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On Elliptic Problem with Singular Cylindrical Potential, a Concave Term, and Critical Caffarelli-Kohn-Nirenberg Exponent

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Abstract

In this paper, we establish the existence of at least four distinct solutions to an elliptic problem with singular cylindrical potential, a concave term, and critical Caffarelli-Kohn-Nirenberg exponent, by using the Nehari manifold and mountain pass theorem.

Keywords

Singular Cylindrical Potential, Concave Term, Critical Caffarelli-Kohn-Nirenberg Exponent, Nehari Manifold, Mountain Pass Theorem

1. Introduction

In this paper, we consider the multiplicity results of nontrivial nonnegative solutions of the following problem $(\mathcal{P}_{\lambda,\mu})$

$$\begin{cases} L_{\mu,a}u = 2_* |y|^{-2_* b} h|u|^{2_*-2} u + \lambda |y|^{-c} f |u|^{q-2} u, \text{ in } \mathbb{R}^N, y \neq 0 \\ u \in \mathcal{D}_a^{1,2} \end{cases}$$

where $L_{\mu,a}v := -div\left(|y|^{-2a} \nabla v\right) - \mu |y|^{-2(a+1)} v$, where each point x in $\mathbb{R}^N$ is written as a pair $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ where k and N are integers such that $N \geq 3$ and k belongs to $\{1,\cdots,N\}$, $-\infty < a < (k-2)/2$, $a \leq b < a+1$, $1 < q < 2$, $2_* = 2N/(N-2+2(b-a))$ is the critical Caffarelli-Kohn-Nirenberg exponent,

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$0 < c = q(a+1) + N(1-q/2)$, $-\infty < \mu < \bar{\mu}_{a,k} := \left(\left(k-2(a+1)\right)/2\right)^2$, λ is a real parameter, $f \in \mathcal{H}'_\mu \cap C(\mathbb{R}^N)$, h is a bounded positive function on $\mathbb{R}^k$. $\mathcal{H}'_\mu$ is the dual of $\mathcal{H}_\mu$, where $\mathcal{H}_\mu$ and $\mathcal{D}_0^{1,2}$ will be defined later.

Some results are already available for $(\mathcal{P}_{\lambda,\mu})$ in the case $k = N$, see for example [1] [2] and the references therein. Wang and Zhou [1] proved that there exist at least two solutions for $(\mathcal{P}_{\lambda,\mu})$ with $a = 0$, $0 < \mu \le \bar{\mu}_{0,N} = \left((N-2)/2\right)^2$ and $h \equiv 1$, under certain conditions on *f*. Bouchekif and Matallah [3] showed the existence of two solutions of $(\mathcal{P}_{\lambda,\mu})$ under certain conditions on functions *f* and *h*, when $0 < \mu \le \bar{\mu}_{0,N}$, $\lambda \in (0, \Lambda_*)$, $-\infty < a < (N-2)/2$ and $a \le b < a+1$, with Λ_* a positive constant.

Concerning existence results in the case $k < N$, we cite [4] [5] and the references therein. Musina [5] considered $(\mathcal{P}_{\lambda,\mu})$ with $-a/2$ instead of a and $\lambda = 0$, also $(\mathcal{P}_{\lambda,\mu})$ with $a = 0$, $b = 0$, $\lambda = 0$, with $h \equiv 1$ and $a \ne 2-k$. She established the existence of a ground state solution when $2 < k \le N$ and $0 < \mu < \bar{\mu}_{a,k} = \left((k-2+a)/2\right)^2$ for $(\mathcal{P}_{\lambda,\mu})$ with $-a/2$ instead of *a* and $\lambda = 0$. She also showed that $(\mathcal{P}_{\lambda,\mu})$ with $a = 0$, $b = 0$, $\lambda = 0$ does not admit ground state solutions. Badiale *et al.* [6] studied $(\mathcal{P}_{\lambda,\mu})$ with $a = 0$, $b = 0$, $\lambda = 0$ and $h \equiv 1$. They proved the existence of at least a nonzero nonnegative weak solution *u*, satisfying $u(y,z) = u(|y|, z)$ when $2 \le k < N$ and $\mu < 0$. Bouchekif and El Mokhtar [7] proved that $(\mathcal{P}_{\lambda,\mu})$ admits two distinct solutions when $2 < k \le N$, $b = N - p(N-2)/2$ with $p \in (2, 2^*]$, $\mu < \bar{\mu}_{0,k}$, and $\lambda \in (0, \Lambda_*)$ where Λ_* is a positive constant. Terracini [8] proved that there is no positive solutions of $(\mathcal{P}_{\lambda,\mu})$ with $b = 0$, $\lambda = 0$ when $a \ne 0$, $h \equiv 1$ and $\mu < 0$. The regular problem corresponding to $a = b = \mu = 0$ and $h \equiv 1$ has been considered on a regular bounded domain Ω by Tarantello [9]. She proved that, for $f \in H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$, not identically zero and satisfying a suitable condition, the problem considered admits two distinct solutions.

