Applied Mathematics, 2015, 6, 2004-2013
Published Online November 2015 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2015.612178

On Meromorphic Functions That Share One Small Function of Differential Polynomials with Their Derivatives

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Received 19 August 2015; accepted 15 November 2015; published 18 November 2015

Abstract

In this paper, we study the problem of meromorphic functions that share one small function of differential polynomial with their derivatives and prove one theorem. The theorem improves the results of Jin-Dong Li and Guang-Xin Huang [1].

Keywords

Uniqueness, Meromorphic Function, Differential Polynomial

1. Introduction and Results

Let $\mathbb{C}$ denote the complex plane and *f* be a nonconstant meromorphic function on $\mathbb{C}$. We assume the reader is familiar with the standard notion used in the Nevanlinna value distribution theory such as $T(r,f), m(r,f)$, $N(r,f)$ (see, e.g., [2] [3]), and $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function *a* is called a small function with respect to *f*, provided that $T(r,a)=S(r,f)$.

Let *f* and *g* be two nonconstant meromorphic functions. Let *a* be a small function of *f* and g. We say that *f*, *g* share *a* counting multiplicities (CM) if $f-a, g-a$ have the same zeros with the same multiplicities and we say that *f*, *g* share *a* ignoring multiplicities (IM) if we do not consider the multiplicities. In addition, we say that *f* and *g* share ∞ CM, if $\frac{1}{f},\frac{1}{g}$ share 0 CM, and we say that *f* and *g* share ∞ IM, if $\frac{1}{f},\frac{1}{g}$ share 0 IM. Suppose that *f* and *g* share *a* IM. Throughout this paper, we denote by $\bar{N}_L\left(r,\frac{1}{f-1}\right)$ the reduced counting function of

How to cite this paper: Waghamore, H.P. and Rajeshwari, S. (2015) On Meromorphic Functions That Share One Small Function of Differential Polynomials with Their Derivatives. *Applied Mathematics*, **6**, 2004-2013.
http://dx.doi.org/10.4236/am.2015.612178

those common a-points of f and g in $|z|<r$, where the multiplicity f each a-point of f is greater than that of the corresponding a-point of g, and denote by $N_{11}\left(r,\frac{1}{f-a}\right)$ the counting function for common simple 1-point of both f and g, and $N_E^{(2}\left(r,\frac{1}{f-1}\right)$ the counting function of those 1-points of f and g where $p=q\geq 2$. In the same way, we can define $N_{11}\left(r,\frac{1}{g-1}\right), N_E^{(2}\left(r,\frac{1}{g-1}\right)$ and $\bar{N}_L\left(r,\frac{1}{g-1}\right)$. If f and g share 1 IM, it is easy to see that

$$\bar{N}\left(r,\frac{1}{f-1}\right)=N_{11}\left(r,\frac{1}{f-1}\right)+\bar{N}_L\left(r,\frac{1}{f-1}\right)+\bar{N}_L\left(r,\frac{1}{g-1}\right)+N_E^{(2}\left(r,\frac{1}{g-1}\right)=\bar{N}\left(r,\frac{1}{g-1}\right)$$

In addition, we need the following definitions:

Definition 1.1. Let f be a non-constant meromorphic function, and let p be a positive integer and $a\in\mathbb{C}\cup\{\infty\}$. Then by $N_{p)}\left(r,\frac{1}{f-a}\right)$, we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater that p, by $\bar{N}_{p)}\left(r,\frac{1}{f-a}\right)$ we denote the corresponding reduced counting function (ignoring multiplicities). By $N_{(p}\left(r,\frac{1}{f-a}\right)$, we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not less than p, by $\bar{N}_{(p}\left(r,\frac{1}{f-a}\right)$ we denote the corresponding reduced counting function (ignoring multiplicities,) where and what follows, $N_{p)}\left(r,\frac{1}{f-a}\right), \bar{N}_{p)}\left(r,\frac{1}{f-a}\right), N_{(p}\left(r,\frac{1}{f-a}\right), \bar{N}_{(p}\left(r,\frac{1}{f-a}\right)$ mean $N_{p)}(r,f), \bar{N}_{p)}(r,f)$, $N_{(p}(r,f)$, $\bar{N}_{(p}(r,f)$, respectively, if $a=\infty$.

