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Local Study of Scalar Curvature of Cyclic Surfaces Obtained by Homothetic Motion of Lorentzian Circle

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Abstract

In this paper we consider the homothetic motion of Lorentzian circle by studying the scalar curvature for the corresponding cyclic surface locally. We prove that if the scalar curvature $\mathcal{K}$ is constant, then $\mathcal{K} = 0$. We describe the equations that govern such surfaces.

Keywords

Minkowski Space, Cyclic Surfaces, Homothetic Motion, Scalar Curvature

1. Introduction

Homothetic motion is general form of Euclidean motion. It is crucial that homothetic motions are regular motions. These motions have been studied in kinematic and differential geometry in recent years. An equiform transformation in the n-dimensional Euclidean space $\mathbb{R}^n$ is an affine transformation whose linear part is composed from an orthogonal transformation and a homothetical transformation add see [1]-[3]. Such an equiform transformation maps points $\mathbf{x} \in \mathbb{R}^n$ according to

$$x \mapsto s\mathcal{A}x + d, \quad \mathcal{A} \in SO(n), s \in \mathbb{R}^+, d \in \mathbb{R}^n. \tag{1}$$

The number s is called the scaling factor. A homothetic motion is defined if the parameters of (1), including s, are given as functions of a time parameter t. Then a smooth one-parameter equiform motion moves a point x via $x(t) = s(t)\mathcal{A}(t)x(t) + d(t)$. The kinematic corresponding to this transformation group is called similarity kinematic. See [4]. Recently, the similarity kinematic geometry has been used in computer vision and reverse engineering of geometric models such as the problem of reconstruction of a computer model from an existing ob-

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ject which is known (a large number of) data points on the surface of the technical object [5] [6]. Abdel-All and Hamdoon studied a cyclic surface in $\mathbb{R}^5$. In this sense, they proved that such surface in $\mathbb{R}^5$ is in general contained in a canal hypersurface [7]. Solouma ([8]-[10]) studied locally some geometric problems on surfaces obtained by the equiform motion up to the first order. In Minkowski (semi-Euclidean) space, hyperbolas (Lorentzian circles) play role in Euclidean space [11].

In this work we consider the homothetic motion of the hyperbolas(Lorentzian circles) c_0. Let Σ^0 and Σ be two copies of Euclidean space $\mathbb{R}^n$. Under a one-parameter homothetic motion of moving space Σ^0 with respect to fixed space Σ, we consider $c_0 \subset \Sigma^0$ which is moved according homothetic motion. The point paths of the Lorentzian circle generate a cyclic surface *X*, containing the position of the starting Lorentzian circle. At any moment, the infinitesimal transformations of the motion will map the points of the Lorentzian circle c_0 into the velocity vectors whose end points will form an affine image of c_0 that will be, in general, a Lorentzian circle in the moving space Σ. Both curves are planar and therefore, they span a subspace W of $\mathbb{R}^n$, with $\dim(W) \le 5$. This is the reason because we restrict our considerations to dimension $n = 5$.

Let $x(\phi)$ be a parametrization of c_0 and $X(t,\phi)$ the resultant surface by the homothetic motion. We consider a certain position of the moving space, given by $t = 0$, and we would like to obtain information about the motion at least during a certain period around $t = 0$ if we know its characteristics for one instant. Then we restrict our study to the properties of the motion for the limit case $t \to 0$. A first choice is then approximate $X(t,\phi)$ by the first derivative of the trajectories. The purpose of this paper is to describe the cyclic surfaces obtained by the homothetic motion of the Lorentzian circle and whose scalar curvature $\mathcal{K}$ is constant.

The proof of our results involves explicit computations of the scalar curvature $\mathcal{K}$ of the surface $X(t,\phi)$. As we shall see, equation $\mathcal{K} = \text{constant}$ reduces to an expression that can be written as a linear combination of the hyperbolic functions $\cosh n\phi$ and $\sinh n\phi$, $n \in \mathbb{N}$, namely, $\sum_{n=1}^{4}(E_n \cosh n\phi + F_n \sinh n\phi) = 0$ and E_n and F_n are functions on the variable *t*. In particular, the coefficients must vanish. The work then is to compute explicitly these coefficients E_n and F_n by successive manipulations. The authors were able to obtain the results using the symbolic program Mathematica to check their work. The computer was used in each calculation several times, giving understandable expressions of the coefficients E_n and F_n.

