Applied Mathematics, 2015, 6, 2211-2216
Published Online November 2015 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2015.613194

Strong Local Non-Determinism of Sub-Fractional Brownian Motion

Nana Luan

School of Insurance and Economics, University of International Business and Economics, Beijing, China
Email: luannana@uibe.edu.cn

Received 30 August 2015; accepted 27 November 2015; published 30 November 2015

Abstract

Let $X^H = \{X^H(t), t \in \mathbb{R}_+\}$ be a subfractional Brownian motion in $\mathbb{R}^d$. We prove that X^H is strongly locally nondeterministic.

Keywords

Sub-Fractional Brownian Motion, Fractional Brownian Motion, Self-Similar Gaussian Processes, Strong Local Non-Determinism

1. Introduction

The fractional Brownian motion (fBm for short) is the best known and most used process with long-dependence property for models in telecommunications, turbulence, image processing and finance. This process is first introduced by [1] and later studied by [2]. The self-similarity and stationarity of the increments are two main properties for which fBm enjoy success as a modeling tool. The fBm is the only continuous Gaussian process which is self-similar and has stationary increments; see [3]. Many authors have also proposed for using more general self-similar Gaussian processes and random fields as stochastic models; see e.g. [4]-[9]. Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. However, in contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes until [10] fills the gap by developing systematic ways to study sample path properties of a class of self-similar Gaussian process, namely, the bifractional Brownian motion. Their main tools are the Lamperti transformation, which provides a powerful connection between self-similar processes and stationary processes; see [11], and the strong local non-determinism of Gaussian processes; see [12]. In particular, for any self-similar Gaussian processes $X = \{X(t), t \in \mathbb{R}\}$, the Lamperti transformation leads to a stochastic integal representation for X.

An extension of Bm which preserves many properties of the fBm, but not the stationarity of the increments, is so called sub-fractional Brownian motion (sub-fBm, in short) introduced by [13]. The sub-fBm is another class of self-similar Gaussian process which has properties analogous to those of fBm; see [13]-[15]. Given a constant $H \in (0,1)$, the sub-fractional Brownian motion in $\mathbb{R}$ is a centered Gaussian process $X_0^H = \{X_0^H(t), t \in \mathbb{R}_+\}$ with covariance function

$$R^H(s,t) := R(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right] \tag{1}$$

and $X_0^H(0) = 0$.

Let $X_1^H, \cdots, X_d^H$ be independent copies of X_0^H. We define the Gaussian process $X^H = \{X^H(t), t \in \mathbb{R}_+\}$ with values in $\mathbb{R}^d$ by

$$X^H(t) = \left(X_1^H(t), \cdots, X_d^H(t)\right), \quad \forall t \in \mathbb{R}_+. \tag{2}$$

By (1), one can verify easily that X^H is a self-similar process with index *H*, that is, for every constant $a > 0$,

$$\{X^H(at), t \in \mathbb{R}_+\} \overset{d}{=} \{a^H X^H(t), t \in \mathbb{R}_+\}, \tag{3}$$

where $X \overset{d}{=} Y$ means that the two processes have the same finite dimensional distributions. Note that X^H does not have stationary increments.

The strong local non-determinism is an important tool to study the sample path properties of self-similar Gaussian process, such as the small ball probability and Chung's law of the iterated logarithm. In this paper, we apply the Lamperti transformation to prove the strong local non-determinism of X_0^H. Throughout this paper, a specified positive and finite constant is denoted by c_i which may depend on *H*.

2. Strong Local Non-Determinism

Theorem 1. *For all constants* $0 < a < b$, X_0^H *is strongly locally* φ*-nondeterministic on* $I = [a,b]$ *with* $\varphi(r) = r^{2H}$. *That is, there exist positive constants* c_1 *and* r_0 *such that for all* $t \in I$ *and all* $0 < r \le \min\{t, r_0\}$,

$$Var\left(X_0^H(t) \mid X_0^H(s) : s \in I, r \le |s-t| \le r_0\right) \ge c_1 \varphi(r). \tag{4}$$

Proof. By Lamperti's transformation (see [11]), we consider the centered stationary Gaussian process $Y_0 = \{Y_0(t), t \in \mathbb{R}\}$ defined by

$$Y_0(t) = \mathrm{e}^{-Ht} X_0^H(\mathrm{e}^t), \quad \text{for every } t \in \mathbb{R}. \tag{5}$$