Before formulating our results, we give some definitions and notation.

We denote by $\mathcal{D}_a^{1,2} = \mathcal{D}_a^{1,2}\left(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k}\right)$ and $\mathcal{H}_\mu = \mathcal{H}_\mu\left(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k}\right)$, the closure of $C_0^\infty\left(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k}\right)$ with respect to the norms

$$\|u\|_{a,0} = \left(\int_{\mathbb{R}^N} |y|^{-2a} |\nabla u|^2 \, \mathrm{d}x\right)^{1/2}$$

and

$$\|u\|_{a,\mu} = \left(\int_{\mathbb{R}^N} \left(|y|^{-2a} |\nabla u|^2 - \mu |y|^{-2(a+1)} |u|^2\right) \mathrm{d}x\right)^{1/2},$$

respectively, with $\mu < \bar{\mu}_{a,k} = \left(\left(k-2(a+1)\right)/2\right)^2$ for $k \ne 2(a+1)$.

From the Hardy-Sobolev-Maz'ya inequality, it is easy to see that the norm $\|u\|_{a,\mu}$ is equivalent to $\|u\|_{a,0}$. More explicitly, we have

$$\left(1 - \left(1/\bar{\mu}_{a,k}\right) \max(\mu, 0)\right)^{1/2} \|u\|_{0,a} \le \|u\|_{\mu,a} \le \left(1 - \left(1/\bar{\mu}_{a,k}\right) \min(\mu, 0)\right)^{1/2} \|u\|_{0,a},$$

for all $u \in \mathcal{H}_\mu$.

We list here a few integral inequalities.

The starting point for studying $(\mathcal{P}_{\lambda,\mu})$, is the Hardy-Sobolev-Maz'ya inequality that is particular to the cylindrical case $k < N$ and that was proved by Maz'ya in [4]. It states that there exists positive constant $C_{a,2_*}$ such that

$$C_{a,2_*} \left(\int_{\mathbb{R}^N} |y|^{-2_* b} |v|^{2_*} \, \mathrm{d}x\right)^{2/2_*} \le \int_{\mathbb{R}^N} \left(|y|^{-2a} |\nabla v|^2 - \mu |y|^{-2(a+1)} v^2\right) \mathrm{d}x, \tag{1.1}$$

for any $v \in C_c^\infty\left(\left(\mathbb{R}^k \setminus \{0\}\right) \times \mathbb{R}^{N-k}\right)$.

The second one that we need is the Hardy inequality with cylindrical weights [5]. It states that

$$\bar{\mu}_{a,k}\int_{\mathbb{R}^N}|y|^{-2(a+1)}v^2\mathrm{d}x \le \int_{\mathbb{R}^N}|y|^{-2a}|\nabla v|^2\,\mathrm{d}x,\ \text{ for all } v\in\mathcal{H}_\mu, \tag{1.2}$$

It is easy to see that (1.1) hold for any $u\in\mathcal{H}_\mu$ in the sense

$$\left(\int_{\mathbb{R}^N}|y|^{-c}|u|^p\,\mathrm{d}x\right)^{1/p} \le C_{a,p}\left(\int_{\mathbb{R}^N}|y|^{-2a}|\nabla v|^2\,\mathrm{d}x\right)^{1/p} \tag{1.3}$$

where $C_{a,p}$ positive constant, $1\le p\le 2N/(N-2)$, $c\le p(a+1)+N(1-p/2)$, and in [10], if $p<2N/(N-2)$ the embedding $\mathcal{H}_\mu \to L_p\left(\mathbb{R}^N,|y|^{-c}\right)$ is compact, where $L_p\left(\mathbb{R}^N,|y|^{-c}\right)$ is the weighted L_p space with norm

$$|u|_{p,c}=\left(\int_{\mathbb{R}^N}|y|^{-c}|u|^p\,\mathrm{d}x\right)^{1/p}.$$

Since our approach is variational, we define the functional J on $\mathcal{H}_\mu$ by

$$J(u):=(1/2)\|u\|_{\mu,a}^2-P(u)-Q(u),$$

with

$$P(u):=2_*\int_{\mathbb{R}^N}|y|^{-2_*b}h|u|^{2_*}\,\mathrm{d}x,\quad Q(u):=(1/q)\int_{\mathbb{R}^N}|y|^{-c}\lambda f|u|^q\,\mathrm{d}x.$$