This paper is organized as follows: In Section 2, we obtain the expression of the scalar curvature $\mathcal{K}$ for the cyclic surfaces obtained by homothetic motion of Lorentzian circle. In successive Sections 3 and 4, we distinguish the cases $\mathcal{K} = 0$ and $\mathcal{K} \ne 0$, respectively. Finally, in Section 5 explicit examples of surfaces with $\mathcal{K} = 0$ and $\mathcal{K} \ne 0$ are given.

2. Scalar Curvature of Cyclic Surfaces

In two copies Σ^0, Σ of semi-Euclidean 5-space $\mathbb{E}^5$, we consider a unit Lorentzian circle c_0 in the x_1x_2-plane of Σ^0 centered at the origin and represented by

$$x(\phi) = (\cosh\phi, \sinh\phi, 0, 0, 0)^{\mathrm{T}}, t, \phi \in \mathbb{R}.$$

Under a one-parameter homothetic motion of c_0 in the moving space Σ^0 with respect to fixed space Σ. The position of a point $x(\phi) \in \Sigma^0$ at "time" *t* may be represented in the fixed system as

$$X(t,\phi) = s(t)\mathcal{A}(t)x(\phi) + d(t), \ t \in I \subset \mathbb{R}, \phi \in \mathbb{R}, \tag{2}$$

where $d(t) = (b_1(t), b_2(t), b_3(t), b_4(t), b_5(t))^{\mathrm{T}}$ describes the position of the origin of Σ^0 at the time *t*, $\mathcal{A}(t) = (a_{ij}(t))$, $1 \le i, j \le 5$ is a semi orthogonal matrix and $s(t)$ provides the scaling factor of the moving system. For varying *t* and fixed $x(\phi)$, $X(t,\phi)$ gives a parametric representation of the path (or trajectory) of $x(\phi)$. Moreover we assume that all involved functions are of class C^1. Using the Taylor's expansion up to the first order, the representation of the cyclic surface is

$$X(t,\phi) = \left\{ s(0)\mathcal{A}(0) + \left[\dot{s}(0)\mathcal{A}(0) + s(0)\dot{\mathcal{A}}(0) \right] t \right\} x(\phi) + d(0) + t\dot{d}(0),$$

where $(\cdot)$ denotes the differentiation with respect to *t*.

As homothetic motion has an invariant point, we can assume without loss of generality that the moving frame Σ^0 and the fixed frame Σ coincide at the zero position $t = 0$. Then we have

$$\mathcal{A}(0)=I, s(0)=1 \text{ and } d(0)=0.$$

Thus

$$X(t,\phi)=\left[I+(s'I+\Omega)t\right]x(\phi)+td',$$

where $\Omega=\dot{\mathcal{A}}(0)=(\omega_k)$, $1\le k\le 10$ is a semi skew-symmetric matrix. In this paper all values of s, b_i and their derivatives are computed at $t=0$ and for simplicity, we write s' and b_i' instead of $\dot{s}(0)$ and $\dot{b}_i(0)$ respectively. In these frames, the representation of $X(t,\phi)$ is given by

$$\begin{pmatrix} X_1\\ X_2\\ X_3\\ X_4\\ X_5 \end{pmatrix}(t,\phi)=\begin{pmatrix} 1+s't & t\omega_1 & t\omega_2 & t\omega_3 & t\omega_4\\ t\omega_1 & 1+s't & t\omega_5 & t\omega_6 & t\omega_7\\ t\omega_2 & -t\omega_5 & 1+s't & t\omega_8 & t\omega_9\\ t\omega_3 & -t\omega_6 & -t\omega_8 & 1+s't & t\omega_{10}\\ t\omega_4 & -t\omega_7 & -t\omega_9 & -t\omega_{10} & 1+s't \end{pmatrix}\begin{pmatrix} \cosh\phi\\ \sinh\phi\\ 0\\ 0\\ 0 \end{pmatrix}+t\begin{pmatrix} b_1'\\ b_2'\\ b_3'\\ b_4'\\ b_5' \end{pmatrix},$$

or in the equivalent form

$$\begin{pmatrix} X_1\\ X_2\\ X_3\\ X_4\\ X_5 \end{pmatrix}(t,\phi)=\begin{pmatrix} 1+s't\\ t\omega_1\\ t\omega_2\\ t\omega_3\\ t\omega_4 \end{pmatrix}\cosh\phi+\begin{pmatrix} t\omega_1\\ 1+s't\\ -t\omega_5\\ -t\omega_6\\ -t\omega_7 \end{pmatrix}\sinh\phi+t\begin{pmatrix} b_1'\\ b_2'\\ b_3'\\ b_4'\\ b_5' \end{pmatrix}. \tag{3}$$