The covariance function $r(t) := \mathbb{E}(Y_0(0) Y_0(t))$ is given by

$$r(t) = \mathrm{e}^{-Ht}\left\{1 + \mathrm{e}^{2Ht} - \frac{1}{2}\left[(\mathrm{e}^t + 1)^{2H} + |\mathrm{e}^t - 1|^{2H}\right]\right\} = \mathrm{e}^{Ht}\left\{\mathrm{e}^{-2Ht} + 1 - \frac{1}{2}\left[(1 + \mathrm{e}^{-t})^{2H} + |1 - \mathrm{e}^{-t}|^{2H}\right]\right\}, \tag{6}$$

where $r(t)$ is an even function. By (6) and Taylor expansion, we verify that $r(t) = O(\mathrm{e}^{-\beta t})$, as $t \to \infty$, where $\beta = \min\{H, 1-H\}$. It follows that $r(\cdot) \in L^1(\mathbb{R})$. Also, by using (6) and the Taylor expansion again, we also have

$$r(t) \sim 2 - \frac{1}{2}\left(2^{2H} + |t|^{2H}\right) \quad \text{as } t \to 0. \tag{7}$$

Using Bochner's theorem, Y_0 has the following stochastic integral representation

$$Y_0(t) = \int_{\mathbb{R}} \mathrm{e}^{i\lambda t} W(\mathrm{d}\lambda), \quad \forall t \in \mathbb{R}, \tag{8}$$

where *W* is a complex Gaussian measure with control measure Δ whose Fourier transform is $r(\cdot)$. The measure Δ is called the spectral measure of Y_0.

Since $r(\cdot) \in L^1(\mathbb{R})$, the spectral measure Δ of Y_0 has a continuous density function $f(\lambda)$ which can be represented as the inverse Fourier transform of $r(\cdot)$:

$$f(\lambda) = \frac{1}{\pi}\int_0^\infty r(t)\cos(t\lambda)\,\mathrm{d}t. \tag{9}$$

We would like to prove that f has the following asymptotic property

$$f(\lambda) \sim c_2 |\lambda|^{-(1+2H)} \quad \text{as } \lambda \to \infty, \tag{10}$$

where $c_2 > 0$ is an explicit constant depending only on H.

In the following we give a direct proof of (10) by using (9) and an Abelian argument similar to that in the proof of **Theorem 1** of [16]. Without loss of generality, we assume that $\lambda > 0$. Applying integration-by-parts to (9), we get

$$f(\lambda) = -\frac{1}{\pi\lambda}\int_0^\infty r'(t)\sin(t\lambda)\,\mathrm{d}t \tag{11}$$

with

$$r'(t) = H\mathrm{e}^{Ht}\left[1-\mathrm{e}^{-2Ht} + \frac{1}{2}\left(1+\mathrm{e}^{-t}\right)^{2H-1}\left(\mathrm{e}^{-t}-1\right) - \frac{1}{2}\left|1-\mathrm{e}^{-t}\right|^{2H-1}\left(1+\mathrm{e}^{-t}\right)\right]. \tag{12}$$

We need to distinguish three cases: $2H < 1$, $2H = 1$ and $2H > 1$. In the first case, it can be verified from (12) that $r(t) = O\left(\mathrm{e}^{-\beta t}\right)$ as $t \to \infty$, hence $r'(t) \in L^1(\mathbb{R})$, and

$$r'(t) \sim -H|t|^{2H-1} \quad \text{as } t \to 0. \tag{13}$$

We will also make use of the properties of higher order derivatives of $r(t)$. It is elementary to compute $r''(t)$ and verify that, when $2H < 1$, we have

$$r''(t) \sim -H(2H-1)|t|^{2H-2} \quad \text{as } t \to 0 \tag{14}$$

and $r''(t) = O\left(\mathrm{e}^{-\beta t}\right)$ as $t \to \infty$ which implies $r''(\cdot) \in L^1(\mathbb{R})$.

