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Itô Formula for Integral Processes Related to Space-Time Lévy Noise

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Abstract

In this article, we give a new proof of the Itô formula for some integral processes related to the space-time Lévy noise introduced in [1] [2] as an alternative for the Gaussian white noise perturbing an SPDE. We discuss two applications of this result, which are useful in the study of SPDEs driven by a space-time Lévy noise with finite variance: a maximal inequality for the *p*-th moment of the stochastic integral, and the Itô representation theorem leading to a chaos expansion similar to the Gaussian case.

Keywords

Lévy Processes, Poisson Random Measure, Stochastic Integral, Itô Formula, Itô Representation Theorem

1. Introduction

Random processes indexed by sets in the space-time domain are useful objects in stochastic analysis, since they can be viewed as mathematical models for the noise perturbing a stochastic partial differential equation (SPDE). In the recent years, a lot of effort has been dedicated to studying the behaviour of the solution of basic equations (like the heat or wave equations), driven by a *Gaussian white noise*. This type of noise was introduced by Walsh in [3] and is defined as a zero-mean Gaussian process $W=\{W(B); B\in\mathcal{B}_b(\mathbb{R}_+\times\mathbb{R}^d)\}$, with covariance $E[W(A)W(B)]=|A\cap B|$, where $|\cdot|$ denotes the Lebesgue measure and $\mathcal{B}_b(\mathbb{R}_+\times\mathbb{R}^d)$ is the class of bounded Borel sets in $\mathbb{R}_+\times\mathbb{R}^d$.

In the recent articles [1] [2], a new process has been introduced as an alternative for the Gaussian white noise perturbing an SPDE, which has a structure similar to a Lévy process. We introduce briefly the definition of this

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process below.

Let N be a Poisson random measure (PRM) on $\mathbb{E}=\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}_0$ of intensity $\mu=\mathrm{d}t\mathrm{d}x\nu(\mathrm{d}z)$ where $\mathbb{R}_0=\mathbb{R}\setminus\{0\}$ and ν is a Lévy measure on $\mathbb{R}$:

$$\int_{\mathbb{R}_0}\left(1\wedge|z|^2\right)\nu(\mathrm{d}z)<\infty \quad \text{and} \quad \nu(\{0\})=0.$$

We denote by $\hat{N}$ the compensated PRM defined by $\hat{N}(A)=N(A)-\mu(A)$ for any Borel set A in $\mathbb{E}$ with $\mu(A)<\infty$. The Lévy-type noise process mentioned above is defined as $Z=\left\{Z(B);B\in\mathcal{B}_b\left(\mathbb{R}_+\times\mathbb{R}^d\right)\right\}$, where

$$Z(B)=a|B|+\int_{B\times\{|z|>1\}}zN(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)+\int_{B\times\{|z|\le 1\}}z\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z),$$

for some $a\in\mathbb{R}$. It was shown in [2] that Z is an "independently scattered random measure" (in the sense of [4]) with characteristic function:

$$E\left(\mathrm{e}^{iuZ(B)}\right)=\exp\left\{|B|\left(a+\int_{\mathbb{R}_0}\left(\mathrm{e}^{iuz}-1-iuz1_{\{|z|\le 1\}}\right)\nu(\mathrm{d}z)\right)\right\}, \quad u\in\mathbb{R}.$$

(In particular, Z can be an α-stable random measure with $\alpha\in(0,2)$, as in Definition 3.3.1 of [5].) One can define the stochastic integral of a process $X=\left\{X(t,x);t\ge 0,x\in\mathbb{R}^d\right\}$ with respect to Z and for a *certain* integrands,

$$\int_0^T\int_{\mathbb{R}^d}X(t,x)Z(\mathrm{d}t,\mathrm{d}x)=a\int_0^T\int_{\mathbb{R}^d}X(t,x)\mathrm{d}t\mathrm{d}x+\int_0^T\int_{\mathbb{R}^d}\int_{\{|z|>1\}}X(t,x)zN(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z)$$
$$+\int_0^T\int_{\mathbb{R}^d}\int_{\{|z|\le 1\}}X(t,x)z\hat{N}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}z).$$

The stochastic integral with respect to $\hat{N}$ (or N) can be defined using classical methods (see e.g. [6]). We review briefly this definition here.

Assume that N is defined on a probability space $(\Omega,\mathcal{F},P)$. On this space, we consider the filtration

$$\mathcal{F}_t=\sigma\left(\left\{N([0,s]\times B\times\Gamma);0\le s\le t,B\in\mathcal{B}_b\left(\mathbb{R}^d\right),\Gamma\in\mathcal{B}_b(\mathbb{R}_0)\right\}\right),$$

where $\mathcal{B}_b\left(\mathbb{R}^d\right)$ is the class of bounded Borel sets in $\mathbb{R}^d$ and $\mathcal{B}_b(\mathbb{R}_0)$ is the class of Borel sets in $\mathbb{R}_0$ which are bounded away from 0.

An *elementary* process on $\Omega\times\mathbb{R}^d\times\mathbb{R}_0$ is a process of the form

$$H(\omega,t,x,z)=X(\omega)1_{(a,b]}(t)1_A(x)1_\Gamma(z),$$

where $0\le a<b$, X is an $\mathcal{F}_a$-measurable bounded random variable, $A\in\mathcal{B}_b\left(\mathbb{R}^d\right)$ and $\Gamma\in\mathcal{B}_b(\mathbb{R}_0)$. A process $H=\left\{H(t,x,z);t\ge 0,x\in\mathbb{R}^d,z\in\mathbb{R}_0\right\}$ is called *predictable* if it is measurable with respect to the σ-field $\mathcal{P}_{\Omega\times\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}_0}$ generated by all linear combinations of elementary processes.

As in the classical theory, for any predictable process H such that

$$E\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}|H(s,x,z)|^2\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s<\infty \quad \text{for all } t>0, \tag{1}$$

we can define the stochastic integral of H with respect to $\hat{N}$ and the process

$$\left\{\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}H(s,x,z)\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z);t\ge 0\right\}$$

is a zero-mean square-integrable martingale which satisfies

$$E\left|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}H(s,x,z)\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)\right|^2=\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}|H(s,x,z)|^2\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s. \tag{2}$$

On the other hand, for any predictable process K such that

$$E\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}\left|K(s,x,z)\right|\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s<\infty \quad \text{for all } t>0,$$

we can define the integral of K with respect to N and this integral satisfies

$$E\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}K(s,x,z)N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)=E\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}K(s,x,z)\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s. \tag{3}$$

In this article, we work with processes whose trajectories are right-continuous with left limits. If x is a right continuous function with left limits, we denote by $x(t-)=\lim_{s\uparrow t}x(s)$ the left limit at time t and $\Delta x(t)=x(t)-x(t-)$ the jump size at time t. We will prove the following result.