A point $u\in\mathcal{H}_\mu$ is a weak solution of the equation $\left(\mathcal{P}_{\lambda,\mu}\right)$ if it satisfies

$$\langle J'(u),\varphi\rangle := R(u)\varphi - S(u)\varphi - T(u)\varphi = 0,\quad \text{for all } \varphi\in\mathcal{H}_\mu$$

with

$$R(u)\varphi := \int_{\mathbb{R}^N}\left(|y|^{-2a}(\nabla u\nabla\varphi)-\mu|y|^{-2(a+1)}(u\varphi)\right)$$

$$S(u)\varphi := 2_*\int_{\mathbb{R}^N}|y|^{-2_*b}h|u|^{2_*}\varphi$$

$$T(u)\varphi := \int_{\mathbb{R}^N}|y|^{-c}\left(\lambda f|u|^{q-1}\varphi\right).$$

Here $\langle .,.\rangle$ denotes the product in the duality $\mathcal{H}'_\mu$, $\mathcal{H}_\mu$ $\left(\mathcal{H}'_\mu \text{ dual of } \mathcal{H}_\mu\right)$.

Let

$$S_\mu := \inf_{u\in\mathcal{H}_\mu\setminus\{0\}}\frac{\|u\|_{\mu,a}^2}{\left(\int_{\mathbb{R}^N}|y|^{-2_*b}|u|^{2_*}\,\mathrm{d}x\right)^{2/2_*}}$$

From [11], S_μ is achieved.

Throughout this work, we consider the following assumptions:

(F) there exist $\nu_0>0$ and $\delta_0>0$ such that $f(x)\ge\nu_0$, for all x in $B(0,2\delta_0)$.

(H) $\lim_{|y|\to 0}h(y)=\lim_{|y|\to\infty}h(y)=h_0>0,\ h(y)\ge h_0,\ y\in\mathbb{R}^k$.

Here, $B(a,r)$ denotes the ball centered at a with radius r.

In our work, we research the critical points as the minimizers of the energy functional associated to the problem $\left(\mathcal{P}_{\lambda,\mu}\right)$ on the constraint defined by the Nehari manifold, which are solutions of our system.

Let Λ_0 be positive number such that

$$\Lambda_0 := \left(C_{a,q}\right)^{-q}\left(h_0\right)^{-1/(2_*-2)}\left(S_\mu\right)^{2_*/2(2_*-2)}L(q),$$

where $L(q):=\left(\dfrac{2_*-2}{2_*-q}\right)^{1/(2-q)}\left[\left(\dfrac{2-q}{2_*(2_*-q)}\right)\right]^{1/(2_*-2)}$.

Now we can state our main results.

Theorem 1. *Assume that* $-\infty < a < (k-2)/2$, $0 < c = q(a+1) + N(1-q/2)$, $-\infty < \mu < \overline{\mu}_{a,k}$, (F) *satisfied and* λ *verifying* $0 < \lambda < \Lambda_0$, *then the system* $(\mathcal{P}_{\lambda,\mu})$ *has at least one positive solution.*

Theorem 2. *In addition to the assumptions of the* Theorem 1, *if* (H) *hold and* λ *satisfying* $0 < \lambda < (1/2)\Lambda_0$, *then* $(\mathcal{P}_{\lambda,\mu})$ *has at least two positive solutions.*

Theorem 3. *In addition to the assumptions of the* Theorem 2, *assuming* $N \geq \max(3, 6(a-b+1))$, *there exists a positive real* Λ_1 *such that, if* λ *satisfy* $0 < \lambda < \min((1/2)\Lambda_0, \Lambda_1)$, *then* $(\mathcal{P}_{\lambda,\mu})$ *has at least two positive solution and two opposite solutions.*

This paper is organized as follows. In Section 2, we give some preliminaries. Sections 3 and 4 are devoted to the proofs of Theorems 1 and 2. In the last Section, we prove the Theorem 3.

2. Preliminaries

Definition 1. *Let* $c \in \mathbb{R}$, *E a Banach space and* $I \in C^1(E, \mathbb{R})$.

i) $(u_n)_n$ is a Palais-Smale sequence at level *c* (in short $(PS)_c$) in *E* for *I* if

$$I(u_n) = c + o_n(1) \text{ and } I'(u_n) = o_n(1),$$

where $o_n(1)$ tends to 0 as *n* goes at infinity.

ii) We say that *I* satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in *E* for *I* has a convergent subsequence.

Lemma 1. *Let X Banach space, and* $J \in C^1(X, \mathbb{R})$ *verifying the Palais-Smale condition. Suppose that* $J(0) = 0$ *and that*:

i) there exist $R > 0$, $r > 0$ such that if $\|u\| = R$, then $J(u) \geq r$;

ii) there exist $(u_0) \in X$ such that $\|u_0\| > R$ and $J(u_0) \leq 0$;

let $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} (J(\gamma(t)))$ where

$$\Gamma = \{\gamma \in C([0,1]; X) \textit{ such that } \gamma(0) = 0 \text{ et } \gamma(1) = u_0\},$$

then *c* is critical value of *J* such that $c \geq r$.

