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Mixed Saddle Point and Its Equivalence with an Efficient Solution under Generalized (V, ρ)-Invexity

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Abstract

The purpose of this paper is to define the concept of mixed saddle point for a vector-valued Lagrangian of the non-smooth multiobjective vector-valued constrained optimization problem and establish the equivalence of the mixed saddle point and an efficient solution under generalized (V, ρ)-invexity assumptions.

Keywords

Nonsmooth Multiobjective Programs, (V, ρ)-Invexity, Mixed Saddle Point, Vector-Valued Mixed Lagrangian Function

1. Introduction

Jeyakumar and Mond [1] have introduced the notion of V-invexity for vector function and discussed its application to a class of multiobjective problems. Mishra and Mukherjee [2] and Liu [3] extended the concept of V-invexity of multiobjective programming to the case of nonsmooth multiobjective programming problems and duality results are also obtained. Jeyakumar [4] introduced ρ-invexity for differentiable scalar-valued functions. Also, Jeyakumar [5] defined ρ-invexity for nonsmooth scalar-valued functions, studied duality theorems for nonsmooth optimization problems, and gave relationship between saddle points and optima. In [6] (Bector), a sufficient optimality theorem is proved for a certain minmax programming problem under the assumptions (B, η)-invexity conditions.

Kuk, Lee and Kim [7] discussed that weak vector saddle-point theorems are obtained under V-ρ-invexity for vector-valued functions. Bhatia and Garg [8] defined (V, ρ)-invexity, (V, ρ)-quasiinvexity and (V, ρ)-pseudo-

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invexity for nonsmooth vector-valued Lipschitz functions using Clarke's generalized subgradients and established duality results for multiobjective programming problems. Bhatia [9] introduced higher order strong convexity for Lipschitz functions. The notion of vector-valued partial Lagrangian is also introduced and equivalence of the mixed saddle points of higher order and higher order minima are provided. In [10]-[13], saddle point theory in terms of Lagrangian functions was introduced. In [14] (Reddy and Mukherjee), some problems consisting of nonsmooth composite multiobjective programs have been treated with (V, ρ)-invexity type conditions and also vector saddle point theorems were obtained for composite programs. Yuan, Liu and Lai [15] defined new vector generalized convexity.

In this paper, we define the concept of mixed saddle point for a vector-valued constrained optimization problem and establish the equivalence of the mixed saddle point and an efficient solution under generalized (V, ρ)-invexity assumptions. Further mixed saddle point theorems are obtained.

2. Preliminaries

In this section we require some definitions and results.

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. Throughout this paper, the following conventions for vectors in R^n will be used:

a) $x > y$ if and only if $x_i > y_i, i = 1, 2, \cdots, n$,

b) $x \geqq y$ if and only if $x_i \geqq y_i, i = 1, 2, \cdots, n$,

c) $x \ngtr y$ is the negation of $x > y$.

The following non-smooth multiobjective programming problem is studied in this paper:

$$\text{MOP} \qquad \text{Minimize } f(x) = \left[f_1(x), f_2(x), \cdots, f_p(x)\right],$$
$$\text{subject to } \quad g_j(x) \leqq 0, \qquad j = 1, 2, \cdots, m,$$
$$x \in R^n.$$

where

1) $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \cdots, p$ and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \cdots, m$ are locally Lipschitz functions on R^n.

2) Let $S = \left\{x \in R^n : g_j(x) \leqq 0,\ j = 1, 2, \cdots, m\right\}$. be the set of feasible solution of problem (MOP). Now let $M = \{1, 2, \cdots, m\}, J \subseteq M$ and $K = M \setminus J$, $|J|$ denotes the cardinality of the index set J and $\tau = \left\{x \in R^n : g_k(x) \leqq 0, k \in K\right\}$

clearly $S \subseteq \tau$.