Theorem 1 (Ito Formula I). *Let* $Y=\{Y(t)\}_{t\ge 0}$ *be a process defined by*

$$Y(t)=\int_0^t G(s)\mathrm{d}s+\int_0^t\int_{\mathbb{R}^d}\int_{\{|z|>1\}}K(s,x,z)N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)+\int_0^t\int_{\mathbb{R}^d}\int_{\{|z|\le 1\}}H(s,x,z)\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z),\quad t\ge 0, \tag{4}$$

where G, K and H are predictable processes which satisfy

$$E\int_0^t\left|G(s)\right|\mathrm{d}s<\infty \quad \text{for all } t>0, \tag{5}$$

$$E\int_0^t\int_{\mathbb{R}^d}\int_{\{|z|>1\}}\left|K(s,x,z)\right|\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s<\infty \quad \text{for all } t>0, \tag{6}$$

$$E\int_0^t\int_{\mathbb{R}^d}\int_{\{|z|\le 1\}}\left|H(s,x,z)\right|^2\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s<\infty \quad \text{for all } t>0. \tag{7}$$

Then there exists a modification of Y (denoted also by Y) whose sample paths are right-continuous with left limits, such that for any function $f\in C^2(\mathbb{R})$ and for any $t>0$, with probability 1,

$$\begin{aligned}
&f(Y(t))-f(Y(0))\\
&=\int_0^t f'(Y(s))G(s)\mathrm{d}s+\int_0^t\int_{\mathbb{R}^d}\int_{\{|z|>1\}}\left[f(Y(s-)+K(s,x,z))-f(Y(s-))\right]N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)\\
&\quad+\int_0^t\int_{\mathbb{R}^d}\int_{\{|z|\le 1\}}\left[f(Y(s-)+H(s,x,z))-f(Y(s-))\right]\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)\\
&\quad+\int_0^t\int_{\mathbb{R}^d}\int_{\{|z|\le 1\}}\left[f(Y(s)+H(s,x,z))-f(Y(s))-H(s,x,z)f'(Y(s))\right]\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{8}$$

Note that since the first two terms on the right-hand side of (4) are processes of finite variation and the last term is a square-integrable martingale, Y is a semimartingale. Therefore, the Itô formula given by Theorem 1 can be derived from the corresponding result for a general semimartingale, assuming that Y has sample paths which are right-continuous with left limits (see e.g. Theorem 2.5 of [7]).

The goal of the present article is to give an alternative proof of this result which contains the explicit construction of the modification of Y for which the Itô formula holds.

We will also give the proof of the following variant of the Itô formula, which will be useful for the applications related to the (finite-variance) Lévy white noise, discussed in Section 4.

Theorem 2 (Ito Formula II). *Let* $Y=\{Y(t)\}_{t\ge 0}$ *be a process defined by*

$$Y(t)=\int_0^t G(s)\mathrm{d}s+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}H(s,x,z)\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z),\quad t\ge 0, \tag{9}$$

where G and H are predictable processes which satisfy (5), respectively (1). Then there exists a càdlàg modification of Y (denoted also by Y) such that for any $t>0$, with probability 1,

$$\begin{aligned}
&f(Y(t))-f(Y(0))\\
&=\int_0^t f'(Y(s))G(s)\mathrm{d}s+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}\left[f(Y(s-)+H(s,x,z))-f(Y(s-))\right]\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)\\
&\quad+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}\left[f(Y(s)+H(s,x,z))-f(Y(s))-H(s,x,z)f'(Y(s))\right]\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

The method that we use for proving Theorems 1 and 2 is similar to the one described in Section 4.4.2 of [6] in the case of classical Lévy processes, the difference being that in our case, N is a PRM on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_0$ instead of $\mathbb{R}_+ \times \mathbb{R}_0$. This method relies on a double "interlacing" technique, which consists in first approximating the set $\{|z| \le 1\}$ of small jumps by sets of the form $\{\varepsilon_n < |z| \le 1\}$ with $\varepsilon_n \downarrow 0$ (in the case when H and K vanish outside a bounded Borel set $B \subset \mathbb{R}^d$), and then approximating the spatial domain $\mathbb{R}^d$ by regions of the form $[-a_n, a_n]^d$ with $a_n \uparrow \infty$. This approximation method is described in Section 2. Section 3 is dedicated to the proofs of Theorems 1 and 2. Finally, in Section 4 we discuss two applications of Theorem 2 in the case of the (finite-variance) Lévy white noise introduced in [1].

2. Approximation by Right-Continuous Processes with Left Limits

In this section, we show that the Lévy-type integral processes given by (4) and (9) have right-continuous modifications with left limits, which are constructed by approximation. These modifications will play an important role in the proof of Itô's formula. Since the process $Y_c(t) = \int_0^t G(s)\,ds$ is continuous, we assume that $G = 0$.

We consider first processes of the form (4). We start by examining the case when both integrands H and K vanish outside a set $B \in \mathcal{B}_b(\mathbb{R}^d)$. Since the process $\left\{\int_0^t \int_B \int_{\{|z|>1\}} K(s,x,z) N(ds,dx,dz); t \ge 0\right\}$ is clearly càdlàg (the integral being a sum with finitely many terms), we need to consider only the integral process which depends on H.

Note that if H vanishes a.e. on $\Omega \times [0,T] \times B \times \{z \in \mathbb{R}_0; |z| \le \varepsilon\}$ for some $T > 0$ and $\varepsilon \in (0,1)$, then

$$\int_0^t \int_B \int_{\{|z|\le 1\}} H(s,x,z) \hat{N}(ds,dx,dz)$$
$$= \int_0^t \int_B \int_{\{\varepsilon<|z|\le 1\}} H(s,x,z) N(ds,dx,dz) - \int_0^t \int_B \int_{\{\varepsilon<|z|\le 1\}} H(s,x,z) \nu(dz)\,dx\,ds$$

is a process whose sample paths are right-continuous with left limits (the first term is a sum with finitely many terms and the second term in continuous). Therefore, we will suppose that H satisfies the following assumption:

Assumption A. It is not possible to find $T > 0$ and $\varepsilon \in (0,1)$ such that

$$H(\omega,s,x,z) = 0 \quad \text{a.e. on} \quad \Omega \times [0,T] \times B \times \{z \in \mathbb{R}_0; |z| \le \varepsilon\}$$

with respect to the measure $P \times \mu$.

