r}\binom{r}{k}(X)^k(-Y)^{r-k}\left(1+\alpha^k\beta^{r-k}x\right)^n.\end{aligned}$$

■

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