Problem (MOP) can be associated to problem (MOP_r):

$$\text{MOP}_\text{r} \quad \text{Minimize } f_r(x)$$
$$\text{subject to } \quad f_i(x) \leqq f_i(\bar{x}), \quad i = 1, 2, \cdots, p, \ i \neq r,$$
$$g_j(x) \leqq 0, \qquad j = 1, 2, \cdots, m,$$
$$x \in R^n.$$

Now, we introduce the following definitions:

Definition 1. *A vector function* $f : X \to R^p$, *locally Lipschitz at* $u \in X$, *is said to be* (V, ρ)*-invex at* u *if there exist functions* $\eta, \psi : X \times X \to R^n$, *a real number* ρ *and* $\theta_i : X \times X \to R^+ \setminus \{0\}, i = 1, 2, 3, \cdots, p$ *such that for all* $x \in X$ *for* $i = 1, 2, \cdots, p$

$$f_i(x) - f_i(u) \geq \theta_i(x, u)\xi_i^{\mathrm{T}}\eta(x, u) + \rho\left\|\psi(x, u)\right\|^2$$

for every $\xi_i \in \partial f_i(u), i = 1, 2, \cdots, p, \forall\, x \in X, x \neq u$ *and for* $i = 1, 2, \cdots, p$

$$f_i(x) - f_i(u) > \theta_i(x, u)\xi_i^{\mathrm{T}}\eta(x, u) + \rho\left\|\psi(x, u)\right\|^2$$

for every $\xi_i \in \partial f_i(u), i = 1, 2, \cdots, p,$ *then* f *is called strictly* (V, ρ)*-invex at* u.

Definition 2. *A vector function* $f : X \to R^p$, *locally Lipschitz at* $u \in X$, *is said to be* (V, ρ)*-pseudoinvex at* u *if there exist functions* $\eta, \psi : X \times X \to R^n$, *a real number* ρ *and* $\phi_i : X \times X \to R^+ \setminus \{0\}, i = 1, 2, 3, \cdots, p$ *such* *that for all* $x \in X$

$$\sum_{i=1}^{p} \xi_i^{\mathrm{T}} \eta(x,u) + \rho \|\psi(x,u)\|^2 \geqq 0$$

$$\Rightarrow \sum_{i=1}^{p} \phi_i(x,u) f_i(x) \geqq \sum_{i=1}^{p} \phi_i(x,u) f_i(u)$$

for every $\xi_i \in \partial f_i(u), i = 1, 2, \cdots, p,$

$$\forall\ x \in X, x \neq u$$

$$\sum_{i=1}^{p} \xi_i^{\mathrm{T}} \eta(x,u) \geqq -\rho \|\psi(x,u)\|^2$$

$$\Rightarrow \sum_{i=1}^{p} \phi_i(x,u) f_i(x) > \sum_{i=1}^{p} \phi_i(x,u) f_i(u)$$

for every $\xi_i \in \partial f_i(u), i = 1, 2, \cdots, p,$ *then the function is strictly* (V, ρ)*-pseudoinvex at* u.

Definition 3. *A vector function* $f : X \to R^p$, *locally Lipschitz at* $u \in X$, *is said to be* (V, ρ)*-quasiinvex at* u *if there exist functions* $\mu, \psi : X \times X \to R^n$, *a real number* ρ *and* $\phi_i : X \times X \to R^+ \setminus \{0\}, i = 1, 2, 3, \cdots, p$ *such that for all* $x \in X$

$$\sum_{i=1}^{p} \phi_i(x,u) f_i(x) \leqq \sum_{i=1}^{p} \phi_i(x,u) f_i(u)$$

$$\Rightarrow \sum_{i=1}^{p} \xi_i^{\mathrm{T}} \eta(x,u) \leqq -\rho \|\psi(x,u)\|^2$$

for every $\xi_i \in \partial f_i(u), i = 1, 2, \cdots, p.$

If f is (V, ρ)-invex at each $u \in X$ then the function is (V, ρ)-invex on X. Similar is the definition of other functions. It is evident that every (V, ρ)-invex function is both (V, ρ)-pseudoinvex and (V, ρ)-quasiinvex with $\theta_i = \dfrac{1}{\phi_i}$ and

$$\sum_{i=1}^{p} \phi_i(x,u) = 1$$

From the definitions it is clear that every strictly (V, ρ)-pseudoinvex on X is (V, ρ)-quasiinvex on X.

Definition 4. *A feasible point* $\bar{x} \in X$ *is said to be efficient solution for MOP if there is no other feasible solution* x *such that for some* $r \in \{1, 2, \cdots, p\}$

$$f_r(x) < f_r(\bar{x})$$

and

$$f_i(x) \leqq f_i(\bar{x})$$

for all $i = 1, 2, \cdots, p; i \neq r$.