Lemma 1. *Let* $Y = \{Y(t)\}_{t\ge 0}$ *be a process defined by*

$$Y(t) = \int_0^t \int_B \int_{\{|z|\le 1\}} H(s,x,z) \hat{N}(ds,dx,dz),$$

where $B \in \mathcal{B}_b(\mathbb{R}^d)$ and H is a predictable process which satisfies Assumption A and

$$E\int_0^t \int_B \int_{\{|z|\le 1\}} |H(s,x,z)|^2 \nu(dz)\,dx\,ds < \infty \quad \text{for all } t > 0. \tag{10}$$

Then, there exists a càdlàg modification $\tilde{Y} = \{\tilde{Y}(t)\}_{t\ge 0}$ of Y such that for all $T > 0$,

$$\sup_{t\le T} |Y_n(t) - \tilde{Y}(t)| \to 0 \quad a.s.,$$

where

$$Y_n(t) = \int_0^t \int_B \int_{\{\varepsilon_n<|z|\le 1\}} H(s,x,z) \hat{N}(dt,dx,dz)$$

for some sequence $(\varepsilon_n)_n$ (depending on T) such that $\varepsilon_n \downarrow 0$.

Proof: We use the same argument as in the proof of Theorem 4.3.4 of [6]. Fix $T > 0$. Let

$$\varepsilon_n = \sup\{\varepsilon > 0; I(\varepsilon) \le 8^{-n}\}$$

where

$$I(\varepsilon)=E\int_0^T\int_B\int_{\{|z|\le\varepsilon\}}|H(s,x,z)|^2\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s.$$

Note that $(\varepsilon_n)_n$ is non-increasing and $\varepsilon_n\downarrow 0$. (If $\varepsilon_n\downarrow\varepsilon_*>0$ then $I(\varepsilon_*)\le I(\varepsilon_n)\le 8^{-n}$ for all *n*. Hence $I(\varepsilon_*)=0$, which contradicts Assumption *A*.)

Note that Y_n is a càdlàg martingale. By Doob's submartingale inequality and relation (2),

$$E\left(\sup_{t\le T}|Y_{n+1}(t)-Y_n(t)|^2\right)\le 4E|Y_{n+1}(T)-Y_n(T)|^2$$
$$=4E\int_0^T\int_B\int_{\{\varepsilon_{n+1}<|z|\le\varepsilon_n\}}|H(s,x,z)|^2\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s\le 4I(\varepsilon_n)\le\frac{4}{8^n}.$$

By Chebyshev's inequality, $P\left(\sup_{t\le T}|Y_{n+1}(t)-Y_n(t)|>2^{-n}\right)\le 2^{-n+2}$. By Borel-Cantelli lemma, with probability 1, the sequence $(Y_n)_n$ is Cauchy in the space $D[0,T]$ of càdlàg functions on $[0,T]$ equipped with the sup-norm. Its limit $\tilde{Y}$ is a modification of *Y* since for any $t\in[0,T]$, $\{Y_n(t)\}_n$ also converges to $Y(t)$ in $L^2(\Omega)$. Finally, we note that the process $\tilde{Y}$ does not depend on *T* (although the approximation sequence $(Y_n)_n$ does). If $\tilde{Y}^{(T)}$ is the modification of *Y* on $[0,T]$ and $\tilde{Y}^{(T')}$ is the modification of *Y* on $[0,T']$ with $T<T'$, then $\tilde{Y}^{(T)}(t)=\tilde{Y}^{(T')}(t)$ a.s. for any $t\in[0,T]$. Hence, $\tilde{Y}$ can be extended to $[0,\infty)$. □

We consider now the case when the at least one of the integrands *H* and *K* do not vanish outside a set $B\in\mathcal{B}_b(\mathbb{R}^d)$. More precisely, we introduce the following assumptions:

Assumption B. It is not possible to find $T>0$ and $B\in\mathcal{B}_b(\mathbb{R}^d)$ such that

$$H(\omega,t,x,z)=0\quad\text{a.e. on }\Omega\times[0,T]\times B^c\times\{z\in\mathbb{R}_0;|z|\le 1\}$$

with respect to the measure $P\times\mu$.

Assumption B'. It is not possible to find $T>0$ and $B\in\mathcal{B}_b(\mathbb{R}^d)$ such that

$$K(\omega,t,x,z)=0\quad\text{a.e. on }\Omega\times[0,T]\times B^c\times\{z\in\mathbb{R}_0;|z|>1\}$$

with respect to the measure $P\times\mu$.

We consider bounded Borel sets in $\mathbb{R}^d$ of the form $K_a=[-a,a]^d, a>0$.

Theorem 3 (Interlacing I). *Let* $Y=\{Y(t)\}_{t\ge0}$ *be a process defined by* (4) *with* $G=0$, *where H and K are predictable processes which satisfy conditions* (7), *respectively* (6), *such that either H satisfies Assumption B, or K satisfies Assumption* B'. *Then, there exists a càdlàg modification* $\tilde{Y}=\{\tilde{Y}(t)\}_{t\ge0}$ *of Y such that for all T* > 0,

$$\sup_{t\le T}|\tilde{Y}_n(t)-\tilde{Y}(t)|\to 0\quad a.s.,\tag{11}$$

where $\tilde{Y}_n$ is a càdlàg modification of the process Y_n defined by

$$Y_n(t)=\int_0^t\int_{E_n}\int_{\{|z|\le1\}}H(s,x,z)\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)+\int_0^t\int_{E_n}\int_{\{|z|>1\}}K(s,x,z)N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)$$

with $E_n=K_{a_n}$ for some sequence $(a_n)_n$ (depending on *T*) such that $a_n\uparrow\infty$.

Proof: Fix $T>0$. Let $a_n=\inf\{a>0;I(a)\le 8^{-n}\}$ where

$$I(a)=E\int_0^T\int_{K_a^c}\int_{\{|z|\le1\}}|H(s,x,z)|^2\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s+E\int_0^T\int_{K_a^c}\int_{\{|z|>1\}}|K(s,x,z)|\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s.$$

Note that $(a_n)_n$ is non-decreasing and $a_n\uparrow\infty$. (If $a_n\uparrow a^*<\infty$ then $I(a^*)\le I(a_n)\le 8^{-n}$ for all *n*, and hence $I(a^*)=0$, which contradicts Assumptions *B* or B'.) Let Y_n be the process given in the statement of the theorem with $E_n=K_{a_n}$. We denote by $Y_n^{(1)}(t)$ and $Y_n^{(2)}(t)$ the two integrals which compose $Y_n(t)$, depending on *H*, respectively *K*.

We denote by $\tilde{Y}_n^{(1)}$ the càdlàg modification of $Y_n^{(1)}$ given by Lemma 1. By Doob's submartingale inequality and relation (2),

$$E\left(\sup_{t\le T}|\tilde{Y}_{n+1}^{(1)}(t)-\tilde{Y}_n^{(1)}(t)|^2\right)\le 4E\int_0^T\int_{E_{n+1}2E_n}\int_{\{|z|\le1\}}|H(s,x,z)|^2\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s\le 4I(a_n)\le\frac{4}{8^n}.$$

By Chebyshev's inequality, $P\left(\sup_{t\le T}\left|\tilde{Y}^{(1)}_{n+1}(t)-\tilde{Y}^{(1)}_{n}(t)\right|>2^{-n-1}\right)\le 2^{-n+4}$.

