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A Note on the Almost Sure Central Limit Theorem for Partial Sums of ρ^{-}-Mixing Sequences

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Abstract

Let $\{X_n\}_{n\in N}$ be a strictly stationary sequence of ρ^{-}-mixing random variables. We proved the almost sure central limit theorem, containing the general weight sequences, for the partial sums S_n/σ_n, where $S_n=\sum_{i=1}^{n}X_i$, $\sigma_n^2=\mathrm{E}S_n^2$. The result generalizes and improves the previous results.

Keywords

ρ^{-}-Mixing Sequences, Partial Sums, Almost Sure Central Limit Theorem

1. Introduction

Let $\mathscr{C}$ be a class of functions which are coordinatewise increasing. For a random variable X, define

$$\|X\|_p=\left(\mathrm{E}|X|^p\right)^{1/p}.$$

For two nonempty disjoint sets $S,T\subset N$, we define $\operatorname{dist}(S,T)$ to be $\min\{|j-k|;j\in S,k\in T\}$. Let $\sigma(S)$ be the σ-field generated by $\{X_k,k\in S\}$, and define $\sigma(T)$ similarly.

A sequence $\{X_n,n\ge 1\}$ is called negatively associated (NA) if for ever pair of disjoint subsets S, T of N,

$$\operatorname{cov}\left\{f\left(X_i,i\in S\right),g\left(X_j,j\in T\right)\right\}\le 0,$$

where $f,g\in\mathscr{C}$. $\{X_n,n\ge 1\}$ is called ρ^{*}-mixing, if

$$\rho^{*}(k)=\sup\left\{\rho(S,T);S,T\subset N,\operatorname{dist}(S,T)\ge k\right\}\to 0,\ k\to\infty,$$

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where

$$\rho(S,T)=\sup\left\{\frac{\left|\mathrm{E}(f-\mathrm{E}f)(g-\mathrm{E}g)\right|}{\|f-\mathrm{E}f\|_2\cdot\|g-\mathrm{E}g\|_2};f\in L_2(\sigma(S)),f\in L_2(\sigma(T))\right\}.$$

Definition 1. [1] *A sequence* $\{X_n,n\geq 1\}$ *is called* ρ^-*-mixing, if*

$$\rho^-(k)=\sup\left\{\rho^-(S,T):\mathrm{dist}(S,T)\geq k,S,T\subset N\right\}\to 0,\text{ as }k\to\infty,$$

where

$$\rho^-(S,T)=0\vee\sup\left\{\frac{\mathrm{cov}\left\{f\left(X_i,i\in S\right),g\left(X_j,t\in T\right)\right\}}{\sqrt{\mathrm{var}\left\{f\left(X_i,i\in S\right)\right\}\mathrm{var}\left\{g\left(X_j,t\in T\right)\right\}}};f,g\in\mathscr{C}\right\}.$$

The definition of NA is given by Joag-Dev and Proschan [2], and the concept of ρ^*-mixing random variables is given by Kolmogorov and Rozanov [3]. In 1999, the concept of ρ^--mixing random variables was introduced initially by Zhang and Wang [1]. Obviously, ρ^--mixing random variables include NA and ρ^*-mixing random variables, which have a lot of applications. Their limit properties have received more and more attention recently, and a number of results have been obtained, such as Zhang and Wang [1] for Rosenthal-type moment inequality and Marcinkiewicz-Zygmund law of large numbers, Zhang [4] for the central limit theorems of random fields, Wang and Lu [5] for the weak convergence theorems.