Definition 5. *The vector valued mixed Lagrangian function* $L : \tau \times R_+^{|J|} \to R^p$ *corresponding to problem* (*MOP*) *is defined as*

$$L(x, \lambda_J) = \left[L_1(x, \lambda_J), \cdots, L_p(x, \lambda_J)\right]$$

where $L_i(x, \lambda_J) = f_i(x) + \lambda_J^{\mathrm{T}} g_J(x), i = 1, 2, \cdots, p, x \in \tau, \lambda_J \in R_+^{|J|}$.

Definition 6. *A vector* $(\bar{x}, \bar{\lambda}_J) \in \tau \times R_+^{|J|}$ *is said to be mixed saddle point of mixed Lagrangian* L *if*

$$L(\bar{x}, \bar{\lambda}_J) \nleq L(\bar{x}, \lambda_J), \quad \forall\ \lambda_J \in R_+^{|J|}$$

$$\text{and} \quad L(x, \bar{\lambda}_J) \nleq L(\bar{x}, \bar{\lambda}_J), \quad \forall\ x \in \tau.$$

Definition 7. *A function* $F : X \times X \times R^n \to R$ *is sublinear if for any* $x, \bar{x} \in X$,

1) $F(x, \bar{x}, a_1 + a_2) \leqq F(x, \bar{x}, a_1) + F(x, \bar{x}, a_2)$,

2) $F(x, \bar{x}, \alpha a) \leqq \alpha F(x, \bar{x}, a)$ *for any* $\alpha \in R, \alpha \geqq 0$ *and* $a \in R^n$.

For $\alpha = 0, F(x, \bar{x}, 0) = 0$.

Now, we have established our main results, to prove equivalence between mixed saddle point and an efficient solution.

3. Main Results

Theorem 1. *Let* $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in S \times R^p \times R^m$ *satisfy the following conditions*

$$0 \in \partial\left(\sum_{i=1}^{p} \bar{\alpha}_i f_i + \sum_{j=1}^{m} \bar{\lambda}_j g_j\right)(\bar{x}), \tag{1}$$

$$\bar{\lambda}_j g_j(\bar{x}) = 0, \quad j = 1, 2, \cdots, m, \tag{2}$$

$$\bar{\alpha} > 0, \bar{\alpha}^{\mathrm{T}} e = 1, \ e = (1, 1, \cdots, 1) \in R^p, \tag{3}$$

$$\bar{\lambda} \geqq 0. \tag{4}$$

Further, let $\left[\sum_{i=1}^{p} \bar{\alpha}_i L_i(., \bar{\lambda}_J)\right]$ *be (V, ρ)-pseudoinvex at* $\bar{x}$ *and* $\left[\sum_{k \in K} \bar{\lambda}_k g_k(.)\right]$ *is (V, ρ)-quasiinvex at* $\bar{x}$ *with* $\rho + \sigma \geqq 0$. *Then* $(\bar{x}, \bar{\lambda}_j)$ *is a mixed saddle point of* L.

Proof. Since $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ satisfies (1), we have

$$0 \in \partial\left(\sum_{i=1}^{p} \bar{\alpha}_i f_i + \sum_{j=1}^{m} \bar{\lambda}_j g_j\right)(\bar{x}) \tag{5}$$

$$\Rightarrow 0 \in \partial\left(\sum_{i=1}^{p} \bar{\alpha}_i f_i + \bar{\lambda}_J^{\mathrm{T}} g_J\right)(\bar{x}) + \partial\left(\sum_{k \in K} \lambda_k g_k\right)(\bar{x}) \tag{6}$$

As $\bar{\alpha}^{\mathrm{T}} e = 1$, from (6), we obtain

$$0 \in \partial\left(\sum_{i=1}^{p} \bar{\alpha}_i L_i(\bar{x}, \bar{\lambda}_J)\right) + \partial\left(\sum_{k \in K} \bar{\lambda}_k g_k\right)(\bar{x}). \tag{7}$$