For any fixed *t* in the above expression (3), we generally get an ellipse centered at the point $t(b_1',b_2',b_3',b_4',b_5')$. The latter ellipse reduce to a Lorentzian circle subject to the following conditions

$$\begin{aligned} &\omega_2\omega_5+\omega_3\omega_6+\omega_4\omega_7=0,\\ &\omega_1^2+\omega_2^2+\omega_3^2+\omega_4^2=\omega_1^2-\omega_5^2-\omega_6^2-\omega_7^2=a, \end{aligned} \tag{4}$$

where $a\in\mathbb{R}^+$. We now compute the scalar curvature of the cyclic surface $X(t,\phi)$. The tangent vectors to the parametric curves of $X(t,\phi)$ are

$$X_t(t,\phi)=(s'I+\Omega)x(\phi)+d', \quad X_\phi(t,\phi)=\left[I+(s'I+\Omega)t\right]x'(\phi).$$

A straightforward computation leads to the coefficients of the first fundamental form defined by $g_{11}=X_tX_t^{\mathrm{T}}$, $g_{12}=X_\phi X_t^{\mathrm{T}}$, $g_{22}=X_\phi X_\phi^{\mathrm{T}}$. The scalar product in the above equation in Lorentzian metric. According to the inner product this equation tends to $g_{11}=X_t\varepsilon X_t^{\mathrm{T}}$, $g_{12}=X_\phi\varepsilon X_t^{\mathrm{T}}$, $g_{22}=X_\phi\varepsilon X_\phi^{\mathrm{T}}$ where

$$\varepsilon=\begin{pmatrix} -1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is the sign matrix. Then we get

$$\begin{aligned} g_{11}&=\left[(s'I-\Omega)x^{\mathrm{T}}(\phi)+d'^{\mathrm{T}}\right]\varepsilon\left[(s'I+\Omega)x(\phi)+d'\right],\\ g_{12}&=x'^{\mathrm{T}}(\phi)\varepsilon\left[(s'I+\Omega)x(\phi)+d'\right],\\ g_{22}&=x'^{\mathrm{T}}(\phi)\varepsilon x'(\phi). \end{aligned}$$

Under the conditions (4) a computation yields

$$\begin{aligned} g_{11} &= \alpha + \beta \cosh \phi + \gamma \sinh \phi, \\ g_{12} &= \omega_1 - \left(b_1' + \frac{1}{2} \beta t \right) \sinh \phi + \left(b_2' - \frac{1}{2} \gamma t \right) \cosh \phi, \\ g_{22} &= 1 + 2s't + t^2 \left(s'^2 - a \right), \end{aligned} \tag{5}$$

and

$$\begin{aligned} \alpha &= -s'^2 - \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + b_1'^2 + b_2'^2 + b_3'^2 + b_4'^2 + b_5'^2 + a, \\ \beta &= 2\left(-s'b_1' + \omega_1 b_2' + \omega_2 b_3' + \omega_3 b_4' + \omega_4 b_5' \right), \\ \gamma &= -2\left(-s'b_2' + \omega_1 b_1' + \omega_5 b_3' + \omega_6 b_4' + \omega_7 b_5' \right). \end{aligned} \tag{6}$$

The Christoffel symbols of the second kind are defined by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{m=1}^{2} g^{km} \left(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m} \right),$$

where $x_i \in \{t, \phi\}$, $\{i, j, k\}$ are indices that take the value 1 or 2 and $\left(g^{lm} \right)$ is the inverse matrix of $\left(g_{ij} \right)$. From here, the scalar curvature of $X(t, \phi)$ is defined by

$$\mathcal{K} = \sum_{i,j,l=1}^{2} g^{ij} \left[\frac{\partial \Gamma_{ij}^l}{\partial x_l} - \frac{\partial \Gamma_{il}^l}{\partial x_j} + \sum_{m=1}^{2} \left(\Gamma_{ij}^l \Gamma_{lm}^m - \Gamma_{il}^m \Gamma_{jm}^l \right) \right].$$

Although the explicit computation of the scalar curvature $\mathcal{K}$ can be obtained, for example, by using the Mathematica programme, its expression is some cumbersome. However, the key in our proofs lies that one can write $\mathcal{K}$ as

$$\mathcal{K} = \frac{\mathcal{P}(\cosh n\phi, \sinh n\phi)}{\mathcal{Q}(\cosh n\phi, \sinh n\phi)} = \frac{\sum_{n=0}^{2} \left(A_n \cosh n\phi + B_n \sinh n\phi \right)}{\sum_{n=0}^{4} \left(C_n \cosh n\phi + D_n \sinh n\phi \right)}. \tag{7}$$