Definition 1.2. Let f be a non-constant meromorphic function, and let a be any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We define

$$\delta_k(a,f)=1-\limsup_{r\to\infty}\frac{N_k\left(r,\frac{1}{f-a}\right)}{T(r,f)},$$

where

$$N_k\left(r,\frac{1}{f-a}\right)=\bar{N}\left(r,\frac{1}{f-a}\right)+\bar{N}_{(2}\left(r,\frac{1}{f-a}\right)+\cdots+\bar{N}_{(k}\left(r,\frac{1}{f-a}\right)$$

Remark 1.1. From the above inequalities, we have

$$0\leq\delta_k(a,f)\leq\delta_{k-1}(a,f)\leq\delta_1(a,f)\leq\Theta(a,f)\leq 1.$$

Definition 1.3. Let f be a non-constant meromorphic function, and let a be any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We define

$$\Theta_{k)}(a,f)=1-\bar{N}_{r\to\infty}\frac{\bar{N}_{k)}\left(r,\frac{1}{f-a}\right)}{T(r,f)}$$

Remark 1.2. From the above inequality, we have

$$0\leq\Theta(a,f)\leq\Theta_{k)}(a,f)\leq\Theta_{k-1)}(a,f)\leq 1.$$

Definition 1.4. (see [4]). Let k be a nonnegative integer or infinity. For $a \in C$ we denote by $E_k(a, f)$ the set of all a-points of *f*, where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that *f*, *g* share the value *a* with weight *k*.

We write *f*, *g* share (a,k) to mean that *f*, *g* share the value *a* with weight *k*; clearly if *f*, *g* share (a,k), then *f*, *g* share (a,p) for all integers p with $0 \le p \le k$. Also, we note that *f*, *g* share a value *a* IM or CM if and only if they share $(a,0)$ or (a,∞), respectively.

R. Bruck [5] first considered the uniqueness problems of an entire function sharing one value with its derivative and proved the following result.

Theorem A. Let *f* be a non-constant entire function satisfying $N\left(r, \frac{1}{f'}\right) = S(r, f)$. If *f* and f' share the value 1 CM, then $\frac{f'-1}{f-1} \equiv c$ for some nonzero constant c.

Bruck [5] further posed the following conjecture.

Conjecture 1.1. Let *f* be a non-constant entire function $\rho_1(f)$ be the first iterated order of *f*. If $\rho_1(f)$ is not a positive integer or infinite, *f* and f' share the value 1 CM, then $\frac{f'-1}{f-1} \equiv c$ for some nonzero constant.

Yang [6] proved that the conjecture is true if *f* is an entire function of finite order. Yu [7] considered the problem of an entire or meromorphic function sharing one small function with its derivative and proved the following two theorems.

Theorem B. Let *f* be a non-constant entire function and $a \equiv a(z) (\not\equiv 0, \infty)$ be a meromorphic small function. If $f - a$ and $f^{(k)} - a$ share 0 CM and $\delta(0, f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.

Theorem C. Let *f* be a non-constant non-entire meromorphic function and $a \equiv a(z) (\not\equiv 0, \infty)$ be a meromorphic small function. If

1) *f* and *a* have no common poles.
2) $f - a$ and $f^{(k)} - a$ share 0 CM.
3) $4\delta(0, f) + 2(8+k)\Theta(\infty, f) > 19 + 2k$,

then $f \equiv f^{(k)}$ where *k* is a positive integer.

In the same paper, Yu [7] posed the following open questions.