Nehari Manifold

It is well known that *J* is of class C^1 in $\mathcal{H}_\mu$ and the solutions of $(\mathcal{P}_{\lambda,\mu})$ are the critical points of *J* which is not bounded below on $\mathcal{H}_\mu$. Consider the following Nehari manifold

$$\mathcal{N} = \{u \in \mathcal{H}_\mu \setminus \{0\} : \langle J'(u), u \rangle = 0\},$$

Thus, $u \in \mathcal{N}$ if and only if

$$\|u\|^2_{\mu,a} - 2_* P(u) - Q(u) = 0. \tag{2.1}$$

Note that $\mathcal{N}$ contains every nontrivial solution of the problem $(\mathcal{P}_{\lambda,\mu})$. Moreover, we have the following results.

Lemma 2. *J is coercive and bounded from below on* $\mathcal{N}$.

Proof. If $u \in \mathcal{N}$, then by (2.1) and the Hölder inequality, we deduce that

$$\begin{aligned} J(u) &= ((2_* - 2)/2_* 2)\|u\|^2_{\mu,a} - ((2_* - q)/2_* q) Q(u) \\ &\geq ((2_* - 2)/2_* 2)\|u\|^2_{\mu,a} - \left(\frac{(2_* - q)}{2_* q}\right)\left(\lambda \|f\|_{\mathcal{H}'_\mu}\right)^{1/(2-q)} (C_{a,p})^q \|u\|^q_{\mu,a}. \end{aligned} \tag{2.2}$$

Thus, *J* is coercive and bounded from below on $\mathcal{N}$.

Define

$$\phi(u) = \langle J'(u), u \rangle.$$

Then, for $u \in \mathcal{N}$

$$\begin{aligned}\langle \phi'(u), u\rangle &= 2\|u\|_{\mu,a}^2 - (2_*)^2 P(u) - qQ(u) \\ &= (2-q)\|u\|_{\mu,a}^2 - 2_*(2_*-q)P(u) \\ &= (2_*-q)Q(u) - (2_*-2)\|u\|_{\mu,a}^2 .\end{aligned} \tag{2.3}$$

Now, we split $\mathcal{N}$ in three parts:

$$\mathcal{N}^+ = \{u \in \mathcal{N} : \langle \phi'(u), u\rangle > 0\}$$

$$\mathcal{N}^0 = \{u \in \mathcal{N} : \langle \phi'(u), u\rangle = 0\}$$

$$\mathcal{N}^- = \{u \in \mathcal{N} : \langle \phi'(u), u\rangle < 0\}.$$

We have the following results.

Lemma 3. *Suppose that* u_0 *is a local minimizer for J on* $\mathcal{N}$. *Then, if* $u_0 \notin \mathcal{N}^0$, u_0 *is a critical point of J.*

Proof. If u_0 is a local minimizer for *J* on $\mathcal{N}$, then u_0 is a solution of the optimization problem

$$\min_{\{u/\phi(u)=0\}} J(u).$$

Hence, there exists a Lagrange multipliers $\theta \in \mathbb{R}$ such that

$$J'(u_0) = \theta\phi'(u_0) \text{ in } \mathcal{H}'$$

Thus,

$$\langle J'(u_0), u_0\rangle = \theta\langle \phi'(u_0), u_0\rangle.$$

But $\langle \phi'(u_0), u_0\rangle \neq 0$, since $u_0 \notin \mathcal{N}^0$. Hence $\theta = 0$. This completes the proof.

Lemma 4. *There exists a positive number* Λ_0 *such that for all* λ, *verifying*

$$0 < \lambda < \Lambda_0,$$

we have $\mathcal{N}^0 = \varnothing$.

Proof. Let us reason by contradiction.

Suppose $\mathcal{N}^0 \neq \varnothing$ such that $0 < \lambda < \Lambda_0$. Then, by (2.3) and for $u \in \mathcal{N}^0$, we have

$$\|u\|_{\mu,a}^2 = 2_*(2_*-q)/(2-q)P(u) = ((2_*-q)/(2_*-2))Q(u) \tag{2.4}$$

Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\|u\|_{\mu,a} \geq (S_\mu)^{2_*/2(2_*-2)}\left[(2-q)/2_*(2_*-q)h_0\right]^{-1/(2_*-2)} \tag{2.5}$$

and

$$\|u\|_{\mu,a} \leq \left[\left(\frac{2_*-q}{2_*-2}\right)^{-1/(2-q)}\left(\lambda^{1/(2-q)}\right)\left(C_{a,q}\right)^q\right]. \tag{2.6}$$

From (2.5) and (2.6), we obtain $\lambda \geq \Lambda_0$, which contradicts an hypothesis.