Note that $Y_n^{(2)}$ is a càdlàg process. For any $t\in[0,T]$,

$$\left|Y^{(2)}_{n+1}(t)-Y^{(2)}_{n}(t)\right|\le\int_0^t\int_{E_{n+1}\setminus E_n}\int_{\{|z|>1\}}\left|K(s,x,z)\right|N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z),$$

and hence, using relation (3),

$$E\left(\sup_{t\le T}\left|Y^{(2)}_{n+1}(t)-Y^{(2)}_{n}(t)\right|\right)\le E\int_0^T\int_{E_{n+1}\setminus E_n}\int_{\{|z|>1\}}\left|K(s,x,z)\right|\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s\le I(a_n)\le\frac{1}{8^n}.$$

By Markov's inequality, $P\left(\sup_{t\le T}\left|Y^{(2)}_{n+1}(t)-Y^{(2)}_{n}(t)\right|>2^{-n-1}\right)\le 2^{-2n+1}$.

Let $\tilde{Y}_n(t)=\tilde{Y}^{(1)}_n(t)+Y^{(2)}_n(t)$. Then $P\left(\sup_{t\le T}\left|\tilde{Y}_{n+1}(t)-\tilde{Y}_n(t)\right|>2^{-n}\right)\le 2^{-n+4}+2^{-2n+1}$, and the conclusion follows by the Borel-Cantelli Lemma, as in the proof of Lemma 1. □

We consider next processes of the form (9) with $G = 0$. Note that if H vanishes a.e. outside a set $B\in\mathcal{B}_b(\mathbb{R}^d)$ then

$$Y(t)=\int_0^t\int_B\int_{\{|z|\le1\}}H(s,x,z)\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)+\int_0^t\int_B\int_{\{|z|>1\}}H(s,x,z)N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)$$
$$-\int_0^t\int_B\int_{\{|z|>1\}}H(s,x,z)\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s,$$

where the first term has a càdlàg modification given by Lemma 1, the second term is càdlàg, and the third term is continuous. Therefore, we will suppose that H satisfies the following assumption:

Assumption C. It is not possible to find $T>0$ and $B\in\mathcal{B}_b(\mathbb{R}^d)$ such that

$$H(\omega,s,x,z)=0\quad\text{a.e.}\quad\Omega\times[0,T]\times B^c\times\mathbb{R}_0$$

with respect to the measure $P\times\mu$.

Theorem 4 (Interlacing II). *Let Y be a process given by* (9) *with* $G=0$, *where H is a predictable process which satisfies* (1) *and Assumption C. Then, there exists a càdlàg modification* $\tilde{Y}=\{\tilde{Y}(t)\}_{t\ge0}$ *of Y such that* (11) *holds, where* $\tilde{Y}_n$ *is a càdlàg modification of the process* Y_n *defined by*:

$$Y_n(t)=\int_0^t\int_{E_n}\int_{\mathbb{R}_0}H(s,x,z)\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z),$$

with $E_n=K_{a_n}$ for some sequence $(a_n)_n$ (depending on T) such that $a_n\uparrow\infty$.

Proof: We proceed as in the proof of Theorem 3. Fix $T>0$. Let $a_n=\inf\{a>0;I(a)\le8^{-n}\}$ where

$$I(a)=\int_0^t\int_{K_a^c}\int_{\mathbb{R}_0}\left|H(s,x,z)\right|^2\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s.$$

By Assumption *C*, $a_n\uparrow\infty$. We write $Y_n(t)$ as the sum of two integrals, corresponding to the regions $\{|z|\le1\}$, and $\{|z|>1\}$. We denote these integrals by $Y^{(1)}_n(t)$, respectively $Y^{(2)}_n(t)$. Note that $Y^{(2)}_n$ is càdlàg. Let $\tilde{Y}^{(1)}_n$ be the càdlàg modification of $Y^{(1)}_n$ given by Lemma 1.

Let $\tilde{Y}_n(t)=\tilde{Y}^{(1)}_n(t)+Y^{(2)}_n(t)$. By Doob's submartingale inequality,

$$E\left(\sup_{t\le T}\left|\tilde{Y}_{n+1}(t)-\tilde{Y}_n(t)\right|^2\right)\le4E\int_0^T\int_{E_{n+1}2E_n}\int_{\mathbb{R}_0}\left|H(s,x,z)\right|^2\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s$$

and the conclusion follows as in the proof of Lemma 1. □

3. Proof of Itô Formula

In this section, we give the proofs of Theorem 1 and Theorem 2.

We start with the simpler case when there are no small jumps (the analogue of Lemma 4.4.6 of [6]).

Lemma 2. *Let*

$$Y(t)=\int_0^t G(s)\,\mathrm{d}s+\int_0^t\int_B\int_{\{|z|>\varepsilon\}}K(s,x,z)\,N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)=:Y_c(t)+Y_d(t),$$

where G is a predictable process which satisfies (5), $B\in\mathcal{B}_b(\mathbb{R}^d)$, $\varepsilon>0$ and K is a predictable process. Then, for any function $f\in C^1(\mathbb{R})$ and for any $t>0$,

$$\begin{aligned}&f(Y(t))-f(Y(0))\\&=\int_0^t f'(Y(s))G(s)\,\mathrm{d}s+\int_0^t\int_B\int_{\{|z|>\varepsilon\}}\left[f(Y(s-)+K(s,x,z))-f(Y(s-))\right]N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z).\end{aligned}$$

Proof: We denote $\Gamma=\{|z|>\varepsilon\}$. By Proposition 5.3 of [8], we may assume that the restriction of N to the set $\mathbb{R}_+\times B\times\Gamma$ has points $(T_i,X_i,Z_i),i\geq 1$, where $T_1<T_2<\cdots$ are the points of a Poisson process on $\mathbb{R}_+$ of intensity $\lambda=|B|\nu(\Gamma)$ and $\{(X_i,Z_i)\}_{i\geq 1}$ are i.i.d. on $B\times\Gamma$ with distribution $\lambda^{-1}\mathrm{d}x\nu(\mathrm{d}z)$, independent of $(T_i)_{i\geq 1}$. We consider two cases.

Case 1: $G=0$. By the representation of N, $Y(t)=\sum_{T_i\leq t}K(T_i,X_i,Z_i)$. So $t\mapsto Y(t)$ is a step function which has a jump of size $K(T_i,X_i,Z_i)$ at each point T_i and $Y(T_i-)=Y(T_{i-1})$. Hence

$$\begin{aligned}f(Y(t))-f(Y(0))&=\sum_{T_i\leq t}\left[f(Y(T_i))-f(Y(T_{i-1}))\right]\\&=\sum_{T_i\leq t}\left[f(Y(T_i-)+K(T_i,X_i,Z_i))-f(Y(T_i-))\right],\end{aligned}$$

and the conclusion follows since N has points (T_i,X_i,Z_i) in $\mathbb{R}_+\times B\times\Gamma$.