1) Can a CM shared be replaced by an IM share value?
2) Can the condition $\delta(0, f) > \frac{3}{4}$ of theorem B be further relaxed?
3) Can the condition 3) in theorem C be further relaxed?
4) Can in general the condition 1) of theorem C be dropped?

In 2004, Liu and Gu [8] improved theorem B and obtained the following results.

Theorem D. Let *f* be a non-constant entire function $a \equiv a(z) (\not\equiv 0, \infty)$ be a meromorphic small function. If $f - a$ and $f^{(k)} - a$ share 0 CM and $\delta(0, f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.

Lahiri and Sarkar [9] gave some affirmative answers to the first three questions improving some restrictions on the zeros and poles of *a*. They obtained the following results.

Theorem E. Let *f* be a non-constant meromorphic function, *k* be a positive integer, and $a \equiv a(z) (\not\equiv 0, \infty)$ be a meromorphic small function. If

1) *a* has no zero (pole) which is also a zero (pole) of *f* or $f^{(k)}$ with the same multiplicity.
2) $f - a$ and $f^{(k)} - a$ share $(0,2)$
3) $2\delta_{2+k}(0, f) + (4+k)\Theta(\infty, f) > 5 + k$ then $f \equiv f^{(k)}$.

In 2005, Zhang [10] improved the above results and proved the following theorems.

Theorem F. Let *f* be a non-constant meromorphic function, $k(\ge 1), l(\ge 0)$ be integers. Also let $a \equiv a(z) (\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0,l)$. If $l \ge 2$ and

$$(3+k)\Theta(\infty, f)+2\delta_{2+k}(0, f)>k+4 \tag{1}$$

or $l=1$ and

$$(4+k)\Theta(\infty, f)+3\delta_{2+k}(0, f)>k+6 \tag{2}$$

or $l=0$ and

$$(6+2k)\Theta(\infty, f)+5\delta_{2+k}(0, f)>2k+10 \tag{3}$$

then $f \equiv f^{(k)}$.

In 2015, Jin-Dong Li and Guang-Xiu Huang proved the following Theorem.

Theorem G. Let f be a non-constant meromorphic function, $k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f-a$ and $f^{(k)}-a$ share $(0, l)$. If $l \geq 2$ and

$$(3+k)\Theta(\infty, f)+\delta_2(0, f)+\delta_{2+k}(0, f)>k+4 \tag{4}$$

$l=1$ and

$$\left(\frac{7}{2}+k\right)\Theta(\infty, f)+\frac{1}{2}\Theta(0, f)+\delta_2(0, f)+\delta_{2+k}(0, f)>k+5 \tag{5}$$

or $l=0$ and

$$(6+2k)\Theta(\infty, f)+2\Theta(\infty, f)+\delta_2(0, f)+\delta_{1+k}(0, f)+\delta_{2+k}(0, f)>2k+10 \tag{6}$$

then $f \equiv f^{(k)}$.

In this paper, we pay our attention to the uniqueness of more generalized form of a function namely $f^n P(f)$ and $\left[f^n P(f)\right]^{(k)}$ sharing a small function.

Theorem 1.1. Let f be a non-constant meromorphic function, $k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f^n P(f)-a$ and $\left[f^n P(f)\right]^{(k)}-a$ share $(0, l)$. If

$l \geq 2$ and

$$(k+3)\Theta(\infty, f)+(k+4)\Theta(0, f)+2m\delta(0, f)>2k+7+m-n \tag{7}$$

$l=1$ and

$$\left(k+\frac{7}{2}\right)\Theta(\infty, f)+\left(k+\frac{9}{2}\right)\Theta(0, f)+2m\delta(0, f)>2k+8+m-n \tag{8}$$

or $l=0$ and

$$(2k+6)\Theta(\infty, f)+(2k+7)\Theta(0, f)+3m\delta(0, f)>4k+13+2m-n \tag{9}$$

then $f^n P(f) \equiv \left[f^n P(f)\right]^{(k)}$.