Thus $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$. Define

$$c := \inf_{u\in\mathcal{N}} J(u),\ c^+ := \inf_{u\in\mathcal{N}^+} J(u) \text{ and } c^- := \inf_{u\in\mathcal{N}^-} J(u).$$

For the sequel, we need the following Lemma.

Lemma 5.

i) For all λ such that $0 < \lambda < \Lambda_0$, one has $c \leq c^+ < 0$.

ii) For all λ such that $0<\lambda<(1/2)\Lambda_0$, one has

$$c^- > C_0 = C_0\left(\lambda_1,\lambda_2,S_\mu,\|f\|_{\mathcal{H}'_\mu}\right)$$
$$=\left(\frac{(2_*-2)}{2_*2}\right)\left[\frac{(2-q)}{2_*(2_*-q)h_0}\right]^{-2/(2_*-2)}\left(S_\mu\right)^{2_*/(2_*-2)}$$
$$-\left(\frac{(2_*-q)}{2_*q}\right)\left(\left(\lambda\|f\|_{\mathcal{H}'_\mu}\right)^{1/(2-q)}\right)\left(C_{a,q}\right)^q.$$

Proof. i) Let $u\in\mathcal{N}^+$. By (2.3), we have

$$\left[(2-q)/2_*(2_*-1)\right]\|u\|_{\mu,a}^2 > P(u)$$

and so

$$J(u)=(-1/2)\|u\|_{\mu,a}^2+(2_*-1)P(u)<-\left[\frac{2_*(2_*-q)-2(2_*-1)(2-q)}{2_*2(2_*-q)}\right]\|u\|_{\mu,a}^2.$$

We conclude that $c\le c^+<0$.

ii) Let $u\in\mathcal{N}^-$. By (2.3), we get

$$\left[(2-q)/2_*(2_*-q)\right]\|u\|_{\mu,a}^2 < P(u).$$

Moreover, by (H) and Sobolev embedding theorem, we have

$$P(u)\le\left(S_\mu\right)^{-2_*/2}h_0\|u\|_{\mu,a}^{2_*}.$$

This implies

$$\|u\|_{\mu,a} > \left(S_\mu\right)^{2_*/2(2_*-2)}\left[\frac{(2-q)}{2_*(2_*-q)h_0}\right]^{-1/(2_*-2)},\quad \text{for all } u\in\mathcal{N}^-. \tag{2.7}$$

By (2.2), we get

$$J(u)\ge\left((2_*-2)/2_*2\right)\|u\|_{\mu,a}^2-\left(\frac{(2_*-q)}{2_*q}\right)\left(\lambda\|f\|_{\mathcal{H}'_\mu}\right)^{1/(2-q)}\left(C_{a,p}\right)^q\|u\|_{\mu,a}^q.$$

Thus, for all λ such that $0<\lambda<(1/2)\Lambda_0$, we have $J(u)\ge C_0$.

For each $u\in\mathcal{H}$ with $\int_{\mathbb{R}^N}|y|^{-2_*b}h|u|^{2_*}\,\mathrm{d}x>0$, we write

$$t_m := t_{\max}(u)=\left[\frac{(2-q)\|u\|_{\mu,a}^2}{2_*(2_*-q)\int_{\mathbb{R}^N}|y|^{-2_*b}h|u|^{2_*}\,\mathrm{d}x}\right]^{(2-q)/2_*(2_*-q)}>0.$$

Lemma 6. *Let λ real parameters such that $0<\lambda<\Lambda_0$. For each $u\in\mathcal{H}$ with $\int_{\mathbb{R}^N}|y|^{-2_*b}h|u|^{2_*}\,\mathrm{d}x>0$, one has the following:*

i) If $Q(u)\le 0$, then there exists a unique $t^->t_m$ such that $t^-u\in\mathcal{N}^-$ and

$$J\left(t^-u\right)=\sup_{t\ge0}(tu).$$

ii) If $Q(u)>0$, then there exist unique t^+ and t^- such that $0<t^+<t_m<t^-$, $\left(t^+u\right)\in\mathcal{N}^+$, $t^-u\in\mathcal{N}^-$,

$$J\left(t^+u\right)=\inf_{0\le t\le t_m}J(tu)\ \text{ and }\ J\left(t^-u\right)=\sup_{t\ge0}J(tu).$$

Proof. With minor modifications, we refer to [12].

Proposition 1 (see [12])

i) For all λ such that $0<\lambda<\Lambda_0$, there exists a $(PS)_{c^+}$ sequence in $\mathcal{N}^+$.

ii) For all λ such that $0<\lambda<(1/2)\Lambda_0$, there exists a a $(PS)_{c^-}$ sequence in $\mathcal{N}^-$.

3. Proof of Theorems 1

Now, taking as a starting point the work of Tarantello [13], we establish the existence of a local minimum for J on $\mathcal{N}^+$.