The behavior of the derivatives of $r(t)$ is simpler when $2H = 1$. (12) becomes

$$r'(t) = -\frac{1}{2}\mathrm{e}^{-\frac{t}{2}}, \tag{15}$$

and

$$r''(t) = \frac{1}{4}\mathrm{e}^{-\frac{t}{2}}. \tag{16}$$

Hence, we have $r'(0) = -\frac{1}{2}$, $r''(0) = \frac{1}{4}$, and both $r'(\cdot)$ and $r''(\cdot)$ are in $L^1(\mathbb{R})$.

When $2H > 1$, it can be shown that (14) still holds, and $r''(t) = O\left(\mathrm{e}^{-\beta t}\right)$ as $t \to \infty$.

Now, we proceed to prove (10). First, we consider the case when $0 < 2H < 1$. By a change of variable, we can write

$$f(\lambda) = -\frac{1}{\pi\lambda^2}\int_0^\infty r'\left(\frac{t}{\lambda}\right)\sin t\,\mathrm{d}t. \tag{17}$$

Hence,

$$\frac{f(\lambda)}{-\left(\pi\lambda^2\right)^{-1} r'(1/\lambda)} = \int_0^\infty \frac{r'(t/\lambda)}{r'(1/\lambda)}\sin t\,\mathrm{d}t. \tag{18}$$

Let $p \in (0,\infty)$ be a fixed constant. It follows from (13) and the dominated convergence theorem that

$$\lim_{\lambda\to\infty}\int_0^p \frac{r'(t/\lambda)}{r'(1/\lambda)}\sin t\,\mathrm{d}t = \int_0^p t^{2H-1}\sin t\,\mathrm{d}t. \tag{19}$$

On the other hand, integration-by-parts yields

$$\int_p^\infty r'(1/\lambda)\sin t\mathrm{d}t = r'(p/\lambda)\cos p + \frac{1}{\lambda}\int_p^\infty r''(t/\lambda)\cos t\mathrm{d}t. \tag{20}$$

By Riemann-Lebesgue lemma,

$$\frac{1}{\lambda}\int_p^\infty r''(t/\lambda)\cos t\mathrm{d}t = \int_{p/\lambda}^\infty r''(x)\cos(\lambda x)\mathrm{d}x = \int_{-\infty}^\infty 1_{\{x\ge p/\lambda\}} r''(x)\cos(\lambda x)\mathrm{d}x \to 0 \quad \text{as } \lambda\to\infty. \tag{21}$$

Moreover, since $r'\left(\frac{p}{\lambda}\right) \sim -H\left(\frac{p}{\lambda}\right)^{2H-1}$ as $\lambda\to\infty$ by (13) and $\left(\frac{p}{\lambda}\right)^{2H-1} \to\infty$ as $\lambda\to\infty$, we have $\left|r'\left(\frac{p}{\lambda}\right)\right| \to\infty$ as $\lambda\to\infty$. It follows that

$$\left|\frac{1}{\lambda}\int_p^\infty r''\left(\frac{t}{\lambda}\right)\cos t\mathrm{d}t\right| \le \left|r'\left(\frac{p}{\lambda}\right)\right| \quad \text{as } \lambda\to\infty. \tag{22}$$

Then for all λ large enough, we derive

$$\left|\int_p^\infty r'(t/\lambda)\sin t\,\mathrm{d}t\right| \le \left|r'(p/\lambda)\cos p\right| + \left|\frac{1}{\lambda}\int_p^\infty r''(t/\lambda)\cos t\mathrm{d}t\right| \le 2\left|r'(p/\lambda)\right|. \tag{23}$$

Hence, we have

$$\limsup_{\lambda\to\infty}\left|\int_p^\infty r'(t/\lambda)\sin t\,\mathrm{d}t\right| \le 2p^{2H-1}. \tag{24}$$

Combining (18), (19), and (24), we have

$$\begin{aligned} \lim_{\lambda\to\infty}\frac{f(\lambda)}{-(\pi\lambda^2)^{-1} r'(1/\lambda)} &= \lim_{\lambda\to\infty}\int_0^\infty \frac{r'(t/\lambda)}{r'(1/\lambda)}\sin t\mathrm{d}t \\ &= \lim_{\lambda\to\infty}\int_0^p \frac{r'(t/\lambda)}{r'(1/\lambda)}\sin t\mathrm{d}t + \lim_{\lambda\to\infty}\int_p^\infty \frac{r'(t/\lambda)}{r'(1/\lambda)}\sin t\mathrm{d}t \\ &\to \int_0^\infty t^{2H-1}\sin t\mathrm{d}t \quad \text{as } p\to\infty. \end{aligned} \tag{25}$$

Then we see that, when $0<2H<1$, (10) holds with $c_2 = H\pi^{-1}\int_0^\infty t^{2H-1}\sin t\mathrm{d}t$.