Corollary 1.2. Let f be a non-constant meromorphic function, $n, m(\geq 1), k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f^n P(f)-a$ and $\left[f^n P(f)\right]^{(k)}-a$ share $(0, l)$. If

$l \geq 2$ and $\delta(0, f)>\frac{1}{3}$

or $l=1$ and $\delta(0, f)>\frac{11}{13}$

or $l=0$ and $\delta(0,f)>\frac{4}{5}-\frac{1}{5}\left[2\Theta(0,f)+2\Theta(\infty,f)-4\delta(0,f)\right]$

then $f^nP(f)\equiv\left[f^nP(f)\right]^{(k)}$.

2. Lemmas

Lemma 2.1 (see [1]**).** Let f be a non-constant meromorphic function, k,p be two positive integers, then

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\le N_{p+k}\left(r,\frac{1}{f}\right)+k\bar{N}(r,f)+S(r,f)$$

clearly $\bar{N}\left(r,\frac{1}{f^{(k)}}\right)=N_1\left(r,\frac{1}{f^{(k)}}\right)$

Lemma 2.2 (see [1]). Let

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right) \tag{10}$$

where F and G are two non constant meromorphic functions. If F and G share 1 IM and $H\not\equiv 0$, then

$$N_{11}\left(r,\frac{1}{F-1}\right)\le N(r,H)+S(r,F)+S(r,G)$$

Lemma 2.3 (see [11]). Let f be a non-constant meromorphic function and let

$$R(f)=\frac{\sum_{k=0}^{n}a_kf^k}{\sum_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients a_k and b_j where $a_n\ne 0$ and $b_m\ne 0$. Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

3. Proof of the Theorem

Proof of Theorem 1.1. Let $F=\frac{f^nP(f)}{a}$ and $G=\frac{\left[f^nP(f)\right]^{(k)}}{a}$. Then F and G share $(1,l)$, except the zeros and poles of $a(z)$. Let H be defined by (10).

Case 1. Let $H\not\equiv 0$.

By our assumptions, H have poles only at zeros of F' and G' and poles of F and G, and those 1-points of F and G whose multiplicities are distinct from the multiplicities of corresponding 1-points of G and F respectively. Thus, we deduce from (10) that

$$\begin{aligned}N(r,H)\le{}&\bar{N}_{(2}\left(r,\frac{1}{H}\right)+\bar{N}_{(2}\left(r,\frac{1}{G}\right)+\bar{N}(r,H)+N_0\left(r,\frac{1}{F'}\right)\\&+N_0\left(r,\frac{1}{G'}\right)+\bar{N}_L\left(r,\frac{1}{F-1}\right)+\bar{N}_L\left(r,\frac{1}{G-1}\right)\end{aligned} \tag{11}$$

here $N_0\left(r,\frac{1}{F'}\right)$ is the counting function which only counts those points such that $F'=0$ but $F(F-1)\ne 0$.

Because F and G share 1 IM, it is easy to see that

$$\overline{N}\left(r,\frac{1}{F-1}\right)=N_{11}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_E^{(2}\left(r,\frac{1}{G-1}\right)=\overline{N}\left(r,\frac{1}{G-1}\right) \tag{12}$$

By the second fundamental theorem, we see that

$$\begin{aligned}T(r,F)+T(r,G)\le{}&\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)\\&+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,F)+S(r,G)\end{aligned} \tag{13}$$

Using Lemma 2.2 and (11), (12) and (13), we get

$$\begin{aligned}T(r,F)+T(r,G)\le{}&3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_{11}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)\\&+3\overline{N}_L\left(r,\frac{1}{F-1}\right)+3\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G)\end{aligned} \tag{14}$$

We discuss the following three sub cases.