Proposition 2. *For all* λ *such that* $0<\lambda<\Lambda_0$, *the functional J has a minimizer* $u_0^+\in\mathcal{N}^+$ *and it satisfies*:

i) $J(u_0^+)=c=c^+$,

ii) (u_0^+) is a nontrivial solution of $(\mathcal{P}_{\lambda,\mu})$.

Proof. If $0<\lambda<\Lambda_0$, then by Proposition 1 (i) there exists a $(u_n)_n$ $(PS)_{c^+}$ sequence in $\mathcal{N}^+$, thus it bounded by Lemma 2. Then, there exists $u_0^+\in\mathcal{H}$ and we can extract a subsequence which will denoted by $(u_n)_n$ such that

$$\begin{aligned} &u_n \rightharpoonup u_0^+ \text{ weakly in } \mathcal{H}\\ &u_n \rightharpoonup u_0^+ \text{ weakly in } L^{2_*}\left(\mathbb{R}^N,|y|^{-2_*b}\right)\\ &u_n \to u_0^+ \text{ strongly in } L^q\left(\mathbb{R}^N,|y|^{-c}\right)\\ &u_n \to u_0^+ \text{ a.e in } \mathbb{R}^N \end{aligned} \tag{3.1}$$

Thus, by (3.1), u_0^+ is a weak nontrivial solution of $(\mathcal{P}_{\lambda,\mu})$. Now, we show that u_n converges to u_0^+ strongly in $\mathcal{H}$. Suppose otherwise. By the lower semi-continuity of the norm, then either $\|u_0^+\|_{\mu,a}<\liminf_{n\to\infty}\|u_n\|_{\mu,a}$ and we obtain

$$c\le J(u_0^+)=((2_*-2)/2_*2)\|u_0^+\|_{\mu,a}^2-((2_*-q)/2_*q)Q(u_0^+)<\liminf_{n\to\infty}J(u_n)=c.$$

We get a contradiction. Therefore, u_n converge to u_0^+ strongly in $\mathcal{H}$. Moreover, we have $u_0^+\in\mathcal{N}^+$. If not, then by Lemma 6, there are two numbers t_0^+ and t_0^-, uniquely defined so that $(t_0^+u_0^+)\in\mathcal{N}^+$ and $(t^-u_0^+)\in\mathcal{N}^-$. In particular, we have $t_0^+<t_0^-=1$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}J(tu_0^+)_{\rfloor t=t_0^+}=0 \text{ and } \frac{\mathrm{d}^2}{\mathrm{d}t^2}J(tu_0^+)_{\rfloor t=t_0^+}>0,$$

there exists $t_0^+<t^-\le t_0^-$ such that $J(t_0^+u_0^+)<J(t^-u_0^+)$. By Lemma 6, we get

$$J(t_0^+u_0^+)<J(t^-u_0^+)<J(t_0^-u_0^+)=J(u_0^+),$$

which contradicts the fact that $J(u_0^+)=c^+$. Since $J(u_0^+)=J(|u_0^+|)$ and $|u_0^+|\in\mathcal{N}^+$, then by Lemma 3, we may assume that u_0^+ is a nontrivial nonnegative solution of $(\mathcal{P}_{\lambda,\mu})$. By the Harnack inequality, we conclude that $u_0^+>0$ and $v_0^+>0$, see for exanmple [14].

4. Proof of Theorem 2

Next, we establish the existence of a local minimum for J on $\mathcal{N}^-$. For this, we require the following Lemma.

Lemma 7. *For all* λ *such that* $0<\lambda<(1/2)\Lambda_0$, *the functional J has a minimizer* u_0^- *in* $\mathcal{N}^-$ *and it satisfies:*

i) $J(u_0^-)=c^->0$,

ii) u_0^- is a nontrivial solution of $(\mathcal{P}_{\lambda,\mu})$ in $\mathcal{H}$.

Proof. If $0<\lambda<(1/2)\Lambda_0$, then by Proposition 1 ii) there exists a $(u_n)_n$, $(PS)_{c^-}$ sequence in $\mathcal{N}^-$, thus it bounded by Lemma 2. Then, there exists $u_0^-\in\mathcal{H}$ and we can extract a subsequence which will denoted by $(u_n)_n$ such that

$$u_n \rightharpoonup u_0^- \text{ weakly in } \mathcal{H}$$

$$u_n \rightharpoonup u_0^- \text{ weakly in } L^{2_*}\left(\mathbb{R}^N, |y|^{-2_* b}\right)$$

$$u_n \to u_0^- \text{ strongly in } L^q\left(\mathbb{R}^N, |y|^{-c}\right)$$

$$u_n \to u_0^- \text{ a.e in } \mathbb{R}^N$$

This implies

$$P(u_n) \to P(u_0^-), \text{as } n \text{ goes to } \infty.$$

Moreover, by (H) and (2.3) we obtain

$$P(u_n) > A(q)\|u_n\|_{\mu,a}^2, \tag{4.1}$$

where, $A(q) := (2-q)/2_*(2_*-q)$. By (2.5) and (4.1) there exists a positive number

