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On the Approximate Solution of Fractional Logistic Differential Equation Using Operational Matrices of Bernstein Polynomials

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Abstract

In this paper, operational matrices of Bernstein polynomials (BPs) are presented for solving the non-linear fractional Logistic differential equation (FLDE). The fractional derivative is described in the Riemann-Liouville sense. The operational matrices for the fractional integration in the Riemann-Liouville sense and the product are used to reduce FLDE to the solution of non-linear system of algebraic equations using Newton iteration method. Numerical results are introduced to satisfy the accuracy and the applicability of the proposed method.

Keywords

1. Introduction

It is well-known that the fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in various applications, such as in fluid mechanics, viscoelasticity, biology, physics and engineering applications, for more details see for example ([1] [2]). Consequently, considerable attention has been given to the efficient numerical solutions of FDEs of physical interest, because it is difficult to find exact solutions. Different numerical methods have been proposed in the literature for solving FDEs ([3]-[6]). Recently, several numerical and approximate methods to solve FDEs have been given, such as variational iteration method [7], homotopy perturbation method [7] and collocation method ([8]-[13]).

The fractional Logistic model can obtain by applying the fractional derivative operator on the Logistic equation. The model is initially published by Pierre Verhulst in 1838 ([14] [15]). The continuous Logistic model is described by first order ordinary differential equation. The discrete Logistic model is simple iterative equation that reveals the chaotic property in certain regions [16]. There are many variations of the population modeling [17]. The Verhulst model is the classic example to illustrate the periodic doubling and chaotic behavior in dynamical system [16]. The model is described the population growth may be limited by certain factors like population density ([15] [17]).

The solution of Logistic equation explains the constant population growth rate which does not include the limitation on food supply or spread of diseases [15]. The solution curve of the model increases exponentially from the multiplication factor up to saturation limit which is maximum carrying capacity [15], $\dfrac{\mathrm{d}N}{\mathrm{d}t} = \rho N\left(1 - \dfrac{N}{K}\right)$ where N is the population with respect to time, ρ is the rate of maximum population growth and K is the carrying capacity. The solution of continuous Logistic equation is in the form of constant growth rate as in formula $N(t) = N_0 e^{\rho t}$ where N_0 is the initial population [18].

In this article, we consider FLDE of the form

$$\frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha} = \rho u(t)(1 - u(t)), \quad t > 0, \quad \rho > 0, \tag{1}$$

the parameter $0 < \alpha \leq 1$ refers to the fractional order derivative

We also assume an initial condition

$$u(0) = u_0, \quad u_0 > 0. \tag{2}$$

The exact solution to this problem at $\alpha = 1$, is $u(t) = \dfrac{u_0}{(1 - u_0)e^{-\rho t} + u_0}$.

The existence and the uniqueness of the proposed problem (1) are introduced in details in ([19] [20]).

Khader and Hendy [21] introduced a new approximate formula of the fractional derivative using Legendre series expansion and used it to solve numerically the fractional delay equation. In this article, we extended this work to study the numerical solution of the non-linear FLDE. An approximate formula of the fractional derivative is presented. Special attention is given to study the convergence analysis and estimate an upper bound of the error of the introduced formula.

2. Preliminaries and Notations

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory and the Bernstein polynomials that will be required in the present paper.

The Fractional Integral and Derivative Operators

We present some necessary definitions and mathematical preliminaries of the fractional calculus theory that will be required in the present paper.

Definition 1.

The Riemann-Liouville fractional integral operator J_a^α of order α is defined on $L_1[a,b]$ in the following form

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \xi)^{\alpha - 1} f(\xi) \mathrm{d}\xi, \quad \alpha > 0, \quad a \leq t \leq b. \tag{3}$$

Definition 2.

The Riemann-Liouville fractional derivative operator D_a^α of order α $(n - 1 < \alpha \leq n)$ is defined in the following form

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t - \xi)^{n - \alpha - 1} f(\xi) \mathrm{d}\xi, \quad n \in \mathbb{N}, \quad \alpha > 0, \quad a \leq t \leq b. \tag{4}$$

Definition 3.