The assumption of the constancy of the scalar curvature $\mathcal{K}$ implies that (7) converts into

$$\mathcal{K} \mathcal{Q}(\cosh n\phi, \sinh n\phi) - \mathcal{P}(\cosh n\phi, \sinh n\phi) = 0. \tag{8}$$

Equation (8) means that if we write it as a linear combination of the functions $\{\cosh n\phi, \sinh n\phi\}$ namely, $\sum_{n=0}^{4} \left(E_n \cosh n\phi + F_n \sinh n\phi \right) = 0$, the corresponding coefficients must vanish. From here, we will be able to describe all cyclic surfaces with constant scalar curvature obtained by the homothetic motion of the Lorentzian circle c_0. As we will see, it is not necessary to give the (long) expression of $\mathcal{K}$ but only the coefficients of higher order for the hyperbolic functions.

We distinguish the cases $\mathcal{K} = 0$ and $\mathcal{K} \neq 0$.

3. Cyclic Surfaces with $\mathcal{K} = 0$

In this section we assume that $\mathcal{K} = 0$ on the surface $X(t, \phi)$. From (7), we have

$$\left. \begin{aligned} \mathcal{P}(\cosh n\phi, \sinh n\phi) &= \sum_{n=0}^{2} \left(A_n \cosh n\phi + B_n \sinh n\phi \right) = 0 \\ \mathcal{Q}(\cosh n\phi, \sinh n\phi) &= \sum_{n=0}^{4} \left(C_n \cosh n\phi + D_n \sinh n\phi \right) \neq 0 \end{aligned} \right\} \tag{9}$$

We distinguish different cases that fill all possible cases (Note that we have all solutions by using the symbolic program Mathematica under the condition $s' \neq 0$).

3.1. Case $b_1' = b_2' = 0$

At $\beta = 0$ and $\gamma = 0$, the coefficients $A_n = B_n = 0$ for $0 \le n \le 2$ and the coefficients $C_n = D_n = 0$ for $0 < n \le 4$. Also, since $\sinh 0 = 0$ implies that $D_0 = 0$. But $C_0 = -4\alpha\omega_1^2 + 2\omega_1^4 = 0$ if and only if $\alpha = \omega_1^2$. That's means $\mathcal{Q}(\cosh n\phi, \sinh n\phi) = 0$ gives contradiction with Equation (9), so we have $\alpha \ne \omega_1^2$. We then conclude the following theorem.

Theorem 3.1. Let $X(t,\phi)$ be a cyclic surfaces obtained by the homothetic motion of Lorentzian circle c_0 and given by (3) under condition (4). Assume $b_1' = b_2' = 0$, then $\mathcal{K} = 0$ on the surface if and only if the following conditions hold:

1) $\alpha \ne \omega_1^2$

2) $b_3'\omega_2 + b_4'\omega_3 + b_5'\omega_4 = 0, b_3'\omega_5 + b_4'\omega_6 + b_5'\omega_7 = 0,$

In particular, if $b_i' = 0$ for $3 \le i \le 5$, then circles generating the cyclic surfaces are coaxial.

3.2. Case $b_1'b_2' = 0$, But either b_1' or b_2' Is Not Zero

We have two possibilities:

1) If $b_1' \ne 0$ and $b_2' = 0$, then we have $\beta = 0$, $\gamma = 0$, the coefficients $A_n = B_n = 0$ for $0 \le n \le 2$ and the coefficients $C_4 = \frac{1}{4} b_1'^4$ that's means the equation $\mathcal{Q}(\cosh n\phi, \sinh n\phi) \ne 0$. From expression (6), we have two conditions

$$b_3'\omega_2 + b_4'\omega_3 + b_5'\omega_4 = b_1's'$$
$$b_1'\omega_1 + b_3'\omega_5 + b_4'\omega_6 + b_5'\omega_7 = 0$$

2) If $b_1' = 0$ and $b_2' \ne 0$, then we have $\beta = 0$, $\gamma = 0$, the coefficients $A_n = B_n = 0$ for $0 \le n \le 2$ and the coefficients $C_4 = \frac{1}{4} b_2'^4$ that's means the equation $\mathcal{Q}(\cosh n\phi, \sinh n\phi) \ne 0$. From expression (6), we have

$$b_3'\omega_5 + b_4'\omega_6 + b_5'\omega_7 = b_2's'$$
$$b_2'\omega_1 + b_3'\omega_2 + b_4'\omega_3 + b_5'\omega_4 = 0$$