Secondly, we consider the case $2H=1$. Since $r'(t)$ is continuous and $r'(0) = -\frac{1}{2}$, (19) becomes

$$\lim_{\lambda\to\infty}\int_0^p r'(t/\lambda)\sin t\mathrm{d}t = r'(0)\int_0^p \sin t\mathrm{d}t = r'(0)(1-\cos p). \tag{26}$$

Using (20) and integration-by-parts again we derive

$$\int_p^\infty r'(t/\lambda)\sin t\mathrm{d}t = r'(p/\lambda)\cos p + \frac{1}{\lambda}\int_p^\infty r''(t/\lambda)\cos t\mathrm{d}t. \tag{27}$$

It follows from the (27), (16) and Riemann-Lebesgue lemma that

$$\lim_{\lambda\to\infty}\int_p^\infty r'(t/\lambda)\sin t\mathrm{d}t = r'(0)\cos p. \tag{28}$$

We see from the above and (17) that

$$f(\lambda) \sim \frac{1}{2\pi}|\lambda|^{-2} \quad \text{as } \lambda\to\infty. \tag{29}$$

This verifies that (10) holds when $2H=1$.

Finally we consider the case $1<2H<2$. Note that (19) and (24) are not useful anymore and we need to modify the above argument. By using integration-by-parts to (11) we obtain

$$f(\lambda)=-\frac{1}{\pi\lambda^2}\int_0^\infty r''(t)\cos(t\lambda)\,\mathrm{d}t. \tag{30}$$

Note that we have $-1<2H-2<0$. Hence $r''(t)$ is integrable in the neighborhood of $t=0$. Consequently, the proof for this case is very similar to the case of $0<2H<1$. From (30) and (14), we can verify that (10) holds as well and the constant c_2 is explicitly determined by *H*. Hence we have proved (10) in general.

It follows from (10) and Lemma 1 of [17] (see also [12] for more general results) that $Y_0=\{Y_0(t),t\in\mathbb{R}\}$ is strongly locally φ-nondeterministic on any interval $J=[-T,T]$ with $\varphi(r)=r^{2H}$ in the following sense: There exist positive constants δ and c_3 such that for all $t\in[-T,T]$ and all $r\in(0,|t|\wedge\delta)$,

$$Var\left(Y_0(t)\mid Y_0(s):s\in J,r\le|s-t|\le\delta\right)\ge c_3\varphi(r). \tag{31}$$

Now we prove the strong local nondeterminism of X_0^H on *I*. To this end, note that $X_0^H(t)=t^HY_0(\log t)$ for all $t>0$. We choose $r_0=a\delta$. Then for all $s,t\in I$ with $r\le|s-t|\le r_0$ we have

$$\frac{r}{b}\le|\log s-\log t|\le\delta. \tag{32}$$

Hence, it follows from (31) and (32) that for all $t\in[a,b]$ and $r<r_0$,

$$\begin{aligned}
&Var\left(X_0^H(t)\mid X_0^H(s):s\in I,r\le|s-t|\le r_0\right)\\
&=Var\left(t^HY_0(\log t)\mid s^HY_0(\log s):s\in I,r\le|s-t|\le r_0\right)\\
&=t^{2H}Var\left(Y_0(\log t)\mid Y_0(\log s):s\in I,r\le|s-t|\le r_0\right)\\
&\ge a^{2H}Var\left(Y_0(\log t)\mid Y_0(\log s):s\in I,\frac{r}{b}\le|\log s-\log t|\le\delta\right)\\
&\ge a^{2H}c_3\left(\frac{r}{b}\right)^{2H}=c_1r^{2H}=c_1\varphi(r),
\end{aligned} \tag{33}$$

where $c_1=c_3\left(\frac{a}{b}\right)^{2H}$. This proves **Theorem 1**.