Starting with Brosamler [6] and Schatte [7], in the last two decades several authors investigated the almost sure central limit theorem (ASCLT) for partial sums S_n/σ_n of random variables. We refer the reader to Brosamler [6], Schatte [7], Lacey and Philipp [8], Ibragimov and Lifshits [9], Berkes and Csáki [10], Hörmann [11] and Wu [12]. The simplest form of the ASCLT [6]-[8] reads as follows: let $\{X_n;n\geq 1\}$ be i.i.d. random variables with mean 0, variance $\sigma^2>0$ and partial sums S_n. Then

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left(\frac{S_k}{\sigma\sqrt{k}}\leq x\right)=\Phi(x)\text{ a.s. for any }x\in R. \tag{1}$$

where I denotes indicator function, and $\Phi(x)$ is the standard normal distribution function. For other version of ρ^--mixing sequences, see [13]-[15].

The purpose of this article is to study and establish the ASCLT, containing the general weight sequences, for partial sums of ρ^--mixing sequence. Our results not only generalize and improve those on ASCLT previously obtained by Brosamler [6], Schatte [7] and Lacey and Philipp [8] from the i.i.d. case to ρ^--mixing sequences, but also expand the scope of the weights from $1/k$ to $\exp\left(\log^\alpha k\right)/k$, $0\leq\alpha<1/2$.

Throughout this paper, $a_n\sim b_n$ means $\lim_{n\to\infty}a_n/b_n=1$; and set the positive absolute constant c to vary from line to line.

Theorem 1. *Let* $\{X_n\}_{n\in N}$ *be a strictly stationary* ρ^-*-mixing sequence with* $\mathrm{E}X_1=0$, $0<\mathrm{E}|X_1|^r<\infty$ *for a certain* $r>2$, *and denote* $S_n=\sum_{i=1}^n X_i$, $\sigma_n^2=\mathrm{E}S_n^2$. *Assume that*

(a) $\sigma^2=\mathrm{E}X_1^2+2\sum_{k=2}^{\infty}\mathrm{cov}(X_1,X_k)>0,$

(b) $\sum_{k=2}^{\infty}\left|\mathrm{cov}(X_1,X_k)\right|<\infty,$

(c) $\sum_{k=1}^{\infty}\frac{\rho^-(k)}{k}<\infty.$

Suppose $0\leq\alpha<1/2$ *and set*

$$d_k=\frac{\exp\left(\log^\alpha k\right)}{k},D_n=\sum_{k=1}^{n}d_k. \tag{2}$$

then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k I\left(\frac{S_k}{\sigma_n}\le x\right)=\Phi(x)\text{ a.s. for any } x\in\mathrm{R}. \tag{3}$$

Remark 1. By the terminology of summation procedures (cf. [16], p. 35), Theorem 1 remains valid if we replace the weight sequence $\{d_k\}_{k\ge1}$ by any $\{d_k^*\}_{k\ge1}$ such that $0\le d_k^*\le d_k$ and $\sum_{k\ge1}d_k^*=\infty$.

Remark 2. ρ^--mixing random variables include NA and ρ^*-mixing random variables, so for NA and ρ^*-mixing random variables sequences Theorem 1 also holds.

Remark 3. Essentially, the open problem that whether Theorem 1 holds for $1/2\le\alpha<1$ still remains open.

2. Some Lemmas

Lemma 1. [4] *Let* $\{X_n, n\ge1\}$ *be a weakly stationary* ρ^-*-mixing sequence with* $\mathrm{E}X_n=0$, $0<\mathrm{E}X_1^2<\infty$, *and* $\sigma^2=\mathrm{E}X_1^2+2\sum_{k=2}^{\infty}\operatorname{cov}(X_1,X_k)>0$, $\sum_{k=2}^{\infty}\left|\operatorname{cov}(X_1,X_k)\right|<\infty$, *then*

$$\frac{\sigma_n^2}{n}\to\sigma^2,\quad \frac{S_n}{\sigma_n}\xrightarrow{d}\mathcal{N},\text{ as } n\to\infty,$$