Case 2: *G is arbitrary*. The map $t\mapsto Y_d(t)$ is a step function which has a jump of size $K(T_i,X_i,Z_i)$ at time T_i. Since Y_c is continuous, the jump times and the jump sizes of Y coincide with those of Y_d, *i.e.* $\Delta Y(T_i)=\Delta Y_d(T_i)=K(T_i,X_i,Z_i)$. We use the decomposition

$$f(Y(t))-f(Y(0))=A(t)+B(t),$$

where A and B are defined as follows: if $T_{n-1}\leq t<T_n$, we let

$$A(t)=\sum_{i=1}^{n-1}\left[f(Y(T_i))-f(Y(T_i-))\right]$$

$$B(t)=\sum_{i=1}^{n-1}\left[f(Y(T_i-))-f(Y(T_{i-1}))\right]+\left[f(Y(t))-f(Y(T_{n-1}))\right].$$

Note that

$$\begin{aligned}A(t)&=\sum_{i=1}^{n}\left[f(Y(T_i-)+K(T_i,X_i,Z_i))-f(Y(T_i-))\right]\\&=\int_0^t\int_B\int_\Gamma\left[f(Y(s-)+K(s,x,z))-f(Y(s-))\right]N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z).\end{aligned}$$

It remains to prove that

$$B(t)=\int_0^t f'(Y(s))G(s)\,\mathrm{d}s. \tag{12}$$

For this, we assume that $T_{n-1}\leq t<T_n$ and we write

$$\int_0^t f'(Y(s))G(s)\,\mathrm{d}s=\sum_{i=1}^{n-1}\int_{T_{i-1}}^{T_i}f'(Y(s))G(s)\,\mathrm{d}s+\int_{T_{n-1}}^t f'(Y(s))G(s)\,\mathrm{d}s.$$

So it suffices to prove that

$$\int_{T_{i-1}}^{T_i}f'(Y(s))G(s)\,\mathrm{d}s=f(Y(T_i-))-f(Y(T_{i-1})) \tag{13}$$

for all $i=1,\cdots,n-1$, and

$$\int_{T_{n-1}}^{t} f'(Y(s))G(s)\mathrm{d}s = f(Y(t)) - f(Y(T_{n-1})). \tag{14}$$

We first prove (13). Fix $i = 1,\cdots,n-1$. For any

$$s \in (T_{i-1}, T_i),\quad Y(s) = Y_c(s) + Y_d(T_{i-1}) := g_i(s) \quad \text{and} \quad g_i'(s) = Y_c'(s) = G(s).$$

We extend g_i by continuity to $[T_{i-1}, T_i]$. Hence

$$\begin{aligned}\int_{T_{i-1}}^{T_i} f'(Y(s))G(s)\mathrm{d}s = \int_{T_{i-1}}^{T_i} f'(g_i(s))g_i'(s)\mathrm{d}s &= f(g_i(T_i)) - f(g_i(T_{i-1})) \\ &= f(Y_c(T_i) + Y_d(T_{i-1})) - f(Y_c(T_{i-1}) + Y_d(T_{i-1})) \\ &= f(Y(T_i-)) - f(Y(T_{i-1})),\end{aligned}$$

where for the last equality we used the fact that $Y_d(T_{i-1}) = Y_d(T_i-)$ and hence

$$Y_c(T_i) + Y_d(T_{i-1}) = Y_c(T_i-) + Y_d(T_i-) = Y(T_i-).$$

This proves (13).

Next, we prove (14). Note that if $t = T_{n-1}$, both terms are zero. So, we assume that $t > T_{n-1}$. For any

$$s \in (T_{n-1}, t),\quad Y(s) = Y_c(s) + Y_d(T_{n-1}) := g(s) \quad \text{and} \quad g'(s) = Y_c'(s) = G(s).$$

Arguing as above, we see that

$$\begin{aligned}\int_{T_{n-1}}^{t} f'(Y(s))G(s)\mathrm{d}s = \int_{T_{n-1}}^{t} f'(g(s))g'(s)\mathrm{d}s &= f(g(t)) - f(g(T_{n-1})) \\ &= f(Y_c(t) + Y_d(T_{n-1})) - f(Y_c(T_{n-1}) + Y_d(T_{n-1})) \\ &= f(Y(t)) - f(Y(T_{n-1})),\end{aligned}$$

where for the last equality we used the fact that $Y_d(T_{n-1}) = Y_d(t)$ and hence

$$Y_c(t) + Y_d((T_{n-1}) = Y_c(t) + Y_d(t) = Y(t).$$

This concludes the proof of (14). □

Proof of Theorem 1: We fix $t > 0$. We assume that f' and f'' are bounded. (Otherwise, we use $\tau_k = \inf\{s > 0; |Y(s)| > k\}$ for $k \ge 1$.)

Case 1: *H and K vanish outside a fixed set* $B \in \mathcal{B}_b(\mathbb{R}^d)$.

If *H* vanishes a.e. on $\Omega \times [0,T] \times B \times \{z \in \mathbb{R}_0; |z| \le \varepsilon\}$ for some $T > 0$ and $\varepsilon \in (0,1)$, the conclusion follows from Lemma 2. Therefore, we suppose that *H* satisfies Assumption *A*. By Lemma 1, there exists a càdlàg modification of *Y* (denoted also by *Y*) such that

$$\sup_{s \le t} |Y_n(s) - Y(s)| \to 0, \tag{15}$$

where the process $\{Y_n(s)\}_{s\in[0,t]}$ is defined by

$$\begin{aligned}Y_n(s) = \int_0^s G(r)\mathrm{d}r + \int_0^s \int_B \int_{\{\varepsilon_n < |z| \le 1\}} H(r,x,z)\hat{N}(\mathrm{d}r,\mathrm{d}x,\mathrm{d}z) \\ + \int_0^s \int_B \int_{\{|z|>1\}} K(r,x,z)N(\mathrm{d}r,\mathrm{d}x,\mathrm{d}z), \quad s \in [0,t],\end{aligned}$$

$(\varepsilon_n)_n$ being the sequence given by Lemma 1 with $T = t$. Consequently,

$$\sup_{s \le t} |Y_n(s-) - Y(s-)| \to 0. \tag{16}$$

Note that

$$Y_n(s) = \int_0^s \bar{G}(r)\mathrm{d}r + \int_0^s \int_B \int_{\{|z|>\varepsilon_n\}} \bar{K}(r,x,z)N(\mathrm{d}r,\mathrm{d}x,\mathrm{d}z),$$

where $\bar{G}(s) = G(s) - \int_B \int_{\{\varepsilon_n < |z| \le 1\}} H(s,x,z)\nu(\mathrm{d}z)\mathrm{d}x$ and $\bar{K}(s,x,z) = H(s,x,z)1_{\{|z|\le 1\}} + K(s,x,z)1_{\{|z|>1\}}$. By the Cauchy-Schwarz inequality, $\bar{G}$ satisfies (5) (since B is a *bounded* set and H satisfies (10)). We apply Lemma 2 to Y_n:

$$f(Y_n(t)) - f(Y_n(0)) = \int_0^t f'(Y_n(s))\bar{G}(s)\mathrm{d}s + \int_0^t \int_B \int_{\{|z|>\varepsilon_n\}} \left[f(Y_n(s-) + \bar{K}(s,x,z)) - f(Y_n(s-)) \right] N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z).$$