Funding

Supported by NSFC (No. 11201068) and "The Fundamental Research Funds for the Central Universities" in UIBE (No. 14YQ07).

References

[1] Kolmogorov, A.N. (1940) Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *C.R.* (*Doklady*) *Acad. Sci. URSS* (*N.S.*), **26**, 115-118.

[2] Mandelbrot, B. and van Ness, J.W. (1968) Fractional Brownian Motions, Fractional Noises and Applications. *SIAM Review*, **10**, 422-437. http://dx.doi.org/10.1137/1010093

[3] Samorodnitsky, G. and Taqqu, M.S. (1994) Stable Non-Gaussian Random Processes. Stochastic Models with Infinite Variance. Stochastic Modeling. Chapman & Hall, New York.

[4] Anh, V.V., Angulo, J.M. and Ruiz-Medina, M.D. (1999) Possible Long-Range Dependence in Fractional Random Fields. *Journal of Statistical Planning and Inference*, **80**, 95-110. http://dx.doi.org/10.1016/S0378-3758(98)00244-4

[5] Benassi, A., Bertrand, P., Cohen, S. and Istas, J. (2000) Identification of the Hurst Index of a Step Fractional Brownian Motion. *Statistical Inference for Stochastic Processes*, **3**, 101-111. http://dx.doi.org/10.1023/A:1009997729317

[6] Benson, D.A., Meerschaert, M.M. and Baeumer, B. (2006) Aquifer Operator-Scaling and the Effect on Solute Mixing and Dispersion. *Water Resources Research*, **42**, W01415. http://dx.doi.org/10.1029/2004wr003755

[7] Bonami, A. and Estrade, A. (2003) Anisotropic Analysis of Some Gaussian Models. *Journal of Fourier Analysis and Applications*, **9**, 215-236. http://dx.doi.org/10.1007/s00041-003-0012-2

[8] Cheridito, P., Kawaguchi, H. and Maejima, M. (2003) Fractional Ornstein-Uhlenbeck Processes. *Electronic Journal of Probability*, **8**, 14 p.

[9] Mannersalo, P. and Norros, I. (2002) A Most Probable Path Approach to Queueing Systems with General Gaussian Input. *Computer Network*, **40**, 399-412. http://dx.doi.org/10.1016/S1389-1286(02)00302-X

[10] Tudor, C.A. and Xiao, Y. (2007) Sample Path Properties of Bifractional Brownian Motion. *Bernoulli*, **13**, 1023-1052. http://dx.doi.org/10.3150/07-BEJ6110

[11] Lamperti, J. (1962) Semi-Stable Stochastic Processes. *Transactions of the American Mathematical Society*, **104**, 62-78. http://dx.doi.org/10.1090/S0002-9947-1962-0138128-7

[12] Xiao, Y. (2007) Strong Local Non-Determinism of Gaussian Random Fields and Its Applications. In: Lai, T.-L., Shao, Q.-M. and Qian, L., Eds., *Asymptotic Theory in Probability and Statistics with Applications*, Higher Education Press, Beijing, 136-176.

[13] Bojdecki, T., Gorostiza, L.G. and Talarczyk, A. (2004) Sub-Fractional Brownian Motion and Its Relation to Occupation Times. *Statistics and Probability Letters*, **69**, 405-419. http://dx.doi.org/10.1016/j.spl.2004.06.035

[14] Dzhaparidze, K. and Van Zanten, H. (2004) A Series Expansion of Fractional Brownian Motion. *Probability Theory and Related Fields*, **103**, 39-55. http://dx.doi.org/10.1007/s00440-003-0310-2

[15] Tudor, C. (2007) Some Properties of the Sub-Fractional Brownian Motion. *Stochastics*, **79**, 431-448. http://dx.doi.org/10.1080/17442500601100331

[16] Pitman, E.J.G. (1968) On the Behavior of the Characteristic Function of a Probability Distribution in the Neighborhood of the Origin. *Journal of the Australian Mathematical Society*, **8** 423-443. http://dx.doi.org/10.1017/S1446788700006121

[17] Cuzick, J. and Du Preez, J.P. (1982) Joint Continuity of Gaussian Local Times. *Annals of Probability*, **10**, 810-817. http://dx.doi.org/10.1214/aop/1176993789