Sub case 1.1. $l\ge 2$. Obviously.

$$\begin{aligned}&N_{11}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+3\overline{N}_L\left(r,\frac{1}{F-1}\right)+3\overline{N}_L\left(r,\frac{1}{G-1}\right)\\&\le N\left(r,\frac{1}{G-1}\right)+S(r,F)\le T(r,G)+S(r,F)+S(r,G)\end{aligned} \tag{15}$$

Combining (14) and (15), we get

$$T(r,F)\le 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+S(r,F) \tag{16}$$

that is

$$(n+m)T(r,f)\le 3\overline{N}\left(r,f^nP(f)\right)+N_2\left(r,\frac{1}{f^nP(f)}\right)+N_2\left(r,\frac{1}{\left[f^nP(f)\right]^{(k)}}\right)+S(r,f)$$

By Lemma 2.1 for $p=2$, we get

$$(n+m)T(r,f)\le(k+4)\overline{N}\left(r,\frac{1}{f}\right)+(k+3)\overline{N}(r,f)+2mN\left(r,\frac{1}{f}\right)+S(r,f)$$

So

$$(k+3)\Theta(\infty,f)+(k+4)\Theta(0,f)+2m\delta(0,f)\le 2k+7+m-n$$

which contradicts with (7).

Sub case 1.2. $l=1$. It is easy to see that

$$\begin{aligned}&N_{11}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+3\overline{N}_L\left(r,\frac{1}{G-1}\right)\\&\le N\left(r,\frac{1}{G-1}\right)+S(r,F)\le T(r,G)+S(r,F)+S(r,G)\end{aligned} \tag{17}$$

and

$$\begin{aligned}\overline{N}_L\left(r,\frac{1}{F-1}\right)&\le\frac{1}{2}N\left(r,\frac{F}{F'}\right)\le\frac{1}{2}N\left(r,\frac{F'}{F}\right)+S(r,F)\\&\le\frac{1}{2}\left[\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)\right]+S(r,F)\end{aligned} \tag{18}$$

Combining (14) and (17) and (18), we get

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\frac{7}{2}\bar{N}(r,F)+\frac{1}{2}\bar{N}\left(r,\frac{1}{F}\right)+S(r,f) \tag{19}$$

that is

$$(n+m)T(r,f) \le N_2\left(r,\frac{1}{f^nP(f)}\right)+N_2\left(r,\frac{1}{\left[f^nP(f)\right]^{(k)}}\right)+\frac{7}{2}\bar{N}\left(r,f^nP(f)\right)$$
$$+\frac{1}{2}\bar{N}\left(r,\frac{1}{f^nP(f)}\right)+S(r,f)$$

By Lemma 2.1 for $p=2$, we get

$$(n+m)T(r,f) \le \left(k+\frac{7}{2}\right)\bar{N}(r,f)+\left(k+\frac{9}{2}\right)\bar{N}\left(r,\frac{1}{f}\right)+2mN\left(r,\frac{1}{f}\right)+S(r,f)$$

So

$$\left(k+\frac{7}{2}\right)\Theta(\infty,f)+\left(k+\frac{9}{2}\right)\Theta(0,f)+2m\delta(0,f) \le 2k+8+m-n$$

which contradicts with (8).

Sub case 1.3. $l=0$. It is easy to see that

$$N_{11}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+\bar{N}_L\left(r,\frac{1}{F-1}\right)+2\bar{N}_L\left(r,\frac{1}{G-1}\right)$$
$$\le N\left(r,\frac{1}{G-1}\right)+S(r,F) \le T(r,G)+S(r,F)+S(r,G) \tag{20}$$

$$\bar{N}_L\left(r,\frac{1}{F-1}\right) \le N\left(r,\frac{1}{F-1}\right)-\bar{N}\left(r,\frac{1}{F-1}\right) \le N\left(r,\frac{F}{F'}\right)$$
$$\le N\left(r,\frac{F'}{F}\right)+S(r,F) \le \bar{N}\left(r,\frac{1}{F}\right)+\bar{N}(r,F)+S(r,F) \tag{21}$$