Theorem 3.2. Let $X(t,\phi)$ be a cyclic surfaces obtained by the homothetic motion of Lorentzian circle c_0 and given by (3) under condition (4) hold:

1) Assume $b_1' \ne 0$ and $b_2' = 0$, then $\mathcal{K} = 0$ on the surface if and only if the following conditions

$$b_3'\omega_2 + b_4'\omega_3 + b_5'\omega_4 = b_1's',$$
$$b_1'\omega_1 + b_3'\omega_5 + b_4'\omega_6 + b_5'\omega_7 = 0.$$

2) Assume $b_1' = 0$ and $b_2' \ne 0$, then $\mathcal{K} = 0$ on the surface if and only if the following conditions

$$b_3'\omega_5 + b_4'\omega_6 + b_5'\omega_7 = b_2's',$$
$$b_2'\omega_1 + b_3'\omega_2 + b_4'\omega_3 + b_5'\omega_4 = 0.$$

3.3. Case $b_1'b_2' \ne 0$

If $b_1'b_2' \ne 0$, then we have $\beta = \gamma = 0$, then coefficients $A_n = B_n = 0$ for $0 \le n \le 2$, $C_n \ne 0$ and $D_n \ne 0$ for $0 \le n \le 4$ that's means the equation (8) hold (*i.e.*, $\mathcal{Q}(\cosh n\phi, \sinh n\phi) \ne 0$). From expression (6), we have the two conditions

$$b_1'\omega_1 + b_3'\omega_2 + b_4'\omega_3 + b_5'\omega_4 = b_1's'$$
$$b_2'\omega_1 + b_3'\omega_5 + b_4'\omega_6 + b_5'\omega_7 = b_2's'$$

Theorem 3.3. Let $X(t,\phi)$ be a cyclic surfaces obtained by the homothetic motion of Lorentzian circle c_0 and given by (3) under condition (4). Assume $b_1'b_2' \ne 0$, then $\mathcal{K} = 0$ on the surface if and only if the following conditions hold:

1) $b_1'\omega_1 + b_3'\omega_2 + b_4'\omega_3 + b_5'\omega_4 = b_1's',$

2) $b_2'\omega_1 + b_3'\omega_5 + b_4'\omega_6 + b_5'\omega_7 = b_2's'$.

4. Cyclic Surfaces with $\mathcal{K} \neq 0$

In this section we assume that the scalar curvature $\mathcal{K}$ of the cyclic surface $X(t,\phi)$ obtained by the homothetic motion of Lorentzian circle c_0 and given by (3) under condition (4) is a non-zero constant. The identity (8) writes then as

$$\sum_{n=0}^{4}\left(E_n(t)\cosh n\phi + F_n(t)\sinh n\phi\right) = 0. \tag{10}$$

Following the same scheme as in the case $\mathcal{K}=0$ studied in Section 3, we begin to compute the coefficients E_n and F_n. Let us put $t=0$.

1) CASE $b_1' = b_2' = 0$. The coefficients E_0, E_2 and F_2 are

$$E_0 = 2\mathcal{K}\left(\alpha^2 - 2\alpha\omega_1^2 + \omega_1^4\right) + \left(\beta^2 - \gamma^2\right)\left(\frac{3}{2} + \mathcal{K}\right),$$

$$E_2 = \left(\beta^2 + \gamma^2\right)\left(\frac{3}{2} + \mathcal{K}\right),$$

$$F_2 = 2\beta\gamma\left(\frac{3}{2} + \mathcal{K}\right).$$

If $E_0 = E_2 = F_2 = 0$, we distinguish different possibilities:

1. $\alpha = \omega_1^2$, $\gamma = -\beta$, $\mathcal{K} = \frac{-3}{2}$, we conclude that

$$2a = -b_3'^2 - b_4'^2 - b_5'^2 + 3\omega_1^2 + s'^2$$
$$b_3'\omega_2 + b_4'\omega_3 + b_5'\omega_4 = b_3'\omega_5 + b_4'\omega_6 + b_5'\omega_7$$

2. $\alpha = \omega_1^2$, $\gamma = \beta$ and $\mathcal{K} = \frac{-3}{2}$, we have the same result as in the above case.

3. $\alpha = \omega_1^2$, $\gamma = \beta = 0$ and $\mathcal{K} = \frac{-3}{2}$, we have the same result as in cases from (1) and (2).