$$C_1 := [A(q)]^{2_*/(2_*-2)}(S_\mu)^{2_*/(2_*-2)},$$

such that

$$P(u_n) > C_1. \tag{4.2}$$

This implies that

$$P(u_0^-) \geq C_1.$$

Now, we prove that $(u_n)_n$ converges to u_0^- strongly in $\mathcal{H}$. Suppose otherwise. Then, either $\|u_0^-\|_{\mu,a} < \liminf_{n\to\infty} \|u_n\|_{\mu,a}$. By Lemma 6 there is a unique t_0^- such that $(t_0^- u_0^-) \in \mathcal{N}^-$. Since

$$u_n \in \mathcal{N}^-, J(u_n) \geq J(tu_n), \text{ for all } t \geq 0,$$

we have

$$J(t_0^- u_0^-) < \lim_{n\to\infty} J(t_0^- u_n) \leq \lim_{n\to\infty} J(u_n) = c^-,$$

and this is a contradiction. Hence,

$$(u_n)_n \to u_0^- \text{ strongly in } \mathcal{H}.$$

Thus,

$$J(u_n) \text{ converges to } J(u_0^-) = c^- \text{ as } n \text{ tends to } +\infty.$$

Since $J(u_0^-) = J(|u_0^-|)$ and $u_0^- \in \mathcal{N}^-$, then by (4.2) and Lemma 3, we may assume that u_0^- is a nontrivial nonnegative solution of $(\mathcal{P}_{\lambda,\mu})$. By the maximum principle, we conclude that $u_0^- > 0$.

Now, we complete the proof of Theorem 2. By Propositions 2 and Lemma 7, we obtain that $(\mathcal{P}_{\lambda,\mu})$ has two positive solutions $u_0^+ \in \mathcal{N}^+$ and $u_0^- \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \varnothing$, this implies that u_0^+ and u_0^- are distinct.

5. Proof of Theorem 3

In this section, we consider the following Nehari submanifold of $\mathcal{N}$

$$\mathcal{N}_\varrho = \left\{u \in \mathcal{H}\setminus\{0\} : \langle J'(u), u\rangle = 0 \text{ and } \|u\|_{\mu,a} \geq \varrho > 0\right\}.$$

Thus, $u \in \mathcal{N}_\varrho$ if and only if

$$\|u\|_{\mu,a}^2 - 2_* P(u) - Q(u) = 0 \text{ and } \|u\|_{\mu,a} \geq \varrho > 0.$$

Firsly, we need the following Lemmas

Lemma 8. *Under the hypothesis of theorem 3, there exist* ϱ_0, $\Lambda_2 > 0$ *such that* $\mathcal{N}_\varrho$ *is nonempty for any* $\lambda \in (0, \Lambda_2)$ *and* $\varrho \in (0, \varrho_0)$.

Proof. Fix $u_0 \in \mathcal{H} \setminus \{0\}$ and let

$$g(t) = \langle J'(tu_0), tu_0 \rangle = t^2 \|u_0\|_{\mu,a}^2 - 2_* t^{2_*} P(u_0) - tQ(u_0).$$

Clearly $g(0) = 0$ and $g(t) \to -\infty$ as $n \to +\infty$. Moreover, we have

$$\begin{aligned} g(1) &= \|u_0\|_{\mu,a}^2 - 2_* P(u_0) - Q(u_0) \\ &\geq \left[\|u_0\|_{\mu,a}^2 - 2_* (S_\mu)^{-2_*/2} h_0 \|u_0\|_{\mu,a}^{2_*} \right] - \left(\left(\lambda \|f\|_{\mathcal{H}'_\mu} \right)^{1/(2-q)} \right) \|u_0\|_{\mu,a}. \end{aligned}$$

If $\|u_0\|_{\mu,a} \geq \varrho > 0$ for $0 < \varrho < \varrho_0 = \left(h_0 2_* (2_* - 1) \right)^{-1/(2_*-2)} (S_\mu)^{2_*/2(2_*-2)}$, $h_0 \in (0, \alpha_0)$ for $\alpha_0 = (S_\mu)^{2_*/2} / \left(2_* (2_* - 1) \right)^{(2_*-1)/2_*}$, then there exists

$$\Lambda_2 := \left[\left(h_0 2_* (2_* - 1) \right) (S_\mu)^{-2_*/2} \right]^{-1/(2_*-2)} - \Theta \times \Phi,$$

where

$$\Theta := \left(2_* (2_* - 1) \right)^{2_*-1} \left((h_0)^{2_*/2} S_\mu \right)^{-(2_*)^2/2}$$

and

$$\Phi := \left[\left(h_0 2_* (2_* - 1) \right) (S_\mu)^{-2_*/2} \right]^{-1/(2_*-2)}$$

and there exists $t_0 > 0$ such that $g(t_0) = 0$. Thus, $(t_0 u_0) \in \mathcal{N}_\varrho$ and $\mathcal{N}_\varrho$ is nonempty for any $\lambda \in (0, \Lambda_2)$.