where $\mathcal{N}$ *denotes the standard normal random variable.*

Lemma 2. [5] *For a positive real number* $q\ge2$, *if* $\{X_n, n\ge1\}$ *is a sequence of* ρ^-*-mixing random variables with* $\mathrm{E}X_i=0$, $\mathrm{E}|X_i|^q<\infty$ *for every* $i\ge1$, *then for all* $n\ge1$, *there is a positive constant* $c=c(q,\rho^-(\cdot))$ *such that*

$$\mathrm{E}\left(\max_{1\le j\le n}\left|S_j\right|^q\right)\le c\left(\sum_{i=1}^{n}\mathrm{E}|X_i|^q+\left(\sum_{i=1}^{n}\mathrm{E}X_i^2\right)^{q/2}\right).$$

Lemma 3. [17] *Let* $\{X_n, n\ge1\}$ *be a weakly stationary* ρ^-*-mixing sequence. Assume* $\sup_n\mathrm{E}|X_n|^r<\infty$. *Then for any bounded Lipschitz function f:* $R\to R$, *We have*

$$\left|\operatorname{cov}\left(f\left(\frac{S_i}{\sigma_i}\right),f\left(\frac{S_j}{\sigma_j}\right)\right)\right|\le c\left|-\frac{1}{\sigma_i\sigma_j}\sum_{l=1}^{i}\sum_{m=2i+1}^{2i+l}\operatorname{cov}(X_l,X_m)+8\rho^-(i)+2\frac{\sigma_{2i}}{\sigma_j}\right|$$

Lemma 4. *Let* $\{\xi,\xi_n\}_{n\in N}$ *be a sequence of uniformly bounded random variables. Assume that* $\sum_{k=1}^{\infty}\frac{\rho^-(k)}{k}<\infty$, *and existing constants* $c>0$ *and* $\varepsilon>0$ *such that*

$$\left|\mathrm{E}\xi_k\xi_l\right|\le c\left(\rho^-(k)+\left(\frac{k}{l}\right)^{\varepsilon}\right),\text{ for } 1\le 2k<l,$$

then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\xi_k=0\text{ a.s.}, \tag{4}$$

where d_k *and* D_n *are defined by* (2).

Proof. Set $T_n=\frac{1}{D_n}\sum_{k=1}^{n}d_k\xi_k$, we get

$$\mathrm{E}T_n^2=\frac{1}{D_n^2}\mathrm{E}\left(\sum_{k=1}^{n}d_k\xi_k\right)^2\le\frac{1}{D_n^2}\sum_{1\le k\le l\le n,2k\ge l}d_kd_l\left|\mathrm{E}\xi_k\xi_l\right|+\frac{1}{D_n^2}\sum_{1\le k\le l\le n,2k<l}d_kd_l\left|\mathrm{E}\xi_k\xi_l\right|$$
$$:=\frac{1}{D_n^2}T_{n1}+\frac{1}{D_n^2}T_{n2}.$$

Firstly we estimate T_{n1}. Since ξ_k is a bounded random variable, we get

$$T_{n1} \le c\sum_{k=1}^{n}\sum_{l=k}^{2k} d_k d_l \le c\exp\left(\log^\alpha n\right)\sum_{k=1}^{n} d_k \sum_{l=k}^{2k}\frac{1}{l} \le cD_n \exp\left(\log^2 n\right).$$

Now we estimate T_{n2}. By the conditions $\left|\mathrm{E}\xi_k\xi_l\right| \le c\left(\rho^-(k)+\left(\frac{k}{l}\right)^\varepsilon\right)$ for $l > 2k$, we get

$$T_{n2} = \sum_{1\le k\le l\le n, 2k<l} d_k d_l \left|\mathrm{E}\xi_k\xi_l\right| \le c\sum_{l=2}^{n}\sum_{k=1}^{l-1} d_k d_l \rho^-(k) + c\sum_{l=2}^{n}\sum_{k=1}^{l-1} d_k d_l \left(\frac{k}{l}\right)^\varepsilon$$
$$\le c\sum_{l=1}^{n} d_l \sum_{k=1}^{n} d_k \rho^-(k) + c\sum_{l=1}^{n}\sum_{k=1}^{l} d_k d_l \left(\frac{k}{l}\right)^\varepsilon := A_1 + A_2.$$