The Caputo fractional derivative operator $^{c}D_{a}^{\alpha}$ of order α $(n-1<\alpha\le n), n\in\mathbb{N}$ is defined in the following form

$$^{c}D_{a}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\xi)^{n-\alpha-1}f^{(n)}(\xi)\mathrm{d}\xi,\quad \alpha>0,\quad a\le t\le b. \tag{5}$$

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^{\alpha}\left(\lambda f(x)+\mu g(x)\right)=\lambda D^{\alpha}f(x)+\mu D^{\alpha}g(x),$$

where λ and μ are constants. For the Caputo's derivative we have [2]

$$D^{\alpha}C=0,\quad C\text{ is a constant,} \tag{6}$$

$$D^{\alpha}x^{n}=\begin{cases}0 & \text{for }n\in\mathbb{N}_{0}\text{ and }n<\lceil\alpha\rceil;\\[2mm]\dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha}, & \text{for }n\in\mathbb{N}_{0}\text{ and }n\ge\lceil\alpha\rceil.\end{cases} \tag{7}$$

We use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α, and $\mathbb{N}_{0}=\{0,1,\cdots\}$. Recall that for $\alpha\in\mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

For more details on fractional derivatives definitions and its properties see ([1] [2]).

Lemma 1.

If $\alpha\ge0, n=\lceil\alpha\rceil$ and $a\le t\le b$, then

$$^{c}D_{a}^{\alpha}J_{a}^{\alpha}f(t)=f(t), \tag{8}$$

$$J_{a}^{\alpha}\,^{c}D_{a}^{\alpha}f(t)=f(t)-\sum_{i=0}^{n-1}\frac{f^{(i)}(a)}{i!}(t-a)^{i}, \tag{9}$$

$$^{c}D_{a}^{\alpha}f(t)=D_{a}^{\alpha}f(t)-\sum_{i=0}^{n-1}\frac{f^{(i)}(a)}{\Gamma(i-\alpha+1)}(t-a)^{i-\alpha}. \tag{10}$$

3. Bernstein Polynomials and Their Properties

Definition 4.

The $(n+1)$ Bernstein polynomials of degree n are defined on the interval $[0,1]$ as [22]

$$B_{i,n}(x)=\binom{n}{i}x^{i}(1-x)^{n-i},\quad i=0,1,\cdots,n, \tag{11}$$

where $\binom{n}{i}$ is a binomial coefficient. The first few Bernstein basis polynomials are:

$$B_{0,0}(x)=1,$$
$$B_{0,1}(x)=1-x,\quad B_{1,1}(x)=x,$$
$$B_{0,2}(x)=(1-x)^{2},\quad B_{1,2}(x)=2x(1-x),\quad B_{2,2}(x)=x^{2},$$
$$B_{0,3}(x)=(1-x)^{3},\quad B_{1,3}(x)=3x(1-x)^{2},\quad B_{2,3}(x)=3x^{2}(1-x),\quad B_{3,3}(x)=x^{3}.$$

The Bernstein polynomials have the following properties

1) $B_{k,n}(x)=0$, if $k<0$ or $k>n$;

2) $B_{k,n}(0)=\delta_{k,0}$ and $B_{k,n}(1)=\delta_{k,n}$ where δ is the Kronecker delta function;

3) $B_{k,n}(x)\ge0$, for $x\in[0,1]$;

4) $\sum_{k=0}^{n} B_{k,n}(x) = 1$;

5) They satisfy symmetry $B_{k,n}(x) = B_{n-k,n}(1-x)$;

6) $B'_{k,n}(x) = n\left(B_{k-1,n-1}(x) - B_{k,n-1}(x)\right)$;

7) $\int_0^1 B_{k,n}(x)\,\mathrm{d}x = \dfrac{1}{n+1}$, $\forall k = 0,1,\cdots,n$.

Since the set $\left\{B_{i,m}(x)\right\}_{i=0}^{m}$ in Hilbert space $L_2[0,1]$ is a complete basis, so, we can write any polynomial $u(x)$ of degree m in terms of linear combination of $\left\{B_{i,m}(x)\right\}_{i=0}^{m}$ as in the following form

$$u(x) \simeq \sum_{k=0}^{m} c_k B_{k,m}(x). \tag{12}$$

We can write $\Psi_m(x) = \left[B_{0,m}(x), B_{1,m}(x), \cdots, B_{m,m}(x)\right]^{\mathrm{T}} = A B_m(x)$, where A is an upper triangular matrix, $B_m(x) = \left[1, x, x^2, \cdots, x^m\right]^{\mathrm{T}}$. For more details about the definition, properties and the convergence analysis of Bernstein polynomials [23].