Lemma 9. *There exist M,* Λ_1 *positive reals such that*

$$\langle \phi'(u), u \rangle < -M < 0, \text{ for } u \in \mathcal{N}_\varrho,$$

and any λ verifying

$$0 < \lambda < \min\left((1/2) \Lambda_0, \Lambda_1 \right).$$

Proof. Let $u \in \mathcal{N}_\varrho$, then by (2.1), (2.3) and the Holder inequality, allows us to write

$$\langle \phi'(u), u \rangle \leq \|u_n\|_{\mu,a}^2 \left[\left(\left(\lambda \|f\|_{\mathcal{H}'_\mu} \right)^{1/(2-q)} \right) B(\varrho, q) - (2_* - 2) \right],$$

where $B(\varrho, q) := (2_* - 1)(C_{a,p})^q \varrho^{q-2}$. Thus, if

$$0 < \lambda < \Lambda_3 = \left[(2_* - 2) / B(\varrho, q) \right],$$

and choosing $\Lambda_1 := \min(\Lambda_2, \Lambda_3)$ with Λ_2 defined in Lemma 8, then we obtain that

$$\langle \phi'(u), u \rangle < 0, \text{ for any } u \in \mathcal{N}_\varrho. \tag{5.1}$$

Lemma 10. *Suppose* $N \geq \max(3, 6(a - b + 1))$ *and* $\int_\Omega |y|^{-2_* b} h |u|^{2_*} \mathrm{d}x > 0$. *Then, there exist r and* η *positive constants such that*

i) we have

$$J(u) \geq \eta > 0 \text{ for } \|u\|_{\mu,a} = r.$$

ii) there exists $\sigma \in \mathcal{N}_{\varrho}$ when $\|\sigma\|_{\mu,a} > r$, with $r = \|u\|_{\mu,a}$, such that $J(\sigma) \le 0$.

Proof. We can suppose that the minima of *J* are realized by (u_0^+) and u_0^-. The geometric conditions of the mountain pass theorem are satisfied. Indeed, we have

i) By (2.3), (5.1) and the fact that $P(u) \le (S_\mu)^{-2_*/2} h_0 \|u\|_{\mu,a}^{2_*}$, we get

$$J(u) \ge \left[(1/2) - (2_* - 2)/(2_* - q)q\right] \|u\|_{\mu,a}^2 - (S_\mu)^{-2_*/2} h_0 \|u\|_{\mu,a}^{2_*},$$

Exploiting the function $l(x) = x(2_* - x)$ and if $N \ge \max(3, 6(a - b + 1))$, we obtain that $\left[(1/2) - (2_* - 2)/(2_* - q)q\right] > 0$ for $1 < q < 2$. Thus, there exist η, $r > 0$ such that

$$J(u) \ge \eta > 0 \text{ when } r = \|u\|_{\mu,a} \text{ small.}$$

ii) Let $t > 0$, then we have for all $\phi \in \mathcal{N}_{\varrho}$

$$J(t\phi) := (t^2/2)\|\phi\|_\mu^2 - (t^{2_*})P(\phi) - (t^q/q)Q(\phi).$$

Letting $\sigma = t\phi$ for *t* large enough. Since

$$P(\phi) := \int_\Omega |y|^{-2_* b} h|\phi|^{2_*} \, dx > 0,$$

we obtain $J(\sigma) \le 0$. For *t* large enough we can ensure $\|\sigma\|_{\mu,a} > r$.

Let Γ and *c* defined by

$$\Gamma := \left\{\gamma : [0,1] \to \mathcal{N}_{\varrho} : \gamma(0) = u_0^- \text{ and } \gamma(1) = u_0^+\right\}$$

and

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \inf_{\gamma \in \Gamma} \left(J(\gamma(t))\right).$$

Proof of Theorem 3.

If

$$\lambda < \min\left((1/2)\Lambda_0, \Lambda_1\right),$$

then, by the Lemmas 2 and Proposition 1 ii), *J* verifying the Palais-Smale condition in $\mathcal{N}_{\varrho}$. Moreover, from the Lemmas 3, 9 and 10, there exists u_c such that

$$J(u_c) = c \text{ and } u_c \in \mathcal{N}_{\varrho}.$$

Thus u_c is the third solution of our system such that $u_c \ne u_0^+$ and $u_c \ne u_0^-$. Since $(\mathcal{P}_{\lambda,\mu})$ is odd with respect *u*, we obtain that $-u_c$ is also a solution of $(\mathcal{P}_{\lambda,\mu})$.

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