Hence, there exist

$$\xi \in \partial\left(\sum_{i=1}^{p} \bar{\alpha}_i L_i(\bar{x}, \bar{\lambda}_J)\right) \text{ and } \eta \in \partial\left(\sum_{k \in K} \bar{\lambda}_k g_k\right)(\bar{x})$$

such that

$$\xi + \eta = 0. \tag{8}$$

Now for any $x \in \tau$

$$g_k(x) \leqq 0, \ k \in K. \tag{9}$$

As $\bar{\lambda}_i \geqq 0$, (9) gives

$$\sum_{k \in K} \bar{\lambda}_k g_k(x) \leqq 0. \tag{10}$$

From (2) and (10) it follows that

$$\sum_{k \in K} \bar{\lambda}_k g_k(x) \leqq \sum_{k \in K} \bar{\lambda}_k g_k(\bar{x}). \tag{11}$$

Using the (*V*, *ρ*)-quasiinvexity of $\sum_{k\in K}\bar{\lambda}_k g_k(.)$ at $\bar{x}$, we get

$$V(x,\bar{x}) \leqq -\sigma\left\|\psi(x,\bar{x})\right\|^2. \tag{12}$$

(12) along with the fact $\rho+\sigma \geqq 0$ gives

$$V(x,\bar{x}) \leqq \rho\left\|\psi(x,\bar{x})\right\|^2. \tag{13}$$

From (8) and (13) and using the sublinearity of V, we have

$$0 = V(x,0) = V(x,\xi+\eta) \leqq V(x,\xi)+V(x,\eta) \leqq V(x,\xi)+\rho\left\|\psi(x,\eta)\right\|^2 \tag{14}$$

$$\Rightarrow V(x,\xi) \geqq -\rho\left\|\psi(x,\eta)\right\|^2, \xi \in \partial\left(\sum_{i=1}^{p}\bar{\alpha}_i L_i(\bar{x},\bar{\lambda}_J)\right) \tag{15}$$

Now using (*V*, *ρ*)-pseudoinvex of $\sum_{i=1}^{p}\bar{\alpha}_i L_i(.,\bar{\lambda}_J)$ at $\bar{x}$ in (15)

$$\sum_{i=1}^{p}\bar{\alpha}_i L_i(x,\bar{\lambda}_J) \geqq \sum_{i=1}^{p}\bar{\alpha}_i L_i(\bar{x},\bar{\lambda}_j). \tag{16}$$

Since $\bar{\alpha} > 0$, we obtain from (16)

$$L(x,\bar{\lambda}_J) \nleqq L(\bar{x},\bar{\lambda}_J), \quad \forall x \in \tau \tag{17}$$

Again for any $\lambda_J \in R_+^{|J|}$ and $\bar{x} \in S$ we have

$$\lambda_J^{\mathrm{T}} g_J(\bar{x}) \leqq 0, \tag{18}$$

(18) along with (2) implies

$$\lambda_J^{\mathrm{T}} g_J(\bar{x}) \leqq \bar{\lambda}_J^{\mathrm{T}} g_J(\bar{x}), \tag{19}$$

Therefore, from (19)

$$L_i(\bar{x},\lambda_J) \leqq L_i(\bar{x},\bar{\lambda}_J), \quad \forall\ i = 1,2,\cdots,p \text{ and } \lambda_J \in R_+^{|J|}. \tag{20}$$

Hence

$$L(\bar{x},\bar{\lambda}_J) \nleqq L(\bar{x},\lambda_J), \ \forall\ \lambda_J \in R_+^{|J|}. \tag{21}$$

From (17) and (21) and the fact that $\bar{x} \in S \subseteq \tau$, it follows that $(\bar{x},\bar{\lambda}_J)$ is a mixed saddle point of L.

Theorem 2. *Let* $(\bar{x},\bar{\alpha},\bar{\lambda}) \in S \times R^p \times R^m$ *satisfy the conditions from* (1) *to* (4). *If* $\sum_{i=1}^{p}\bar{\alpha}_i L_i(.,\bar{\lambda}_J)$ *is* (*V*, *ρ*)-*quasiinvex at* $\bar{x}$ *and* $\sum_{k\in K}\lambda_k g_k(.)$ *is strictly* (*V*, *ρ*)-*pseudoinvex at* $\bar{x}$ *with* $\rho+\sigma \geqq 0$ *then* $(\bar{x},\bar{\lambda}_J)$ *is mixed saddle point.*

Proof. Since $(\bar{x},\bar{\alpha},\bar{\lambda})$ satisfies (1), proceeding in the same manner as in the Theorem (1), we have

$$\xi+\eta = 0 \tag{22}$$

where $\xi \in \partial\left(\sum_{i=1}^{p}\bar{\alpha}_i L_i(\bar{x},\bar{\lambda}_J)\right)$ and $\eta \in \partial\left(\sum_{k\in K}\bar{\lambda}_k g_k\right)(\bar{x})$.