From (1), (2) and (3) we have $\alpha = \omega_1^2$, $\gamma = \pm\beta$, $\mathcal{K} = \frac{-3}{2}$ under the following conditions

$$2a = -b_3' - b_4'^2 - b_5'^2 + 3\omega_1^2 + s'^2$$
$$b_3'\omega_2 + b_4'\omega_3 + b_5'\omega_4 = b_3'\omega_5 + b_4'\omega_6 + b_5'\omega_7$$

4. $\alpha = \omega_1^2$, $\gamma = \beta = 0$, $\mathcal{K} = -1$. The coefficients E_1 and F_1 are

$$E_1 = 4\beta(1+\mathcal{K})\left(\alpha - \omega_1^2\right),$$

$$F_1 = 4\gamma(1+\mathcal{K})\left(\alpha - \omega_1^2\right).$$

If $E_1 = F_1 = 0$, we have the following conditions

$$2a = -b_3' - b_4'^2 - b_5'^2 + 3\omega_1^2 + s'^2$$
$$b_3'\omega_2 + b_4'\omega_3 + b_5'\omega_4 = b_3'\omega_5 + b_4'\omega_6 + b_5'\omega_7$$

2) CASE $b_1'b_2' = 0$, but either b_1' or b_2' is not zero. We have two possibilities:

1. If $b_1' \neq 0$ and $b_2' = 0$, then the coefficient $E_4 = \frac{1}{4}b_1'^4\mathcal{K} = 0$, implies that $b_1' = 0$: contradiction

2. If $b_2' \neq 0$ and $b_1' = 0$, then the coefficient $E_4 = \frac{1}{4} b_2'^4 \mathcal{K} = 0$, implies that $b_2' = 0$ which gives a contradiction also.

3) CASE $b_1' b_2' \neq 0$. The computations of $F_4 = -b_1' b_2' \left(b_1'^2 + b_2'^2\right) \mathcal{K} = 0$ implies that $b_1' = b_2' = 0$, contradiction. As conclusion of the above reasoning, we conclude the following theorem.

Theorem 4.1. Let $X(t,\phi)$ be a cyclic surfaces obtained by the homothetic motion of Lorentzian circle c_0 and given by (3) under condition (4). Assume that $b_1' = b_2' = 0$, then the scalar curvature $\mathcal{K} = -1$ or $\mathcal{K} = \frac{-3}{2}$ on the surface if and only if the following conditions hold:

$$2a = -b_3' - b_4'^2 - b_5'^2 + 3\omega_1^2 + s'^2$$
$$b_3'\omega_2 + b_4'\omega_3 + b_5'\omega_4 = b_3'\omega_5 + b_4'\omega_6 + b_5'\omega_7$$

5. Examples of a Cyclic Surfaces with $\mathcal{K} = 0$ and $\mathcal{K} \neq 0$

In this section, we construct two examples of a cyclic surfaces $X(t,\phi)$ with constant scalar curvature $\mathcal{K} = 0$ and $\mathcal{K} \neq 0$. The first example corresponds $\mathcal{K} = 0$ with the case $b_1' b_2' \neq 0$. In the second example, we assume $\mathcal{K} \neq 0$ and $b_1' = b_2' = 0$.

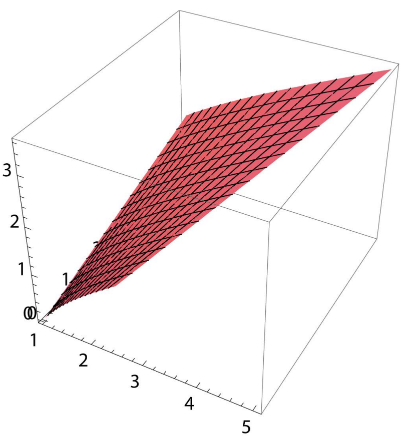
Example 1. Case $b_1' b_2' \neq 0$. Let now the semi orthogonal matrix

$$\mathcal{A}(t) = \begin{pmatrix} \cosh t & \sinh t & 0 & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 & 0 \\ \sin t \sinh t & 0 & \cos t & \sin t \cosh t & 0 \\ 0 & 0 & -\sin t & \cos^2 t & -\sin t \cos t \\ 0 & 0 & 0 & \sin t & \cos t \end{pmatrix}, \tag{11}$$

We assume $s(t) = \mathrm{e}^t$ and $d(t) = (t,t,0,0,0)^{\mathrm{T}}$, then

$$\omega_1 = \omega_8 = 1,\ \omega_{10} = -1 \text{ and } \omega_k = 0 \text{ for } k = 2,3,4,5,6,7,9,$$
$$s' = 1,$$
$$b_1' = b_2' = 1; b_3' = b_4' = b_5' = 0.$$