4. BPs Operational Matrix of Riemann-Liouville Fractional Integration

Theorem 1. [23]

The Bernstein polynomials operational matrix F_α from order $(m+1) \times (m+1)$ for the Riemann-Liouville fractional integral is defined as follows

$$J_a^\alpha \Psi_m(x) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} \Psi_m(\xi)\,\mathrm{d}\xi \approx F_\alpha \Psi_m(x). \tag{13}$$

Definition 5.

We can define the dual matrix $Q_{(m+1)\times(m+1)}$ on the basis of Bernstein polynomials of mth degree as follows

$$Q = \int_0^1 \Psi(x)\left(\Psi(x)\right)^{\mathrm{T}}\,\mathrm{d}x, \tag{14}$$

where

$$(Q)_{i+1,j+1} = \int_0^1 B_{i,m}(x) B_{j,m}(x)\,\mathrm{d}x = \frac{(m!)^2 (2m-i-j)!(i+j)!}{(2m+1)(2m)!(m-i)!(m-j)!i!j!}, \quad i,j = 0,1,\cdots,m. \tag{15}$$

Lemma 2. [24]

Let $L_2[0,1]$ be a Hilbert space with the inner product $\langle f,g\rangle = \int_0^1 f(x) g(x)\,\mathrm{d}x$ and $u(x) \in L_2[0,1]$. Then, we can find the unique vector $c = \left[c_0, c_1, \cdots, c_m\right]^{\mathrm{T}}$ such that $C^{\mathrm{T}}\Psi_m(x)$ is the best approximation of $u(x)$ from space $S_m = Span\left[B_{0,m}(x), B_{1,m}(x), \cdots, B_{m,m}(x)\right]$. Moreover, one can get $Q^{-1}\langle u(x), \Psi_m(x)\rangle$ such that $\langle u(x), \Psi_m(x)\rangle = \left[\langle u, B_{0,m}(x)\rangle, \langle u, B_{1,m}(x)\rangle, \cdots, \langle u, B_{m,m}(x)\rangle\right]^{\mathrm{T}}$.

Definition 6.

Let $u(x)$ be a continuous function on the interval $[0,1]$. Then we can approximate it in the following polynomial in Bernstein form of degree n

$$\overline{B}_n(u)(x) = \sum_{i=0}^{n} u\left(\frac{i}{n}\right) B_{i,n}(x).$$

It can be shown that is uniformly convergent on the interval $[0,1]$,

$$\lim_{n\to\infty} \overline{B}_n(u)(x) = u(x).$$

Theorem 2.

Given a function $u(x) \in C[0,1]$ and any $\delta > 0$, there exists an integer N such that

$$\left|u(x) - \overline{B}_n(u)(x)\right| < \delta, \quad \forall n > N, \ x \in [0,1]. \tag{16}$$

The Bernstein polynomials operational matrix are used for solving many class of fractional differential equations, they used to solve numerically the fractional heat-and wave-like equations [25] and the multi-term orders fractional differential equations [26] and others [27].

5. Implementation of Bernstein Polynomials Operational Matrix for Solving FLDE

In this section, we introduce a numerical algorithm using Bernstein polynomials operational matrix method for solving the fractional Logistic differential equation of the form (1).

The proposed technique will apply as in the following steps:

1) We use the initial condition (2) to reduce the given problem (1) to a problem with zero initial condition. So, we define

$$u(t) = \hat{u}(t) + v(t), \tag{17}$$

where $\hat{u}(t)$ is some known function that satisfied the initial condition (2) and $v(t)$ is a new unknown function.