Now, for any $x \in \tau$, $g_k(x) \leqq 0$, $k \in K$ which along with (2) gives

$$\sum_{k\in K}\bar{\lambda}_k g_k(x) \leqq \sum_{k\in K}\bar{\lambda}_k g_k(\bar{x}). \tag{23}$$

Using strict (*V*, *ρ*)-pseudoinvexity of $\sum_{k\in K}\bar{\lambda}_k g_k(.)$ at $\bar{x}$ in (23) we get

$$V(x,\bar{x}) < -\sigma\left\|\psi(x,\bar{x})\right\|^2. \tag{24}$$

The fact of $\rho+\sigma \geqq 0$ and (24) gives

$$V(x,\bar{x})<\rho\|\psi(x,\bar{x})\|^{2}. \tag{25}$$

From the sublinearty of V

$$0=V(x,\xi+\eta)\leqq V(x,\xi)+V(x,\eta). \tag{26}$$

(25) along with (26) gives

$$V(x,\xi)>-\rho\|\psi(x,\eta)\|^{2}. \tag{27}$$

From (V, ρ)-quasiinvexity of $\sum_{i=1}^{p}\bar{\alpha}_i L_i(.,\bar{\lambda}_J)$ at $\bar{x}$ and (27) it follows that

$$\sum_{i=1}^{p}\bar{\alpha}_i L_i(x,\bar{\lambda}_J)\geqq \sum_{i=1}^{p}\bar{\alpha}_i L_i(\bar{x},\bar{\lambda}_J). \tag{28}$$

From (28), proceeding in the same manner as in Theorem (1) we obtain that $(\bar{x},\bar{\alpha},\bar{\lambda})$ is the mixed saddle point of L.

Theorem 3. *Let* $\bar{x}$ *be an efficient solution for the problem* (*MOP*) *and let the functions* f,g *be regular at* $\bar{x}$. *Assume that for at least one r,* (MOP_r) *is calm at* $\bar{x}$. *Then there exit* $\bar{\alpha}\in R^p$ *and* $\bar{\lambda}\in R^m$ *such that* $(\bar{x},\bar{\alpha},\bar{\lambda})$ *satisfies conditions from* (1) *to* (4). *Further let* $\sum_{i=1}^{p}\bar{\alpha}_i L_i(.,\bar{\lambda}_J)$ *be strictly* (V, ρ)*-pseudoinvex at* $\bar{x}$ *and* $\sum_{k\in K}\lambda_k g_k(.)$ *be* (V, ρ)*-quasiinvex at* $\bar{x}$ *with* $\rho+\sigma\geqq 0$ *then* $(\bar{x},\bar{\lambda}_J)$ *is a mixed saddle point of* L.

Proof. Since $\bar{x}$ is an efficient solution of (1) and Clarke's calmness constraint qualification holds. It follows from Fritz John type necessary optimality conditions that $\exists\,\bar{\lambda}\in R^m$, $\bar{\alpha}\in R^p$ such that

$$0\in\sum_{i=1}^{p}\bar{\alpha}_i\partial f_i(\bar{x})+\sum_{j=1}^{m}\bar{\lambda}_j\partial g_j(\bar{x}), \tag{29}$$

$$\bar{\lambda}_j g_j(\bar{x})=0,\ j=1,2,\cdots,m, \tag{30}$$

$$\bar{\alpha}>0,\ \bar{\alpha}^{\mathrm{T}}e=1,\ e=(1,1,\cdots,1)\in R^p,\bar{\lambda}\geqq 0. \tag{31}$$

Now as f,g are regular at $\bar{x}$, (29) gives

$$0\in\partial\left(\sum_{i=1}^{p}\bar{\alpha}_i f_i+\sum_{j=1}^{m}\bar{\lambda}_j g_j\right)(\bar{x}). \tag{32}$$

(30), (31) and (32) imply that conditions (1) to (4) are satisfied. As $(\bar{x},\bar{\alpha},\bar{\lambda})$ satisfies (1) to (4), proceeding in the same manner as in Theorem (1), we obtain (15).