Theorem 3.3 says that $\mathcal{K} = 0$. In **Figure 1**, we display a piece of $X(t,\phi)$ of Example 1 in axonometric viewpoint $Y(t,\phi)$. For this, the unit vectors $E_4 = (0,0,0,1,0)$ and $E_5 = (0,0,0,0,1)$ are mapped onto the vectors $(1,1,0)$ and $(0,1,1)$ respectively [2]. Then



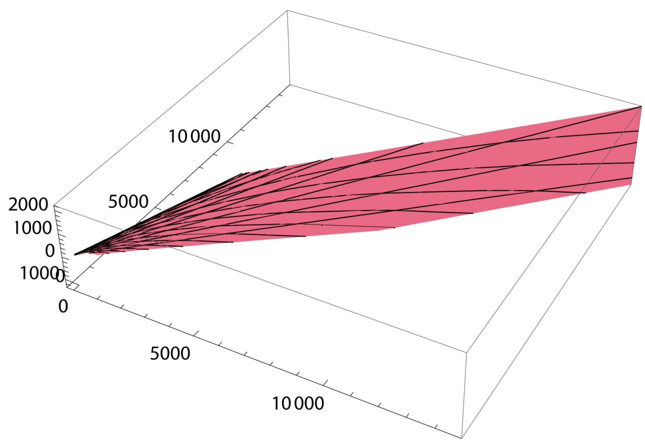


Figure 1. In (a), we have a piece of a cyclic surface foliated by a Lorentzian circle in axonometric view $Y(t,\phi)$ with zero scalar curvature $(\mathcal{K} = 0)$; in (b) we have the corresponding surface $X(t,\phi)$ with Equation (2) that approximates.

$$X(t,\phi)=\begin{pmatrix}t\\t\\0\\0\\0\end{pmatrix}+\begin{pmatrix}1+t\\t\\0\\0\\0\end{pmatrix}\cosh\phi+\begin{pmatrix}t\\1+t\\0\\0\\0\end{pmatrix}\sinh\phi$$

and

$$Y(t,\phi)=\begin{pmatrix}t\\t\\0\end{pmatrix}+\begin{pmatrix}1+t\\t\\0\end{pmatrix}\cosh\phi+\begin{pmatrix}t\\1+t\\0\end{pmatrix}\sinh\phi$$

and both $X(t,\phi)$ and $Y(t,\phi)$ parametrize domains of the x_1x_2-plane.

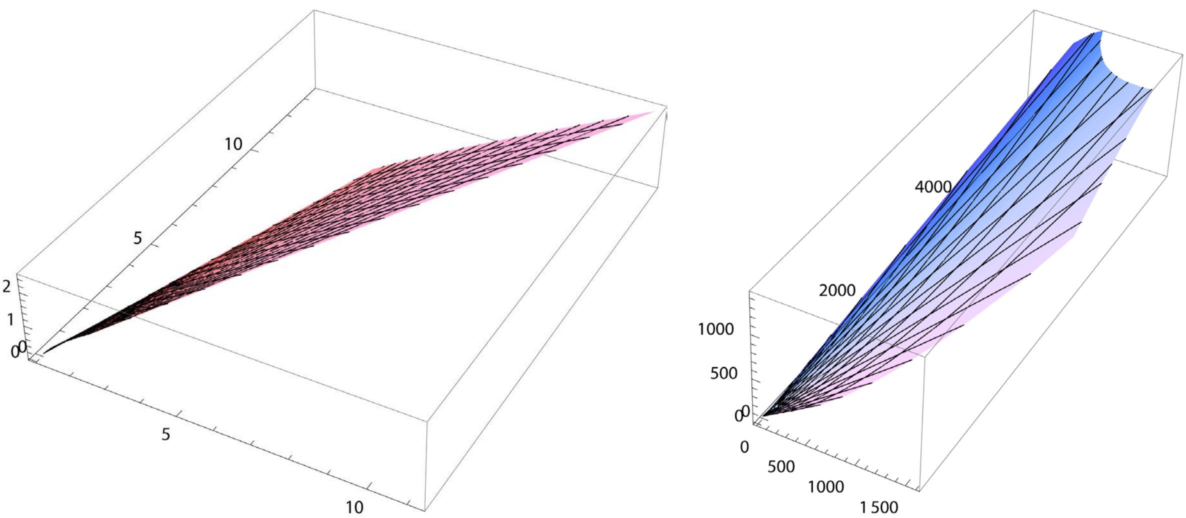
Example 2. Case $b_1'=b_2'=0$. Consider the semi orthogonal matrix