2) Substituting (17) in (1) and (2), we have an initial-value problem as follows

$$D_0^\alpha v(t) = \rho\left(p(t) + q(t)v(t) - v^2(t)\right), \tag{18}$$

where $p(t) = \hat{u}(t) - \hat{u}^2(t)$ and $q(t) = 1 - 2\hat{u}(t)$ subject to the initial condition

$$v(0) = 0. \tag{19}$$

3) Using (10) in Lemma 1 we can write

$$^c D_0^\alpha v(t) = D_0^\alpha v(t). \tag{20}$$

4) Using Lemma 3.3 in [23], the inputs $p(t), q(t)$ and $D_0^\alpha v(t)$ can be approximated as follows

$$p(t) \approx P^T \Psi_m(t), \quad q(t) \approx Q^T \Psi_m(t), \quad D_0^\alpha v(t) \approx C^T \Psi_m(t), \tag{21}$$

where P and Q are known $(m+1) \times 1$ column vectors and C is an unknown $(m+1) \times 1$ column vector.

5) From (9), (13), (19), (20) and (21), we have

$$v(t) = J_0^\alpha {}^c D_0^\alpha v(t) + \sum_{i=0}^{n-1} \frac{v^{(i)}(0)}{i!} t^i = J_0^\alpha {}^c D_0^\alpha v(t) = J_0^\alpha D_0^\alpha v(t)$$
$$= J_0^\alpha \left(C^T \Psi_m(t)\right) \approx C^T J_0^\alpha \Psi_m(t) \approx C^T F_\alpha \Psi_m(t) = C_\alpha^T \Psi_m(t), \tag{22}$$

where $C_\alpha^T = C^T F_\alpha$.

6) By substituting (21) and (22) into (18), we obtain

$$C^T \Psi_m(t) = \rho\left[P^T \Psi_m(t) + Q^T \Psi_m(t)\left(\Psi_m(t)\right)^T F_\alpha^T C - C^T F_\alpha \Psi_m(t)\left(\Psi_m(t)\right)^T F_\alpha^T C\right]$$
$$= \rho\left[P^T \Psi_m(t) + Q^T \Psi_m(t)\left(\Psi_m(t)\right)^T C_\alpha - C_\alpha^T \Psi_m(t)\left(\Psi_m(t)\right)^T C_\alpha\right]. \tag{23}$$

7) Then, from Lemma 3.5 in [23] we have

$$C_\alpha^T \Psi_m(t)\left(\Psi_m(t)\right)^T = \left(\Psi_m(t)\right)^T \hat{C}_\alpha, \tag{24}$$

$$P^T \Psi_m(t)\left(\Psi_m(t)\right)^T = \left(\Psi_m(t)\right)^T \hat{P}, \tag{25}$$

$$Q^T \Psi_m(t)\left(\Psi_m(t)\right)^T = \left(\Psi_m(t)\right)^T \hat{Q}. \tag{26}$$

Therefore we can reduce (23) by (24)-(26) as

$$C^T \Psi_m(t) = \rho\left[P^T \Psi_m(t) + \left(\Psi_m(t)\right)^T \hat{Q} C_\alpha - \left(\Psi_m(t)\right)^T \hat{C}_\alpha C_\alpha\right]. \tag{27}$$

We obtain the following non-linear system of algebraic equations

$$C = \rho \left[P + \hat{Q}C_\alpha - \hat{C}_\alpha C_\alpha \right]. \tag{28}$$

8) By solving this system we can obtain the vector *C*. Then, we can get

$$u(t) \approx \hat{u}(t) + C^{\mathrm{T}} F_\alpha \Psi_m(t). \tag{29}$$

The numerical results of the proposed problem (1) are given in **Figure 1** and **Figure 2** with different values of α in the interval $[0,1]$ with $\rho = 0.5$ and $u_0 = 0.25$. Where in **Figure 1**, we presented a comparison between the behavior of the exact solution and the approximate solution using the introduced technique at $\alpha = 1$ (left), and the behavior of the approximate solution using the proposed method at $\alpha = 0.9$ (right). But, in **Figure 2** we presented the behavior of the approximate solution with different values of α ($\alpha = 0.7$ (left) and $\alpha = 0.6$ (right)).

From these figures we can conclude that the obtained numerical solutions are in excellent agreement with the exact solution.