By condition $\sum_{k=1}^{\infty}\frac{\rho^-(k)}{k} < \infty$, we obtain

$$A_1 \le c\exp\left(\log^\alpha n\right)\sum_{l=1}^{n} d_l \sum_{k=1}^{n}\frac{\rho^-(k)}{k} \le cD_n \exp\left(\log^\alpha n\right).$$

and

$$A_2 \le c\sum_{l=2}^{n}\sum_{k=1}^{l}\frac{\exp\left(\log^\alpha l\right)}{l^{1+\varepsilon}}\cdot\frac{\exp\left(\log^\alpha k\right)}{k^{1-\varepsilon}} \le c\exp\left(\log^\alpha n\right)\sum_{l=1}^{n}\frac{\exp\left(\log^\alpha l\right)}{l^{1+\varepsilon}}\cdot\frac{l^\varepsilon}{\varepsilon} \le cD_n \exp\left(\log^\alpha n\right).$$

Since $D_n \sim \frac{1}{\alpha}\log^{1-\alpha} n \exp\left(\log^\alpha n\right)$ and $\log D_n \sim \log^\alpha n$ for $0 < \alpha < 1/2$ from the proof of Lemma 2.2 in Wu [18], we have, as $n \to \infty$,

$$\exp\left(\log^\alpha n\right) \sim \frac{\alpha D_n}{\left(\log D_n\right)^{(1-\alpha)/\alpha}} \sim \frac{\alpha D_n}{\log^{1-\alpha} n},$$

Thus

$$\mathrm{E}T_n^2 \le \frac{1}{D_n^2}\left(T_{n1} + A_1 + A_2\right) = c\frac{\exp\left(\log^\alpha n\right)}{D_n} \le \frac{c}{\log^{1-\alpha} n}.$$

Let $n_k = \exp\left(k^\tau\right)$, $\tau > 1/(1-\alpha)$, we get

$$\sum_{K=1}^{\infty} P\left(\left|T_{n_k}\right| > \varepsilon\right) \le c\sum_{k=1}^{\infty}\mathrm{E}T_{n_k}^2 \le c\sum_{k=1}^{\infty}\frac{1}{k^{(1-\alpha)\tau}} < \infty.$$

By Borel-Cantelli lemma,

$$T_{n_k} \to 0 \text{ a.s.}, k \to \infty.$$

For any n, existing n_k and n_{k+1} such that $n_k < n \le n_{k+1}$, then, by $\left|\xi_i\right| \le c$ for any i,

$$\left|T_n\right| \le \left|\frac{1}{D_{n_k}}\sum_{i=1}^{n_k} d_i\xi_i\right| + \frac{1}{D_{n_k}}\sum_{i=n_k+1}^{n_{k+1}} d_i\left|\xi_i\right| \le \left|T_{n_k}\right| + c\left(\frac{D_{n_{k+1}} - D_{n_k}}{D_{n_k}}\right) \to 0 \text{ a.s. } n \to \infty,$$

from $\frac{D_{n_{k+1}}}{D_{n_k}} \sim \frac{\exp\left((k+1)^\tau\right)}{\exp\left(k^\tau\right)} \sim \exp\left((k+1)^\tau\left(1-\left(\frac{k}{k+1}\right)^\tau\right)\right) \to 1$. *i.e.*, (4) holds. This completes the proof of Lemma 4.