Now, using strict (V, ρ)-pseudoinvexity of $\sum_{i=1}^{p}\bar{\alpha}_i L_i(.,\bar{\lambda}_J)$ at $\bar{x}$, we get

$$\sum_{i=1}^{p}\bar{\alpha}_i L_i(x,\bar{\lambda}_J)>\sum_{i=1}^{p}\bar{\alpha}_i L_i(\bar{x},\bar{\lambda}_J). \tag{33}$$

Since, $\bar{\alpha}>0$, we obtain from (33)

$$L(x,\bar{\lambda}_J)\nleq L(\bar{x},\bar{\lambda}_J),\quad \forall\ x\in\tau.$$

Again, proceeding in the same manner as in Theorem (1), it is proved that $(\bar{x},\bar{\lambda}_J)$ is a mixed saddle point of L.

In the next theorem no invexity or generalized invexity is used.

Theorem 4. *If* $(\bar{x},\bar{\lambda}_J)$ *is a mixed saddle point of mixed Lagrangian then* $\bar{x}$ *is an efficient solution of the problem* (*MOP*).

Proof: Since $(\bar{x},\bar{\lambda}_J)$ is a mixed saddle point of L, we have $\bar{x}\in\tau$ and

$$L(\bar{x},\bar{\lambda}_J)\nleq L(\bar{x},\lambda_J),\quad \forall\ \lambda_J\in R_J^{|J|}. \tag{34}$$

From (34), we get

$$\left(\bar{\lambda}_J - \lambda_J\right)^{\mathrm{T}} g_J\left(\bar{x}\right) \geqq 0, \quad \forall\, \lambda_J \in R_+^{|J|}. \tag{35}$$

Taking $\lambda_J = \bar{\lambda}_J + u$ in (35), where $u \in R_+^{|J|}$ is a vector having unity at the j^{th} position and zero elsewhere, we get

$$g_J\left(\bar{x}\right) \leqq 0, \quad \forall\, j \in J.$$

Moreover, $\bar{x} \in \tau$, hence

$$g_k\left(\bar{x}\right) \leqq 0, \quad \forall\, k \in K. \tag{36}$$

Thus, we have

$$g_j\left(\bar{x}\right) \leqq 0, \quad j = 1, 2, \cdots, m.$$

Hence, $\bar{x}$ is feasible for the problem (MOP). Further, taking $\lambda_J = 0$ in (35), we get

$$\bar{\lambda}_J^{\mathrm{T}} g_J\left(\bar{x}\right) \geqq 0. \tag{37}$$

But as $\bar{x} \in S$ and $\bar{\lambda}_J \geqq 0$, from (37), we obtain

$$\bar{\lambda}_J^{\mathrm{T}} g_J\left(\bar{x}\right) = 0. \tag{38}$$

Now contrary to the result, let $\bar{x}$ be not an efficient solution of the problem (MOP). Then there exist $x \in S$ and an index $r, 1 \leqq r \leqq p$, such that

$$f_r\left(x\right) < f_r\left(\bar{x}\right) \tag{39}$$

and

$$f_i\left(x\right) \leqq f_i\left(\bar{x}\right). \tag{40}$$

(39) and (40) along with (38) give

$$f_r\left(x\right) + \bar{\lambda}_J^{\mathrm{T}} g_J\left(x\right) < f_r\left(\bar{x}\right) + \bar{\lambda}_J^{\mathrm{T}} g_J\left(\bar{x}\right) \tag{41}$$

and

$$f_i\left(x\right) + \bar{\lambda}_J^{\mathrm{T}} g_J\left(x\right) \leqq f_i\left(x\right) + \bar{\lambda}_J^{\mathrm{T}} g_J\left(\bar{x}\right) \tag{42}$$

that is

$$L_r\left(x, \bar{\lambda}_J\right) < L_r\left(\bar{x}, \bar{\lambda}_J\right). \tag{43}$$

$$L_i\left(x, \bar{\lambda}_J\right) \leqq L_i\left(\bar{x}, \bar{\lambda}_J\right), \quad \forall\, i = 1, 2, \cdots, p, \; i \neq r. \tag{44}$$

(43) and (44) are contradiction to the fact that

$$L\left(x, \bar{\lambda}_J\right) \nleq L\left(\bar{x}, \bar{\lambda}_J\right), \quad \forall\, x \in \tau.$$

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