6. Conclusion and Remarks

In this article, we used operational matrices of the Riemann-Liouville fractional integral and the product by Bernstein polynomials for solving the fractional Logistic differential equation. The properties of these operational
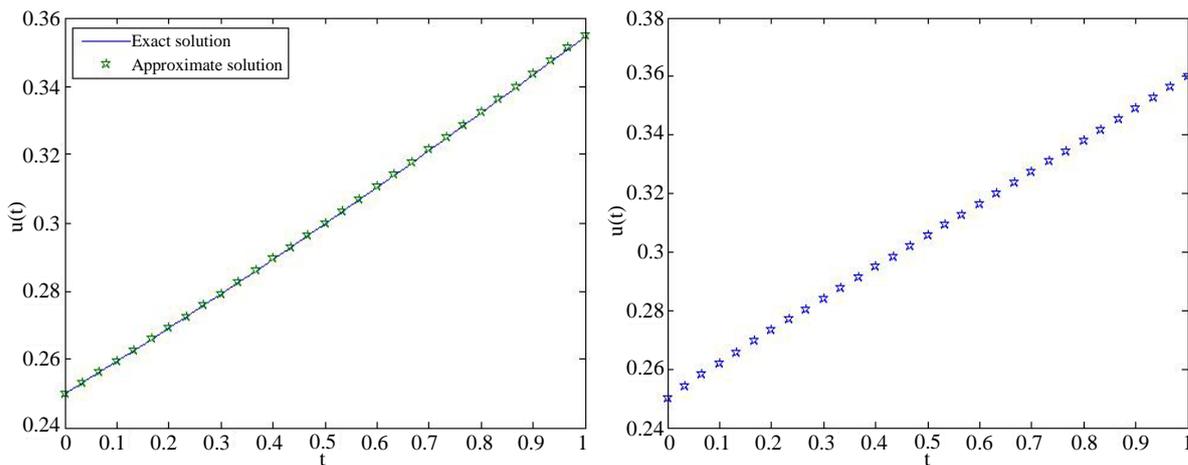

Figure 1. A comparison between the approximate solution and the exact solution at $\alpha = 1$ (left). The behavior of the approximate solution using the proposed method at $\alpha = 0.9$ (right).
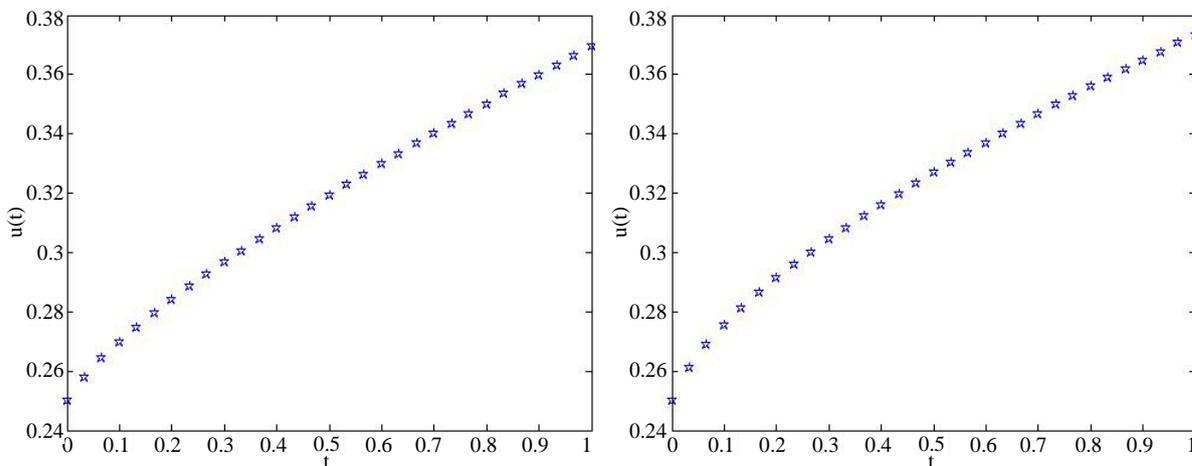

Figure 2. The behavior of the approximate solution using the proposed method at $\alpha = 0.7$ (left) and at $\alpha = 0.6$ (right).

matrices are used to reduce FLDE to a non-linear system of algebraic equations which solved by Newton iteration method. From the obtained numerical results, we can conclude that this method gives results with an excellent agreement with the exact solution. All numerical results are obtained using Matlab program 8.

Acknowledgements

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