After using the definitions of $\bar{G}$ and $\bar{K}$, as well as adding and subtracting

$$\int_0^t \int_B \int_{\{\varepsilon_n < |z| \le 1\}} \left[f(Y_n(s) + H(s,x,z)) - f(Y_n(s)) \right] \nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s,$$

we obtain that:

$$\begin{aligned} & f(Y_n(t)) - f(Y_n(0)) \\ &= \int_0^t f'(Y_n(s))G(s)\mathrm{d}s + \int_0^t \int_B \int_{\{|z|>1\}} \left[f(Y_n(s-) + K(s,x,z)) - f(Y_n(s-)) \right] N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z) \\ &\quad + \int_0^t \int_B \int_{\{|z|\le 1\}} \left[f(Y_n(s-) + H(s,x,z)) - f(Y_n(s-)) \right] \hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z) \\ &\quad + \int_0^t \int_B \int_{\{|z|\le 1\}} \left[f(Y_n(s) + H(s,x,z)) - f(Y_n(s)) - H(s,x,z) f'(Y_n(s)) \right] \nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s \\ &:= T_{1,n} + T_{2,n} + T_{3,n} + T_{4,n}. \end{aligned} \tag{17}$$

We denote by T_1, T_2, T_3, respectively T_4 the four terms on the right-hand side of (8). The conclusion will follow by taking the limit as $n \to \infty$ in (17). The left-hand side converges to $f(Y(t)) - f(Y(0))$, by (15).

We treat separately the four terms in the right-hand side. By the dominated convergence theorem,

$$E|T_{1,n} - T_1| \le E\int_0^t \left| f'(Y_n(s)) - f'(Y(s)) \right| |G(s)| \mathrm{d}s \to 0.$$

Since $T_{2,n}$ is a sum with a finite number of terms, using (15) and the continuity of *f*, we see that $T_{2,n} \to T_2$ a.s. For the third term, note that $E|T_{3,n} - T_3|^2 \le 2(A_n + B_n)$, where

$$A_n = E\int_0^t \int_B \int_{\{\varepsilon_n < |z| \le 1\}} |V_n(s,x,z) - V(s,x,z)|^2 \nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s,$$

$$B_n = E\int_0^t \int_B \int_{\{|z| \le \varepsilon_n\}} |V(s,x,z)|^2 \nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s,$$

and $V_n(s,x,z) := f(Y_n(s) + H(s,x,z)) - f(Y_n(s)) \to V(s,x,z) := f(Y(s) + H(s,x,z)) - f(Y(s))$ a.s., by (15) and the continuity of *f*. By the dominated convergence theorem, $A_n \to 0$ and $B_n \to 0$. To justify the application of this theorem, we use Taylor's formula of the first order:

$$f(b) - f(a) = (b-a)\int_0^1 f'(a + \theta(b-a))\mathrm{d}\theta, \tag{18}$$

and the fact that f' is bounded. This proves that $T_{3,n} \to T_3$ in $L^2(\Omega)$.

Finally, $E|T_{4,n} - T_4| \le C_n + D_n$, where

$$C_n = E\int_0^t \int_B \int_{\{\varepsilon_n < |z| \le 1\}} |U_n(s,x,z) - U(s,x,z)| \nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s,$$

$$D_n = E\int_0^t \int_B \int_{|z| \le \varepsilon_n} |U(s,x,z)| \nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s,$$

and

$$\begin{aligned} U_n(s,x,z) &:= f(Y_n(s) + H(s,x,z)) - f(Y_n(s)) - H(s,x,z) f'(Y_n(s)) \\ &\to U(s,x,z) := f(Y(s) + H(s,x,z)) - f(Y(s)) - H(s,x,z) f'(Y_n(s)) \end{aligned}$$

a.s., by (16) and the continuity of *f*. By the dominated convergence theorem, $C_n \to 0$ and $D_n \to 0$. To justify the application of this theorem, we use Taylor's formula of second order:

$$f(b)-f(a)=(b-a)f'(a)+(b-a)^2\int_0^1 f''(a+\theta(b-a))(1-\theta)\mathrm{d}\theta, \tag{19}$$

and the fact that f'' is bounded. This proves that $T_{4,n} \to T_4$ in $L^1(\Omega)$.

Case 2. *H satisfies Assumption B or K satisfies Assumption* B'.

By Theorem 3, there exists a càdlàg approximation of *Y* (denoted also by *Y*) such that (15) holds, where $\{Y_n(s)\}_{s\in[0,t]}$ is a càdlàg modification of

$$Y_n(s)=\int_0^s G(r)\mathrm{d}r+\int_0^s\int_{E_n}\int_{\{|z|\le 1\}} H(r,x,z)\hat{N}(\mathrm{d}r,\mathrm{d}x,\mathrm{d}z)$$
$$+\int_0^s\int_{E_n}\int_{\{|z|>1\}} K(r,x,z)N(\mathrm{d}r,\mathrm{d}x,\mathrm{d}z), \quad s\in[0,t],$$

$(E_n)_n \subset \mathcal{B}_b(\mathbb{R}^d)$ being the sequence given by Theorem 3 with $T=t$. Using the result of Case 1 for the process Y_n, we obtain

$$f(Y_n(t))-f(Y_n(0))$$
$$=\int_0^t f'(Y_n(s))G(s)\mathrm{d}s+\int_0^t\int_{E_n}\int_{\{|z|>1\}}\left[f(Y_n(s-)+K(s,x,z))-f(Y_n(s-))\right]N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)$$
$$+\int_0^t\int_{E_n}\int_{\{|z|\le 1\}}\left[f(Y_n(s-)+H(s,x,z))-f(Y_n(s-))\right]\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)$$
$$+\int_0^t\int_{E_n}\int_{\{|z|\le 1\}}\left[f(Y_n(s)+H(s,x,z))-f(Y_n(s))-H(s,x,z)f'(Y_n(s))\right]\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s.$$

The conclusion follows letting $n\to\infty$ as in Case 1. □

Proof of Theorem 2: We assume that f' and f'' are bounded. We fix *t*.