3. Proof

Proof of Theorem 1. By Lemma 1, we have

$$\frac{S_k}{\sigma_k} \xrightarrow{d} \mathcal{N}, \text{ as } k \to \infty.$$

This implies that for any $g(x)$ which is a bounded function with bounded continuous derivatives,

$$\mathrm{E}g\left(\frac{S_k}{\sigma_k}\right) \to \mathrm{E}g(\mathcal{N}), \text{ as } k \to \infty,$$

Hence, by the Toeplitz lemma, we obtain

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \mathrm{E}g\left(\frac{S_k}{\sigma_k}\right) = \mathrm{E}g(\mathcal{N}).$$

In the other hand, from Theorem 7.1 of Billingsley [19] and Section 2 of Peligrad and Shao [20], we know that (3) is equivalent to

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k g\left(\frac{S_k}{\sigma_k}\right) = \mathrm{E}g(\mathcal{N}) \text{ a.s..}$$

Hence, to prove (3), it suffices to prove

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left(g\left(\frac{S_k}{\sigma_k}\right) - \mathrm{E}g\left(\frac{S_k}{\sigma_k}\right)\right) = 0 \text{ a.s.,} \tag{5}$$

for any $g(x)$ which is a bounded function with bounded continuous derivatives.

Let $k \ge 1$, define

$$\xi_k = g\left(\frac{S_k}{\sigma_k}\right) - \mathrm{E}g\left(\frac{S_k}{\sigma_k}\right).$$

For any $1 \le 2k < l$, we get,

$$\begin{aligned} \left|\mathrm{E}\xi_k\xi_l\right| &= \left|\mathrm{cov}\left(g\left(\frac{S_k}{\sigma_k}\right), g\left(\frac{S_l}{\sigma_l}\right)\right)\right| \\ &= \left|\mathrm{cov}\left(g\left(\frac{S_k}{\sigma_k}\right), g\left(\frac{S_l}{\sigma_l}\right) - g\left(\frac{\sum_{i=2k+1}^{l} X_i}{\sigma_l}\right)\right)\right| + \left|\mathrm{cov}\left(g\left(\frac{S_k}{\sigma_k}\right), g\left(\frac{\sum_{i=2k+1}^{l} X_i}{\sigma_l}\right)\right)\right| \\ &:= I_1 + I_2. \end{aligned} \tag{6}$$

Firstly we estimate I_1. By Lemma 1 $\frac{\sigma_n^2}{n} \to \sigma^2$, we note that certain $n_0 \in N$, $0 < \varepsilon < \sigma$ exist such that $\frac{1}{\sigma_n} \le \frac{1}{(\sigma - \varepsilon)\sqrt{n}}$ as $n > n_0$. Since g is a bounded Lipschitz function, *i.e.*, there exists a constant $c > 0$ such that $|g(x)| \le c$, $|g(x) - g(y)| \le c|x - y|$ for any $x, y \in \mathrm{R}$. By Jensen inequality, Lemma 2 and $\sigma < \infty$, we obtain that

$$I_1 \le c \frac{\mathrm{E}\left|\sum_{i=1}^{2k} X_i\right|}{\sqrt{l}} \le c \frac{\sqrt{\mathrm{E}\left(\sum_{i=1}^{2k} X_i\right)^2}}{\sqrt{l}} \le c \frac{\sqrt{\left(\sum_{i=1}^{2k} \mathrm{E}X_i^2\right)}}{\sqrt{l}} \le c \frac{\sqrt{\left(\sum_{i=1}^{2k} \mathrm{E}X_1^2\right)}}{\sqrt{l}} \le c \left(\frac{k}{l}\right)^{1/2}. \tag{7}$$

Now we estimate I_2. Note that g is a bounded function with bounded continuous derivatives, so, by Lemma

3, we have

$$I_2 \le c\rho^-(k). \tag{8}$$

So if $l > 2k$, combining with (6), (7), (8), we obtain

$$\left|\mathrm{E}\xi_k\xi_l\right| \le c\left(\left(\frac{k}{l}\right)^{1/2} + \rho^-(k)\right).$$

By Lemma 4, (5) holds.
This completes the proof of Theorem 1.1.

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