$$\mathcal{A}(t)=\begin{pmatrix}\cosh t & \cos t\sinh t & 0 & \sin t\sinh t & 0\\ \sinh t & \cosh t & 0 & 0 & 0\\ 0 & 0 & \cos^2 t & \sin t & \sin t\cos t\\ 0 & 0 & -\sin t & \cos t & 0\\ \sin t\sinh t & 0 & -\sin t\cosh t & 0 & \cos t\end{pmatrix}, \tag{12}$$

Let $s(t)=\mathrm{e}^t$ and $d(t)=(0,0,t,t,t)^{\mathrm{T}}$, then

$$\omega_1=\omega_8=\omega_9=1 \text{ and } \omega_k=0 \text{ for } k=2,3,4,5,6,7,10,$$
$$s'=1,$$
$$b_1'=b_2'=0; b_3'=b_4'=b_5'=1.$$

Theorem 4.1 says that $\mathcal{K}=-1$ or $\mathcal{K}=\frac{-3}{2}$. In **Figure 2**, we display a piece of $X(t,\phi)$ of Example 2 in axonometric viewpoint $Y(t,\phi)$. Then



(a) (b)

Figure 2. In (a), we have a piece of a cyclic surface foliated by a Lorentzian circle in axonometric view $Y(t,\phi)$ with non-zero scalar curvature $(\mathcal{K}=-1)$; in (b) we have the corresponding surface $X(t,\phi)$ with Equation (2) that approximates.

$$X(t,\phi)=\begin{pmatrix}0\\0\\t\\t\\t\end{pmatrix}+\begin{pmatrix}1+t\\t\\0\\0\\0\end{pmatrix}\cosh\phi+\begin{pmatrix}t\\1+t\\0\\0\\0\end{pmatrix}\sinh\phi$$

and

$$Y(t,\phi)=t\begin{pmatrix}1\\2\\1\end{pmatrix}+\begin{pmatrix}1+t\\t\\0\end{pmatrix}\cosh\phi+\begin{pmatrix}t\\1+t\\0\end{pmatrix}\sinh\phi$$

and both $X(t,\phi)$ and $Y(t,\phi)$ parametrize domains of the x_1x_2-plane.

References

[1] Do Carmo, M. (1976) Differential Geometry of Curves and Surfaces. Prentice-Hall Inc. Englewood Cliffs, New Jersey.

[2] Gordon, V.O. and Sement Sov, M.A. (1980) A Course in Descriptive Geometry. Mir Publishers, Moscow.

[3] Jagy, W. (1998) Sphere Foliated Constant Mean Curvature Submanifolds. *The Rocky Mountain Journal of Mathematics*, **28**, 983-1015. http://dx.doi.org/10.1216/rmjm/1181071750

[4] Bottema, O. and Roth, B. (1990) Theoretical Kinematic. Dover Publications Inc., New York.

[5] Odehnal, B., Pottmann, H. and Wallner, J. (2006) Equiform Kinematics and the Geometry of Line Elements. *Beiträge zur Algebra und Geometrie*, **47**, 567-582.

[6] Pottmann, H. and Wallner, J. (2001) Computational Line Geometry. Springer Heidelberg Dordrecht, London, New York.

[7] Abdel-All, N.H. and Hamdoon, F.M. (2004) Cyclic Surfaces in E^5 Generated by Equiform Motions. *Journal of Geometry*, **79**, 1-11. http://dx.doi.org/10.1007/s00022-003-1682-2

[8] Solouma, E.M. (2015) Three Dimensional Surfaces Foliated by an Equiform Motion of Pseudohyperbolic Surfaces in $\mathbb{E}^7$. *JP Journal of Geometry and Topology*, Accepted (To appear).

[9] Solouma, E.M. (2012) Local Study of Scalar Curvature of Two-Dimensional Surfaces Obtained by the Motion of Circle. *Applied Mathematics and Computation*, **219**, 3385-3394. http://dx.doi.org/10.1016/j.amc.2012.09.066

[10] Solouma, E.M., *et al.* (2007) Three Dimensional Surfaces Foliated by Two Dimensional Spheres. *Journal of the Egyptian Mathematical Society*, **1**, 101-110.

[11] O'Neill, B. (1983) Semi-Riemannian Geometry with Application to Relativity. Academic Press, New York and London.