Case 1. *H vanishes outside a set* $B\in\mathcal{B}_b(\mathbb{R}^d)$. We write

$$Y(t)=\int_0^t\bar{G}(s)\mathrm{d}s+\int_0^t\int_B\int_{\{|z|\le 1\}}H(s,x,z)\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)+\int_0^t\int_B\int_{\{|z|>1\}}H(s,x,z)N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z),$$

where $\bar{G}(s)=G(s)-\int_B\int_{\{|z|>1\}}H(s,x,z)\nu(\mathrm{d}z)\mathrm{d}x$. By the Cauchy-Schwarz inequality, $\bar{G}$ satisfies (5) (since *B* is a *bounded* set). By Theorem 1, there exists a càdlàg modification of *Y* (denoted also by *Y*) such that

$$f(Y(t))-f(Y(0))$$
$$=\int_0^t f'(Y(s))\bar{G}(s)\mathrm{d}s+\int_0^t\int_B\int_{\{|z|>1\}}\left[f(Y(s-)+H(s,x,z))-f(Y(s-))\right]N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)$$
$$+\int_0^t\int_B\int_{\{|z|\le 1\}}\left[f(Y(s-)+H(s,x,z))-f(Y(s-))\right]\hat{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}z)$$
$$+\int_0^t\int_B\int_{\{|z|\le 1\}}\left[f(Y(s)+H(s,x,z))-f(Y(s))-H(s,x,z)f'(Y(s))\right]\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s.$$

We add and subtract $\int_0^t\int_B\int_{\{|z|>1\}}\left[f(Y(s)+H(s,x,z))-f(Y(s))\right]\nu(\mathrm{d}z)\mathrm{d}x\mathrm{d}s$. The conclusion follows by rearranging the terms.

Case 2. *H satisfies Assumption C*.

By Theorem 4, there exists a càdlàg modification of *Y* (denoted also by *Y*) such that (15) holds, where $\{Y_n(s)\}_{s\in[0,t]}$ is a càdlàg modification of

$$Y_n(s)=\int_0^s G(r)\mathrm{d}r+\int_0^s\int_{E_n}\int_{\mathbb{R}_0}H(r,x,z)\hat{N}(\mathrm{d}r,\mathrm{d}x,\mathrm{d}z), \quad s\in[0,t],$$

$(E_n)_n$ being the sequence given by Theorem 4 with $T=t$. We write the Itô formula for the process Y_n (using Case 1) and we let $n\to\infty$. □

4. Applications

In this section, we assume that the Lévy measure ν satisfies the condition:

$$v := \int_{\mathbb{R}_0} z^2 \nu(\mathrm{d}z) < \infty.$$

As in [1], we consider the process $L = \left\{ L(B); t \geq 0, B \in \mathcal{B}_b\left(\mathbb{R}_+ \times \mathbb{R}^d\right) \right\}$ defined by:

$$L(B) = \int_{B \times \mathbb{R}_0} z\hat{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}z).$$

For any predictable process $X = \left\{ X(t,x); t \geq 0, x \in \mathbb{R}^d \right\}$ such that

$$E\int_0^T \int_{\mathbb{R}^d} \left| X(t,x) \right|^2 \mathrm{d}x\mathrm{d}t < \infty \quad \text{for any } T > 0, \tag{20}$$

we can define the stochastic integral of X with respect to L and this integral satisfies:

$$\int_0^T \int_{\mathbb{R}^d} X(t,x) L(\mathrm{d}t, \mathrm{d}x) = \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}_0} X(t,x) z\hat{N}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}z).$$

By (2), this integral has the following isometry property:

$$E\left| \int_0^T \int_{\mathbb{R}^d} X(t,x) L(\mathrm{d}t, \mathrm{d}x) \right|^2 = vE\int_0^T \int_{\mathbb{R}^d} \left| X(t,x) \right|^2 \mathrm{d}x\mathrm{d}t.$$

When used as a noise process perturbing an SPDE, L behaves very similarly to the Gaussian white noise. For this reason, L was called a *Lévy white noise* in [1].

4.1. Kunita Inequality

The following maximal inequality is due to Kunita (see Theorem 2.11 of [7]). In problems related to SPDEs with noise L, this result plays the same role as the Burkholder-Davis-Gundy inequality for SPDEs with Gaussian white noise.

Theorem 5 (Kunita Inequality). *Let* $Y = \left\{ Y(s) \right\}_{t \geq 0}$ *be a process given by*

$$Y(t) = \int_0^t \int_{\mathbb{R}^d} X(s,x) L(\mathrm{d}s, \mathrm{d}x), \quad t \geq 0,$$

where X is a predictable process which satisfies (20).

If $m_p = \int_{\mathbb{R}_0} |z|^p \nu(\mathrm{d}z) < \infty$ for some $p \geq 2$, then for any $t > 0$,

$$E\left(\sup_{s \leq t} \left| Y(s) \right|^p \right) \leq C_p \left\{ E\left(\int_0^t \int_{\mathbb{R}^d} \left| X(s,x) \right|^2 \mathrm{d}x\mathrm{d}s \right)^{p/2} + E\int_0^t \int_{\mathbb{R}^d} \left| X(s,x) \right|^p \mathrm{d}x\mathrm{d}s \right\},$$

where $C_p = K_p \max\left(v^{p/2}, m_p \right)$ and K_p is the constant in Theorem 2.11 of [7].

Proof: We apply Theorem 2 with $f(x) = |x|^p$ and $H(s,x,z) = X(s,x)z$. The proof is identical to that of Theorem 2.11 of [7]. We omit the details. □

Remark 1. Kunita's constant K_p cannot be computed explicitly. Theorem 5 is proved in [9] using a different method which shows that K_p is directly related to the constant B_p in Rosenthal's inequality, which is $O(p/\ln p)$.

4.2. Itô Representation Theorem and Chaos Expansion

In this section, we give an application to Theorem 2 to exponential martingales, which leads to Itô representation theorem and a chaos expansion (similarly to Sections 5.3 and 5.4 of [6]).

For any $h \in L^2\left(\mathbb{R}_+ \times \mathbb{R}^d\right)$ we let $L_h(t) = \int_0^t \int_{\mathbb{R}^d} h(s,x) L(\mathrm{d}s, \mathrm{d}x)$ for $t \geq 0$. We work with the càdlàg modification of the process L_h given by Theorem 4. By Lemma 2.4 of [1],

$$E\left(\mathrm{e}^{iL_h(t)} \right) = \exp\left\{ \int_0^t \int_{\mathbb{R}^d} \Psi\left(h(s,x) \right) \mathrm{d}x\mathrm{d}s \right\},$$

where

$$\Psi(u)=\int_{\mathbb{R}_0}\left(e^{iuz}-1-iuz\right)\nu(dz),\quad u\in\mathbb{R}.$$

Hence $E\left(M_h(t)\right)=1$ for all $t\geq 0$, where

$$M_h(t)=\exp\left\{iL_h(t)-\int_0^t\int_{\mathbb{R}^d}\Psi\left(h(s,x)\right)dxds\right\},\quad t\geq 0.$$

The following result is the analogue of Lemma 5.3.3 of [6].