Similarly we have

$$\bar{N}_L\left(r,\frac{1}{G-1}\right) \le \bar{N}\left(r,\frac{1}{G}\right)+\bar{N}(r,G)+S(r,F)$$
$$\le N_1\left(r,\frac{1}{G}\right)+\bar{N}(r,F)+S(r,F) \tag{22}$$

Combining (14) and (20)-(22), we get

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2\bar{N}\left(r,\frac{1}{F}\right)+6\bar{N}(r,F)+N_1\left(r,\frac{1}{G}\right)+S(r,F) \tag{23}$$

that is

$$(n+m)T(r,f) \le N_2\left(r,\frac{1}{f^nP(f)}\right)+N_2\left(r,\frac{1}{\left[f^nP(f)\right]^{(k)}}\right)+2\bar{N}\left(r,\frac{1}{f^nP(f)}\right)$$
$$+6\bar{N}\left(r,\frac{1}{f^nP(f)}\right)+N_1\left(r,\frac{1}{\left[f^nP(f)\right]^{(k)}}\right)+S(r,f)$$

By Lemma 2.1 for $p=2$ and for $p=1$ respectively, we get

$$(n+m)T(r,f)\le(2k+7)\bar{N}\left(r,\frac{1}{f}\right)+(2k+6)\bar{N}(r,f)+3mN\left(r,\frac{1}{f}\right)$$

So

$$(2k+6)\Theta(\infty,f)+(2k+7)\Theta\left(0,f+2m\delta(0,f)\right)\le 4k+13+2m-n$$

which contradicts with (9).

Case 2. Let $H\equiv 0$.

on integration we get from (10)

$$\frac{1}{F-1}\equiv\frac{C}{G-1}+D, \tag{24}$$

where C, D are constants and $c\ne 0$. we will prove that $D=0$.

Sub case 2.1. Suppose $D\ne 0$. If z_0 be a pole of f with multiplicity p such that $a(z_0)\ne 0,\infty$, then it is a pole of G with multiplicity $(n+m)p+k$ respectively. This contradicts (24). It follows that $N(r,f)=S(r,f)$ and hence $\Theta(\infty,f)=1$. Also it is clear that $\bar{N}(r,f)=\bar{N}(r,G)=S(r,f)$. From (7)-(9) we know respectively

$$(k+4)\Theta(0,f)+2m\delta(0,f)>k+4+m-n \tag{25}$$

$$\left(k+\frac{9}{2}\right)\Theta(0,f)+2m\delta(0,f)>k+\frac{9}{2}+m-n \tag{26}$$

and

$$(2k+7)\Theta(0,f)+2m\delta(0,f)>2k+7+m-n \tag{27}$$

Since $D\ne 0$, from (24) we get

$$\bar{N}\left(r,\frac{1}{F-\left(1+\frac{1}{D}\right)}\right)=\bar{N}(r,G)=S(r,f),$$

Suppose $D\ne -1$.

Using the second fundamental theorem for F we get

$$T(r,F)\le\bar{N}(r,F)+\bar{N}\left(r,\frac{1}{F}\right)+\bar{N}\left(r,\frac{1}{F-\left(1+\frac{1}{D}\right)}\right)\le\bar{N}\left(r,\frac{1}{F}\right)+S(r,f)$$

i.e.,

$$(n+m)T(r,F)\le\bar{N}\left(r,\frac{1}{F}\right)+S(r,f)\le(n+m)T(r,f)+S(r,f).$$

So, we have $(n+m)T(r,f)=\bar{N}\left(r,\frac{1}{f}\right)$ and so $\Theta(0,f)=1-(n+m)$. which contradicts (25)-(27).