Lemma 3. *For any* $h\in L^2\left(\mathbb{R}_+\times\mathbb{R}^d\right)$ *and* $t>0$, *with probability* 1,

$$M_h(t)=1+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}\left(e^{ih(s,x)z}-1\right)M_h(s-)\hat{N}(ds,dx,dz).$$

Proof: We apply Theorem 2 to the function $f(x)=e^{ix}$ and the process

$$Y(t)=L_h(t)+i\int_0^t\int_{\mathbb{R}^d}\Psi\left(h(s,x)\right)dxds.$$

Hence, $H(s,x,z)=h(s,x)z$ and $G(s)=i\int_{\mathbb{R}^d}\Psi\left(h(s,x)\right)dx$. We obtain:

$$\begin{aligned}M_h(t)-1=&\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}\left(e^{iY(s-)+ih(s,x)z}-e^{iY(s-)}\right)\hat{N}(ds,dx,dz)\\&+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}\left(e^{iY(s)+ih(s,x)z}-e^{iY(s)}-izh(s,x)e^{iY(s)}\right)\nu(dz)dxds\\&+\int_0^t ie^{iY(s)}\left(i\int_{\mathbb{R}^d}\Psi\left(h(s,x)\right)dx\right)ds.\end{aligned}$$

Since the sum of the last two integrals is 0, the conclusion follows. □

We fix $T>0$. We let $\mathcal{F}_t^L=\sigma\left(\left\{L_s(B);0\leq s\leq t,B\in\mathcal{B}_b\left(\mathbb{R}^d\right)\right\}\right)$. We denote by $L^2_{\mathbb{C}}\left(\Omega,\mathcal{F}_T^L,P\right)$ be the space of $\mathbb{C}$-valued square-integrable random variables which are measurable with respect to $\mathcal{F}_T^L$.

Lemma 4. *The linear span of the set* $\mathcal{A}=\left\{M_h(T);h\in L^2\left(\mathbb{R}_+\times\mathbb{R}^d\right)\right\}$ *is dense in* $L^2_{\mathbb{C}}\left(\Omega,\mathcal{F}_T^L,P\right)$.

Proof: The proof is similar to that of Lemma 5.3.4 of [6]. We omit the details. □

Theorem 6 (Ito Representation Theorem). *For any* $F\in L^2_{\mathbb{C}}\left(\Omega,\mathcal{F}_T^L,P\right)$, *there exists a unique predictable* $\mathbb{C}$*-valued process* $\psi=\left\{\psi(t,x,z);t\in[0,T],x\in\mathbb{R}^d,z\in\mathbb{R}_0\right\}$ *satisfying*

$$E\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}\left|\psi(t,x,z)\right|^2\nu(dz)dxdt<\infty \tag{21}$$

such that

$$F=E(F)+\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}_0}\psi(t,x,z)\hat{N}(dt,dx,dz). \tag{22}$$

Proof: By Lemma 3, relation (22) holds for $F=M_h(T)$ with $\psi(t,x,z)=\left(e^{ih(t,x)z}-1\right)M_h(t-)$. The conclusion follows by an approximation argument using Lemma 4. □

The multiple (and iterated) integral with respect $\hat{N}$ can be defined similarly to the Gaussian white-noise case (see e.g. Section 5.4 of [6]).

More precisely, we consider the Hilbert space $\mathcal{H}=L^2(U,\mathcal{U},\mu)$, where

$$U=[0,T]\times\mathbb{R}^d\times\mathbb{R}_0,\quad \mathcal{U}=\mathcal{B}([0,T])\times\mathcal{B}\left(\mathbb{R}^d\right)\times\mathcal{B}\left(\mathbb{R}_0\right)\text{ and }\mu=dtdx\nu(dz).$$

For any integer $n\geq 1$, we consider the n-th tensor product space $\mathcal{H}^{\otimes n}=L^2\left(U^n,\mathcal{U}^n,\mu^n\right)$. The n-th multiple integral $I_n(f)$ with respect to $\hat{N}$ can be constructed for any function $f\in\mathcal{H}^{\otimes n}$, and this integral has the isometry property:

$$E\left|I_n(f)\right|^2=n!\|f\|^2_{\mathcal{H}^{\otimes n}}.$$

Moreover, if $n\neq m$, then $E\left[I_n(f)I_m(g)\right]=0$ for all $f\in\mathcal{H}^{\otimes n}$ and $g\in\mathcal{H}^{\otimes m}$.

We have the following result.

Theorem 7 (Chaos Expansion). *For any* $F \in L^2(\Omega, \mathcal{F}_T^L, P)$, *there exist some symmetric functions* $f_n \in \mathcal{H}^{\otimes n}$, $n \geq 1$ *such that*

$$F = E(F) + \sum_{n\geq 1} I_n(f_n) \quad \text{in } L^2(\Omega).$$

In particular,

$$E|F|^2 = |E(F)|^2 + \sum_{n\geq 1} n! \|f_n\|^2_{\mathcal{H}^{\otimes n}}.$$

Proof: We use the same argument as in the classical case, when $\hat{N}$ is a PRM on $\mathbb{R}_+ \times \mathbb{R}_0$ and

$$L(t) = \int_0^t \int_{\mathbb{R}_0} z\hat{N}(\mathrm{d}s, \mathrm{d}z), t \geq 0$$

is a square-integrable Lévy process (see Theorem 5.4.6 of [6] or Theorem 10.2 of [10]). By Theorem 6, there exists a predictable process ψ_1 satisfying (1) such that

$$F = E(F) + \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}_0} \psi_1(t_1, x_1, z_1) \hat{N}(\mathrm{d}t_1, \mathrm{d}x_1, \mathrm{d}z_1). \tag{23}$$

By (21), $E|\psi_1(t_1, x_1, z_1)|^2 < \infty$ for almost all (t_1, x_1, z_1). For such (t_1, x_1, z_1) fixed, we apply Theorem 6 again to the variable $\psi_1(t_1, x_1, z_1)$. Hence, there exists a predictable process

$$\psi_2 = \{\psi_2(t_2, x_2, z_2); t_2 \in [0, t_1], x_2 \in \mathbb{R}^d, z_2 \in \mathbb{R}_0\}$$

Satisfying

$$E\int_0^{t_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}_0} |\psi_1(t_1, x_1, z_1)|^2 \nu(\mathrm{d}z_1) \mathrm{d}x_1 \mathrm{d}t_1 < \infty$$

such that

$$\psi_1(t_1, x_1, z_1) = E(\psi_1(t_1, x_1, z_1)) + \int_0^{t_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}_0} \psi_2(t_2, x_2, z_2) \hat{N}(\mathrm{d}t_2, \mathrm{d}x_2, \mathrm{d}z_2).$$

We substitute this into (23) and iterate the procedure. We omit the details. □

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