If $D=-1$, then

$$\frac{F}{F-1}\equiv C\frac{1}{G-1} \tag{28}$$

and from which we know $\bar{N}\left(r,\frac{1}{F}\right)=\bar{N}(r,G)=S(r,f)$ and hence, $\bar{N}\left(r,\frac{1}{f}\right)=S(r,f)$. If $C\ne -1$, We know from (28) that

$$\bar{N}\left(r,\frac{1}{G-(1+C)}\right)=\bar{N}(r,F)=S(r,f)$$

So from Lemma 2.1 and the second fundamental theorem we get

$$T\left(r,\left[f^nP(f)\right]^{(k)}\right)\le\bar{N}(r,G)+\bar{N}\left(r,\frac{1}{G}\right)+\bar{N}\left(r,\frac{1}{G-(1+C)}\right)+S(r,f)$$

$$\le\bar{N}\left(r,\frac{1}{\left[f^nP(f)\right]^{(k)}}\right)+S(r,f)$$

$$(n+m)T(r,f)\le(k+1)\bar{N}\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f}\right)+k\bar{N}(r,f)+S(r,f),$$

which is absurd. So $C=-1$ and we get from (28) that $FG\equiv1$, which implies $\left[\frac{\left[f^nP(f)\right]^{(k)}}{f^nP(f)}\right]=\frac{a^2}{f^{n+m}}.$

In view of the first fundamental theorem, we get from above

$$(n+m)T(r,f)\le k\left[\bar{N}(r,f)+\bar{N}\left(r,\frac{1}{f}\right)\right]+S(r,f)=S(r,f),$$

which is impossible.

Sub case 2.2. $D=0$ and so from (24) we get

$$G-1\equiv C(F-1).$$

If $C\ne1$, then

$$G\equiv C\left(F-1+\frac{1}{C}\right)$$

and $\bar{N}\left(r,\frac{1}{G}\right)=\bar{N}\left(r,\frac{1}{F-\left(1-\frac{1}{C}\right)}\right).$

By the second fundamental theorem and Lemma 2.1 for $p=1$ and Lemma 2.3 we have

$$(n+m)T(r,f)+S(r,f)=T(r,F)$$

$$\le\bar{N}(r,F)+\bar{N}\left(r,\frac{1}{F}\right)+\left(r,\frac{1}{F-\left(1-\frac{1}{C}\right)}\right)+S(r,G)$$

$$\le\bar{N}\left(r,f^nP(f)\right)+\bar{N}\left(r,\frac{1}{f^nP(f)}\right)+\bar{N}\left(r,\frac{1}{\left[f^nP(f)\right]^{(k)}}\right)+S(r,f)$$

$$\le\bar{N}(r,f)+\bar{N}\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f}\right)+(k+1)\bar{N}\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f}\right)+k\bar{N}(r,f)+S(r,f)$$

$$\le(k+2)\bar{N}\left(r,\frac{1}{f}\right)+(k+1)\bar{N}(r,f)+2mN\left(r,\frac{1}{f}\right)+S(r,f).$$

Hence

$$\left(k+1\right)\Theta\left(\infty, f\right)+\left(k+2\right)\Theta\left(0, f\right)+2m\delta\left(0, f\right)\le 2k+3+m-n$$

So, it follows that

$$\left(k+3\right)\Theta\left(\infty, f\right)+\left(k+4\right)\Theta\left(0, f\right)+2m\delta\left(0, f\right)\le 2k+7+m-n$$

$$\left(k+\frac{7}{2}\right)\Theta\left(\infty, f\right)+\left(k+\frac{9}{2}\right)\Theta\left(0, f\right)+3m\delta\left(0, f\right)\le 2k+8+m-n,$$

and

$$\left(2k+6\right)\Theta\left(\infty, f\right)+\left(2k+7\right)\Theta\left(0, f\right)+3m\delta\left(0, f\right)\le 4k+13+m-n.$$

This contradicts (7)-(9). Hence $C=1$ and so $F\equiv G$, that is $f^nP\left(f\right)\equiv\left[f^nP\left(f\right)\right]^{(k)}$. This completes the proof of the theorem.

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